Analytic description of moduli spaces - Gluing

$\boxed{\text{Goal:}}$ Describe $\overline{\mathcal{M}}$ locally by Fredholm sections

* not necc. transverse

$$s \curvearrowright \begin{matrix} \mathcal{E} \\ \downarrow \\ \mathcal{B} \end{matrix}$$

* $s^{-1}(0) \underset{\text{locally homeo}}{\longleftrightarrow} \overline{\mathcal{M}}$ not $\cancel{s^{-1}(0)/\mathrm{Aut}}$

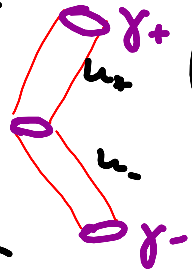
because generally do not expect Aut-equiv. $\pitchfork$

References:

* Salamon, Lecture Notes on Floer homology §3
* McDuff - Salamon, J-hol. curves and Symplectic Topology §10, A3

Tools for describing neighbourhoods of broken/nodal curves in moduli spaces

⓪ broken/nodal curves as fiber products

$\{$  $\}$

$$= \mathcal{M}(\gamma_-,\cdot) \times_{fib} \mathcal{M}(\cdot,\gamma_+) = \bigcup_{\gamma\in Per} \mathcal{M}(\gamma_-,\gamma)\times\mathcal{M}(\gamma,\gamma_+)$$

$[u_-] \in \mathcal{M}(\gamma_-,\cdot)$, $[u_+] \in \mathcal{M}(\cdot,\gamma_+)$

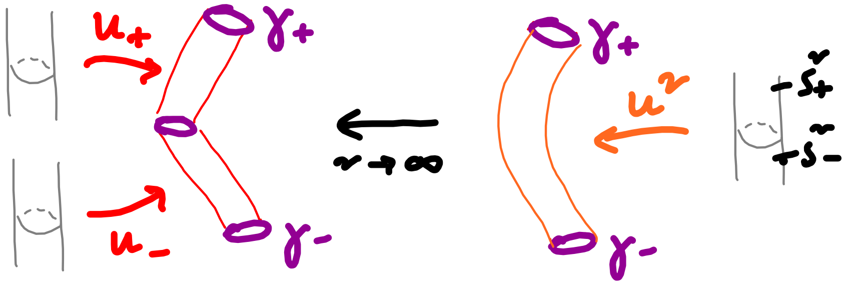
$[u_-] \mapsto \lim_{+\infty} u_-$ → periodic orbits ← $\lim_{-\infty} u_+ \mapsfrom [u_+]$

simplify:
- $\mathcal{M}(..)$ cut out transversally near u_-, u_+
- isolated broken curve

Recall: $\overline{\mathcal{M}} = \underbrace{\tilde{\mathcal{M}}/Aut}_{\mathcal{M}} \cup_{glue} \{\text{broken/nodal curves}\}$

⓪ broken / nodal curves as fiber products

⓪' broken / nodal curves as limits of smooth curves (maps modulo reparametrization)



$E(u^r) = E > 0$

$\exists\, s_+^r > s_-^r : \underline{s_+^r - s_-^r} \to \infty$

$u^r(s_\pm^r + \cdot, \cdot) \xrightarrow[r \to \infty]{} u^\pm$ $\quad C^\infty_{loc}$

$\Updownarrow$ elliptic estimates

$W^{k,p}_{loc}$

$kp > 2$

$E = \underbrace{E(u_+)}_{>0} + \underbrace{E(u_-)}_{>0}$

from diverging parts

$\Leftrightarrow$ convergence

$\overline{\mathcal{M}}$

<u>Goal 1</u>: compact metrizable topology on $\overline{\mathcal{M}}$ with

Goal 2:

$$\begin{tikzcd} \underbrace{\mathcal{M}\times\mathcal{M}}_{\text{fiber}}\times\{\text{gluing param}\} \arrow[r, "\text{pregluing}"] \arrow[dr, "\text{gluing}"'] & \text{maps } u \text{ with } \|\bar\partial_J u\| \text{ small} \arrow[d, "②"] \\ & \overline{\mathcal{M}} \supset \{\text{broken/nodal curves}\} \end{tikzcd}$$

$\{\text{broken/nodal curves}\} \times \{0\}$

② Newton Iteration ⇒ Prop^n [A.3.4 in McDuff-Salamon]

