Analytic description of moduli spaces - Gluing

References:

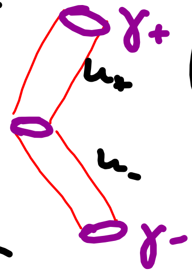
* Salamon, Lecture Notes on Floer homology §3

* McDuff - Salamon, J-hol. curves and Symplectic Topology §10, A3

Tools for describing neighbourhoods of broken/nodal curves in moduli spaces

(0) broken/nodal curves as fiber products

$$\left\{ \text{picture} \right\}$$



$$= \mathcal{M}(\gamma_-,\cdot) \times_{\text{fib}} \mathcal{M}(\cdot,\gamma_+) = \bigcup_{\gamma\in \text{Per}} \underbrace{\mathcal{M}(\gamma_-,\gamma)}_{=\bar\partial_J^{-1}(0)/\mathbb{R}} \times \underbrace{\mathcal{M}(\gamma,\gamma_+)}_{=\bar\partial_J^{-1}(0)/\mathbb{R}}$$

$\mathcal{M}(\gamma_-,\cdot) \ni [u_-] \mapsto \lim_{+\infty} u_-$ $\longrightarrow$ periodic orbits (finite) $\longleftarrow$ $\lim_{-\infty} u_+ \leftarrow\!\!\shortmid [u_+] \in \mathcal{M}(\cdot,\gamma_+)$

$$\left\{ \text{picture} \right\} = \bigcup_{A=A_++A_-} \mathcal{M}_{0,1}(A_-,J) \times \mathcal{M}_{0,2}(A_+,J) \underset{\text{Locally}}{\simeq} s^{-1}(0)$$

finite by Gromov comp.

$$\mathcal{M}_{0,1}(A_-,J) \xrightarrow{ev} M \xleftarrow{ev_1} \mathcal{M}_{0,2}(A_+,J)$$

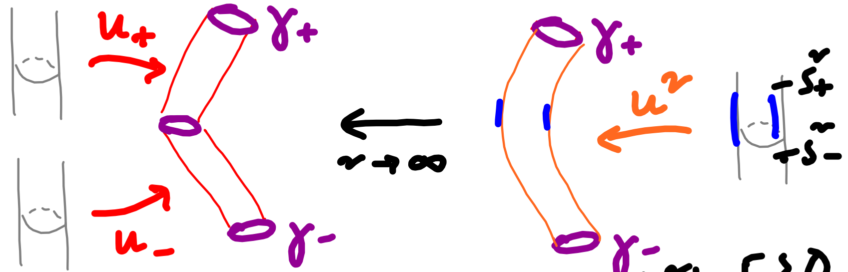
$$s(u_-,u_+) = \begin{pmatrix} \bar\partial_J u_- \\ \bar\partial_J u_+ \\ \text{"}u_-(\text{marked point}) - u_+(\text{marked point})\text{"} \end{pmatrix}$$

in local coordinates

Recall: $\overline{\mathcal{M}} = \underbrace{\tilde{\mathcal{M}}/\mathrm{Aut}}_{\mathcal{M}} \cup_{\text{glue}} \{\text{broken/nodal curves}\}$

⓪ broken/nodal curves as fiber products

⓪' broken/nodal curves as limits of smooth curves
(maps modulo reparametrization)



$E(u^\nu) = E > 0$

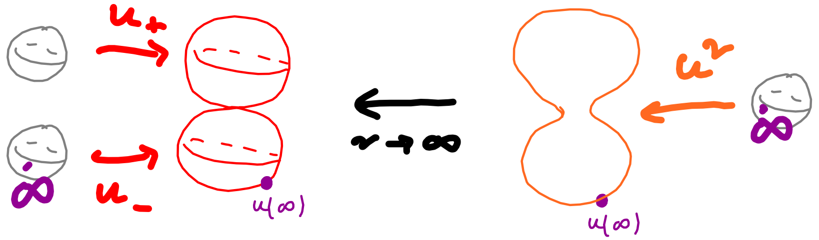
$\exists\ s_+^\nu > s_-^\nu$:

$u^\nu(s_\pm^\nu + \cdot, \cdot) \xrightarrow[\nu\to\infty]{} u_\pm$ C^∞_{loc}

$s_+^\nu - s_-^\nu \to \infty$

$E = E(u_+) + E(u_-)$

$\Rightarrow$ $\operatorname{im} u^\nu \downarrow$ Gromov-Hausdorff $\operatorname{im} u_+ \cup \operatorname{im} u_-$



$\exists\ \psi_+^\nu, \psi_-^\nu \in \mathrm{Aut}(\mathbb{P}^1)$ $\psi_-^\nu(\infty) = \infty$

$u^\nu \circ \psi_\pm^\nu \to u_\pm$ $C^\infty_{loc}(\mathbb{P}^1 \setminus \underset{\uparrow \text{ node}}{pt})$

$\int u^{\nu *}\omega = \int u_+^*\omega + \int u_-^*\omega$

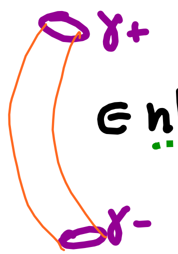
"Pregluing" is used to construct topology on $\overline{\mathcal{M}} = \tilde{\mathcal{M}}/\mathrm{Aut} \cup \left\{\begin{matrix}\text{broken/nodal}\\ \text{curves}\end{matrix}\right\}$

$[u^\nu] \xrightarrow[\nu\to\infty]{} [u_-, u_+]$

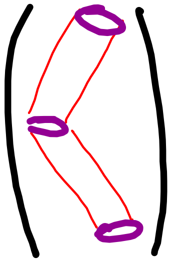
"Gluing" constructs "smooth structure" near $[u_-, u_+] \in \overline{\mathcal{M}}$

Goal 1

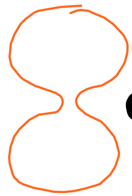
topology on $\overline{\mathcal{M}}$



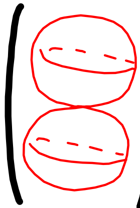
$\in$ nbhd $\Big($



$\Big) \subset \overline{\mathcal{M}}$



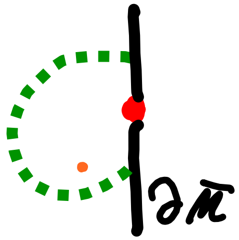
$\in$ nbhd $\Big($



$\Big) \subset \overline{\mathcal{M}}$

$\left\{\begin{matrix}\text{gluing}\\ \text{parameters}\end{matrix}\right\} = (R_0, \infty] \simeq [0, e^{-R_0})$

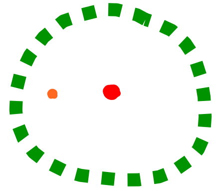
$\Uparrow$ boundary



$\left\{\begin{matrix}\text{gluing}\\ \text{parameters}\end{matrix}\right\} = (R_0, \infty) \times S^1 \cup \{\infty\}$

$\simeq \{ z \in \mathbb{C} \mid |z| < e^{-R_0} \}$

$\Uparrow$ no boundary



① pregluing

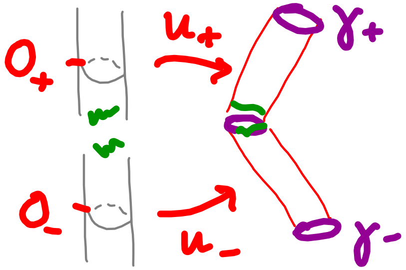
$$\underbrace{\mathcal{M} \underset{\text{fiber product}}{\times} \mathcal{M}}_{\text{broken/nodal}} \times \underset{\substack{\rotatebox{90}{$\in$} \\ 0}}{\left[\begin{matrix}\text{gluing}\\ \text{parameters}\end{matrix}\right]} \longrightarrow \{ \bar\partial_J u = \text{small} \}$$