X, Y Banach spaces, $U \subset X$ open, $x_0 \in U$

$f: U \to Y$ $\mathcal{C}^1$, $df(x_0): X \to Y$ has bounded right inverse

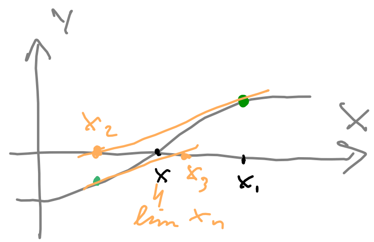
($Q: Y \to X$ s.t. $df(x_0)\circ Q = id_Y$)

$c, \delta > 0$ s.t. $B_\delta(x_0) \subset \mathcal{U}$

$$\left.\begin{array}{l} \|df(x) - df(x_0)\| \leq \frac{1}{2c} \quad \forall x \in B_\delta(x_0) \\ \|Qy\| \leq c\|y\| \quad \forall y \in Y \end{array}\right\} \Rightarrow \begin{array}{l} df(x) \text{ has} \\ \text{right inverse} \\ Q_x \approx Q \end{array}$$

$x_1 \in X$ "almost solution" $\left(\|x_1 - x_0\| \leq \delta/8, \ \|f(x_1)\| < \delta/4c \right)$

$\Rightarrow \exists!\ x \in X$ "nearby solution" $\left(\begin{array}{l} f(x) = 0 \\ x - x_1 \in \operatorname{im} Q \\ \|x - x_0\| \leq \delta \end{array} \right)$



In fact, $\|x - x_1\| \leq 2c\|f(x_1)\|$

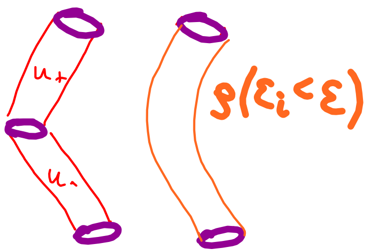
Goal 2: $\mathcal{M} \times_{\text{fibr}} \mathcal{M} \times \{\text{gluing param}\} \xrightarrow{\text{pregluing}}$ maps u with $\|\tilde{\partial}_J u\|$ small

$\downarrow$ ②

Goal 3: $\xrightarrow{g \text{ gluing map}} \overline{\mathcal{M}}$

- g injective
- g locally surjective : $[w]$ near $([u_-],[u_+]) \Rightarrow [w] \in \operatorname{im} g([u_-][u_+], \cdot)$
- Gromov compactness : $\overline{\mathcal{M}}' \setminus \operatorname{im} g$ is compact
- g compatible with smooth structure on $\mathcal{M} = \tilde{\mathcal{M}}/\text{Aut} \subset \overline{\mathcal{M}}$

$\Rightarrow$ **End result** (in Hamiltonian Floer theory, assuming $\pitchfork$)

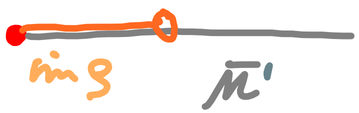


$([u_-],[u_+]) \in \bigcup_{\gamma} \mathcal{M}^\circ(\gamma_-,\gamma) \times \mathcal{M}^\circ(\gamma,\gamma_+)$

locally $g : [0,\varepsilon) \hookrightarrow \overline{\mathcal{M}}'(\gamma_-,\gamma_+)$, $\operatorname{im} g = \text{nbhd}([u_-][u_+])$

$\Rightarrow \overline{\mathcal{M}}' \setminus \operatorname{im} g$ compact 1-manifold

(or use g as chart to make $\overline{\mathcal{M}}'$ a compact 1-manifold)



$\Rightarrow 0 = \#\partial(\overline{\mathcal{M}}' \setminus \operatorname{im} g)$

$= \#\{\text{conn. components of } \operatorname{im} g\}$

$= \#\{\text{broken trajectories}\}$

$=$ contribution to $\partial^2 \gamma_-$ in γ_+

① pregluing

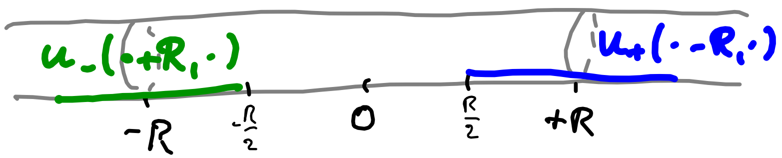
$\mathcal{M}\times_{\text{fiber}}\mathcal{M}\times\{\text{gluing param}\}\to$ maps u with $\|\bar\partial_J u\|$ small