$$([u_-], [u_+], 0) \longmapsto ([u_-], [u_+])$$

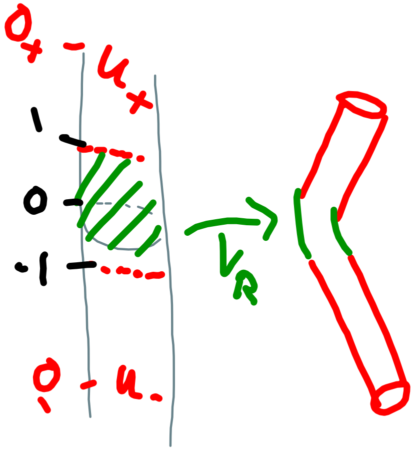
① pregluing

$\mathcal{M} \times_{\text{fiber product}} \mathcal{M} \times \left\{ \begin{smallmatrix}\text{gluing}\\ \text{parameters}\end{smallmatrix} \right\} \longrightarrow$ almost $\tilde{\mathcal{M}}$

$$\{\text{gluing parameters}\} \supset [0,\varepsilon) \cong (R_0, \infty] \ni R, \qquad \varepsilon = e^{-R_0}$$



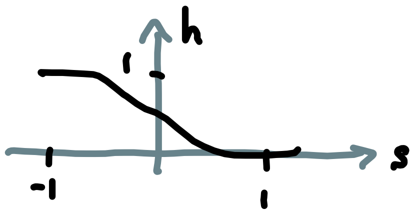
$$(u_-, u_+, R_{<\infty}) \longmapsto \underbrace{\#_R(u_-, u_+)}_{v_R} : \mathbb{R}\times S^1 \longrightarrow M$$



$$v_R(s,t) = \begin{cases} u_+(s-\frac{R}{2}, t) & ; \; s \geq 1 \\ h(s,t)\, u_-(s+\frac{R}{2}, t) + (1-h(s,t))\, u_+(s-\frac{R}{2}, t) & ; \; -1 < s < 1 \\ u_-(s+\frac{R}{2}, t) & ; \; s \leq 1 \end{cases}$$

in local coordinates (in fibers of $\gamma^* TM$)

Note
- $\to \bar\partial_J v_R = 0$
- $\to du_-, du_+, \bar\partial_J v_R$ all exp. small
- $\to \bar\partial_J v_R = 0$

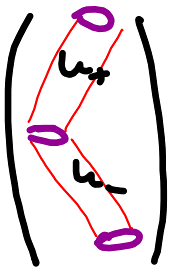


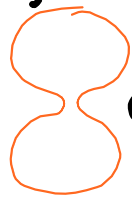
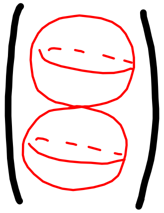
⚠ this is one of many possible interpolation schemes
key is to shift u_- & u_+ apart by R

(0) broken/nodal curves as fiber products $\overline{\mathcal{M}} \supset \tilde{\mathcal{M}}/\mathrm{Aut} = \overline{\mathcal{M}} \underset{\text{fiber product}}{\times} \overline{\mathcal{M}}$

(1) pregluing $\overline{\mathcal{M}} \underset{\text{fiber product}}{\times} \overline{\mathcal{M}} \times \left\{ \begin{smallmatrix} \text{gluing} \\ \text{parameters} \end{smallmatrix} \right\} \to$ almost $\overline{\mathcal{M}}$

$\to$ define (compact, <u>metrizable</u>) Gromov topology on $\overline{\mathcal{M}}$ by

$[V]$  γ_- $\in \underset{\cdots\cdots}{\varepsilon\text{-nbhd}}$  $\subset \overline{\mathcal{M}}$