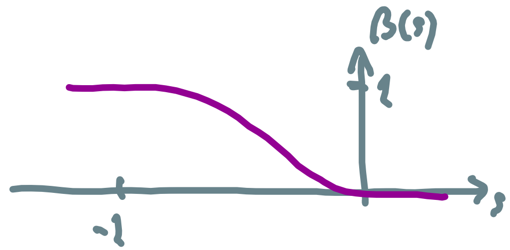
not quite

$$\bigcup_{\gamma} \mathcal{M}^\circ(\gamma_-,\gamma)\times\mathcal{M}^\circ(\gamma,\gamma_+)\times[0,e^{-R_0}) \longrightarrow \overline{\mathcal{M}}'(\gamma_-,\gamma_+)$$

$$([u_-],[u_+],e^{-R}) \longrightarrow \begin{cases}([u_-],[u_+]) & ;\ R=\infty\ (e^{-R}=0)\\ [\underbrace{\#_R(u_-,u_+)}_{w_R}] & ;\ R_0<R<\infty\end{cases}$$

$u_-|_{[R/2,\infty)}=\exp_\gamma(\eta_-)$ $\qquad$ $u_+|_{(-\infty,-R/2]}=\exp_\gamma(\eta_+)$ $\qquad$ $\eta_\pm(s,t)\in T_{\gamma(t)}M$





$$w_R(s,t)=\begin{cases} u_+(s+R,t) & ;\ s\geq \frac{R}{2}+1\\ \exp_{\gamma(t)}\left(\beta(-s-\frac{R}{2})\eta_+(s+R,t)\right) & ;\ \frac{R}{2}\leq s\leq \frac{R}{2}+1\\ \gamma(t) & ;\ -\frac{R}{2}\leq s\leq \frac{R}{2}\\ \exp_{\gamma(t)}\left(\beta(s+\frac{R}{2})\eta_-(s-R,t)\right) & ;\ -\frac{R}{2}-1\leq s\leq -\frac{R}{2}\\ u_-(s-R,t) & ;\ s\leq -\frac{R}{2}-1\end{cases}$$

① ✓ pregluing $\mathcal{M}\times_{\text{fdm}}\mathcal{M}\times\{\text{gluing param}\}\to$ maps u with $\|\bar\partial_J u\|$ small

② $\underline{\text{Prop}^{\underline{n}}}$: $f_R : X \supset U \to Y$ $\mathcal{C}^1$, $\underset{=0}{x_0} \in U = B_\Delta \subset W^{1,p}(w_R^* TM)$

$\forall R \geq R_1$ $\qquad \zeta \mapsto \bar\partial_{J,H}(\exp_{w_R}(\zeta))$

$df(x_0) : X \to Y$ has bounded right inverse $Q : Y \to X$ s.t. $df(x_0)\circ Q = id_Y$

"$D_{w_R}\bar\partial_{J,H}$"

$c, \Delta > 0$ s.t. $\|Q\| \leq c$, $B_\Delta(x_0) \subset \mathcal{U}$, $\|df(x) - df(x_0)\| \leq \frac{1}{2c}$ $\forall x \in B_\Delta(x_0)$

$\underset{0=}{x_1} \in X$ "almost solution" $\left(\|x_1 - x_0\| \leq \Delta/8 \checkmark,\ \|\underbrace{f(x_1)}_{\bar\partial_{J,H} w_R}\| < \Delta/4c\right)$

$\Rightarrow \exists!\ \underset{\zeta_R}{x} \in X$ "nearby solution" $\left(\begin{array}{ll} \underbrace{f(x)=0}_{\bar\partial_J \exp_{w_R}(\zeta_R)=0} & \zeta = x - x_1 \in \operatorname{im} Q \\ & \|x - \underset{=\zeta}{x_0}\| \leq \Delta \end{array}\right)$

$\|\underbrace{x - x_1}_{\zeta_R}\| \leq 2c\,\|\underbrace{f(x_1)}_{\bar\partial_{JH} w_R}\|$

Gluing in Hamiltonian Floer Theory

$$u_\pm : \mathbb{R}\times S^1 \to M, \quad \bar\partial_{J,H} u_\pm = 0, \quad \lim_{s\to+\infty} u_-(s,\cdot) = \gamma = \lim_{s\to-\infty} u_+(s,\cdot)$$