$\}\to$  $\in \underset{\cdots\cdots}{\text{nbhd}}$  $\subset \overline{\mathcal{M}}$

$\overset{?}{\Updownarrow}$

$\exists\, V_-, V_+ : \mathbb{R}\times S^1 \to M,\quad d_{C^\infty}(V_-, u_-) < \varepsilon,\quad [V] = [\#_R(V_-, V_+)]$

$R > R_0 = -\ln\varepsilon \qquad d_{C^\infty}(V_+, u_+) < \varepsilon$

$\Downarrow$

note that $V|_{(-\infty,-\frac{R}{2}-1]}(\cdot + \tfrac{R}{2}, \cdot) = V_-|_{(-\infty,-1]} \overset{C^\infty}{\approx} u_-|_{(-\infty,-1]}$

$V|_{[\frac{R}{2}+1,\infty)}(\cdot - \tfrac{R}{2}, \cdot) = V_+|_{[1,\infty)} \overset{C^\infty}{\approx} u_+|_{[1,\infty)}$

as in convergence notion of (0')

Goal 2:

$$\begin{tikzcd} \underbrace{\mathcal{M}\times\mathcal{M}}_{\text{fiber}}\times\{\text{gluing param}\} \arrow[r, "\text{pregluing}"] \arrow[dr, "\text{gluing}"'] & \text{maps } u \text{ with } \|\bar\partial_J u\| \text{ small} \arrow[d, "2"] \\ & \overline{\mathcal{M}} \supset \{\text{broken/nodal curves}\} \end{tikzcd}$$

$\{\text{broken/nodal curves}\} \times \{0\}$

(2) Newton Iteration $\Rightarrow$ Prop^{n} [A.3.4 in McDuff-Salamon]

X, Y Banach spaces, $U \subset X$ open, $x_0 \in U$

$f : U \to Y$ $\mathcal{C}^1$, $df(x_0) : X \to Y$ has bounded right inverse

$(Q : Y \to X$ s.t. $df(x_0) \circ Q = id_Y)$

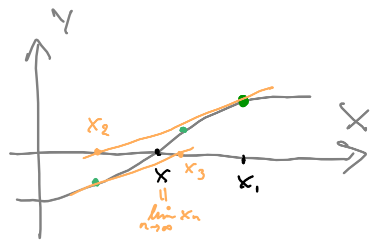
$c, \delta > 0$ s.t. $B_\delta(x_0) \subset \mathcal{U}$

$$\|df(x) - df(x_0)\| \leq \frac{1}{2c} \quad \forall x \in B_\delta(x_0)$$

$$\|Qy\| \leq c\|y\| \quad \forall y \in Y$$

$x_1 \in X$ "almost solution" $\left(\|x_1 - x_0\| \leq \delta/8 \, , \; \|f(x_1)\| < \delta/4c \right)$

$\Rightarrow \exists!\ x \in X$ "nearby solution" $\left(\begin{array}{l} f(x) = 0 \\ x - x_1 \in \operatorname{im} Q \\ \|x - x_0\| \leq \delta \end{array} \right)$



In fact, $\|x - x_1\| \leq 2c\,\|f(x_1)\|$

Goal 2: $$\begin{tikzcd} \mathcal{M} \times_{\text{fiber}} \mathcal{M} \times \{\text{gluing param}\} \arrow[r, "\text{pregluing}"] \arrow[dr, "g \text{ gluing map}"'] & \text{maps } u \text{ with } \|\bar{\partial}_J u\| \text{ small} \arrow[d, "2"] \\ & \overline{\mathcal{M}} \end{tikzcd}$$

Goal 3:

- g injective
- g locally surjective
- g compatible with smooth structure on $\mathcal{M} = \tilde{\mathcal{M}}/\mathrm{Aut} \subset \overline{\mathcal{M}}$