$\bar\partial_{J,H} = \partial_s + J(\partial_t - X_H)$

exp. decay: $|\partial_s u_\pm| \leq C e^{-\delta|s|}, \delta>0$ $\quad \sim |\eta_\pm|$

$$D_\pm := D_{u_\pm}\bar\partial_{J,H} : W^{1,p}(\mathbb{R}\times S^1, u_\pm^* TM) \to L^p(\mathbb{R}\times S^1, u_\pm^* TM)$$

surjective

$\ker D_\pm = \mathbb{R}\,\partial_s u_\pm$

$\Updownarrow$

index 1

isolated mod $\mathbb{R}$

$$\xi \mapsto \nabla_s \xi + J(u)\nabla_t \xi + \nabla_\xi J(u)\partial_t u - \nabla_\xi \nabla H(u)$$

$D_\pm^* : L^{p^*} \to (W^{1,p})^*$

$\quad\;\cup \qquad\quad \cup$

$W^{2,p} \to W^{1,p}$

$$D_\pm^* \xi = -\nabla_s \xi + J(u)\nabla_t \xi + \ldots$$

$$\Rightarrow D^* D = -\partial_s^2 - \partial_t^2 + \ldots$$

im D^* complement to $\ker D$

$$w_R = \#_R(u_-, u_+)$$

$$f_R : \underbrace{W^{1,p}(\mathbb{R}\times S^1, w_R^* TM)}_{X} \to \underbrace{L^p(\mathbb{R}\times S^1, w_R^* TM)}_{Y}$$

$$\xi \mapsto \Phi(\xi)^{-1}\left(\bar\partial_{J,H} \exp_{w_R}(\xi)\right)$$

Φ: parallel transport along $\exp_{w_R}(\tau\cdot\xi)$ $\tau\in[0,1]$

Claim: $\exists c_0, c_1, c_2 : \forall R \geq R_1$

(a) $\| f_R(0) \|_{L^p} \leq c_0 e^{-\delta R}$ almost solution

(b) $\| \xi \|_{W^{1,p}} \leq c_1 \| df_R(0) \xi \|_{L^p} \quad \forall \xi \in \operatorname{im} D_R^*$ $\exists$ right inverse

(c) $\| f_R(\xi + \hat{\xi}) - f_R(\xi) - df_R(\xi)\hat{\xi} \|_{L^p} \leq c_2 \| \hat{\xi} \|^2_{W^{1,p}} \quad \forall \|\xi\|_{W^{1,p}}, \|\hat{\xi}\|_{W^{1,p}} \leq 1$

$\Downarrow$ "quadratic estimate" $\Rightarrow f_R \; C^1$

(c') $\| df_R(\xi)\hat{\xi} - df_R(0)\hat{\xi} \|_{L^p} \leq c_2 \|\xi\|_{W^{1,p}} \|\hat{\xi}\|_{W^{1,p}}$

To prove note

(a): $\bar{\partial}_{J,H} w_R = 0$ except $\bar{\partial}_{J,H} \exp_\gamma(\beta \cdot \eta_\pm(s \pm R))$, $\|\eta_\pm\| < c e^{-\delta |s|}$

(b) notes by Salamon

Newton Iterations ($\forall R$): $f_R : W^{1,p} \supset B_\Delta \to L^p$ $\quad \mathcal{C}^1$, $x_0 = 0 \subset B_\Delta$

$df_R(0)$ has bounded right inverse $\|Q\| \le c = C_1$

$\|df_R(\zeta) - df_R(0)\| \le \frac{1}{2c}$ $\forall \zeta \in B_\Delta$ by $(\mathcal{C}')$ with $C_2 \|\zeta\| \le {}^1/_{2C_1}$

$\le \Delta \longrightarrow$ pick $\Delta = \frac{1}{2C_1C_2}$

$\|f_R(x_1 = 0)\| \le \Delta/4c$ by (a) with $e^{-\delta R} \le \frac{1}{8C_1^2C_2C_0}$

$\Rightarrow \exists!\, \zeta_R \in \mathrm{im}\, Q \subset W^{1,p}$: $f_R(\zeta_R) = 0$, $\|\zeta_R\| \le \Delta$

take $R \ge R_1 := \delta^{-1} \ln 8C_1^2C_2C_0$

in fact $\|\zeta_R\| \le 2c\, \|f_R(0)\|$

$\Rightarrow$ gluing map $g(([u_-],[u_+]), R) = \exp_{w_R}(\zeta_R)$