Polyfold-Fredholm theory

M-polyfolds
- retracts and splicings
- scale smooth maps between them
- a "finite dimensional" example
- pregluing as M-polyfold chart

literature:
- Hofer-Wysocki-Zehnder
- Hofer - surveys
- Fabert-Fish-Golovko-Wehrheim: "Polyfolds - A first and second look"

Definition 5.0.3. *An* **M-polyfold** *is a second countable and metrizable space* $\mathcal{X}$ *together with an open covering by the images of M-polyfold charts (see Definition 5.1.1), which are compatible in the sense that the transition map induced by the intersection of the images of any two charts is scale smooth (see Definition 5.2.3).*

$$
\begin{tikzcd}
\mathcal{X} = \bigcup_i \underset{\text{open}}{U_i} & \mathcal{O}_i = r_i(\mathcal{U}_i) \arrow[l, "\simeq"', "\phi_i"] \\
& \mathcal{U}_i \subset_{\text{open}} \mathbb{E}_i \arrow[u, "r_i"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
& \mathcal{O}_i & \mathcal{U}_i \subset_{\text{open}} \mathbb{E}_i \arrow[l] \\
& \cup_{\text{open}} & \cup_{\text{open}} \\
& \varphi_i^{-1}(U_j) \arrow[dl, "\varphi_i"'] & r_i^{-1}\varphi_i^{-1}(U_j) \arrow[l, "r_i"] \arrow[dddl, "\iota_j\circ\varphi_j^{-1}\circ\varphi_i\circ r_i"] \\
U_i \cap U_j & & \\
& \varphi_j^{-1}(U_i) \arrow[ul, "\varphi_j"] & \\
& \cap_{\text{open}} & \\
& \mathcal{O}_j \subset_{\iota_j} \mathbb{E}_j &
\end{tikzcd}
$$

Definition 5.1.1. *An* **M-polyfold chart** [illegible] *is a triple* $(U, \phi, \mathcal{O})$ *consisting of an open subset* $U \subset \mathcal{X}$*, an sc-retract* $\mathcal{O} \subset \mathbb{E}$ *(see Definition 5.1.2) in an sc-Banach space* $\mathbb{E}$*, and a homeomorphism* $\phi : U \to \mathcal{O}$.

Definition 5.1.2. *A* **scale smooth retraction** *(for short* **sc-retraction***) on an sc-Banach space* $\mathbb{E}$ *is an* sc^∞ *map* $r : \mathcal{U} \to \mathcal{U} \subset \mathbb{E}$ *defined on an open subset* $\mathcal{U} \subset \mathbb{E}$*, such that* $r \circ r = r$*, and hence* $r|_{r(\mathcal{U})} = \mathrm{id}|_{r(\mathcal{U})}$.

A **sc-retract** *in* $\mathbb{E}$ *is a subset* $\mathcal{O} \subset \mathbb{E}$ *that is the image* $r(\mathcal{U}) = \mathcal{O}$ *of an sc-retraction on* $\mathbb{E}$. [illegible]

Definition 5.2.3. *Let* $f : \mathcal{O} \to \mathcal{R}$ *be a map between sc-retracts* $\mathcal{O} \subset \mathbb{E}$ *and* $\mathcal{R} \subset \mathbb{F}$*, and let* $\iota_{\mathcal{R}} : \mathcal{R} \to \mathbb{F}$ *denote the inclusion map. Then we say that* f *is* sc^k *for* $k \in \mathbb{N}$ *or* $k = \infty$ *if* $\iota_{\mathcal{R}} \circ f \circ r : \mathcal{U} \to \mathbb{F}$ *is* sc^k *for some choice of sc-retraction* $r : \mathcal{U} \to \mathcal{U} \subset \mathbb{E}$ *with* $r(\mathcal{U}) = \mathcal{O}$. [illegible]

Definition 5.2.1. *The* **sc-tangent bundle** *of an sc-retract* $\mathcal{O} \subset \mathbb{E}$ *is the image* $\mathrm{T}\mathcal{O} := \mathrm{T}r(\mathrm{T}\mathcal{U}) \subset \mathrm{T}\mathbb{E}$ *of the tangent map for any choice of retraction* $r : \mathcal{U} \to \mathcal{U} \subset \mathbb{E}$ *with* $r(\mathcal{U}) = \mathcal{O}$. *In particular, its fibers are the tangent spaces*[22]

$$\mathrm{T}_p\mathcal{O} := \mathrm{T}r(\{p\} \times E_0) = \{p\} \times \operatorname{im} \mathrm{D}_p r \subset \{p\} \times E_0.$$

$$E_0 = \operatorname{im} D_p r \oplus \underbrace{\ker D_p r}_{= \operatorname{im}(\mathrm{id} - D_p r)}$$

Dr $\quad$ $\mathrm{id} - Dr$

complementary projections

Lemma:

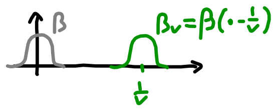
(a) r, r' sc-retractions, $r(\mathcal{U}) = \mathcal{O} = r'(\mathcal{U}')$ $\Rightarrow$ $\operatorname{im} D_p r = \operatorname{im} D_p r'$ $\quad$ "$= T_p\mathcal{O}$" well defined

(b) $T_p\mathcal{O} = \{ \dot\gamma(0) \mid \gamma : (-\varepsilon, \varepsilon) \to \mathcal{O} \subset \mathbb{E} \ sc^1, \gamma(0) = p \}$

Examples of sc-retractions:

$$\varrho: \begin{array}{l} \mathbb{R}^n \times \mathbb{E}' \to \mathbb{R}^n \times \mathbb{E}' \\ (v,f) \mapsto (v, \pi_v f) \end{array}$$

$$(\pi_v): \begin{array}{l} \mathbb{R} \times \mathbb{L}^2(\mathbb{R}) \to \mathbb{L}^2(\mathbb{R}) \\ (v,f) \mapsto \begin{cases} \langle f, \beta_v \rangle \beta_v & ; v>0 \\ 0 & ; v \le 0 \end{cases} \end{array}$$



$\|\beta\|_{L^2}=1$

Definition 5.1.3. *A* **sc-smooth splicing** *on an sc-Banach space* $\mathbb{E}'$ *is a family of linear projections* $(\pi_v : \mathbb{E}' \to \mathbb{E}')_{v \in U}$, *that is* $\pi_v \circ \pi_v = \pi_v$, *that are parametrized by an open subset* $U \subset \mathbb{R}^d$ *in a finite dimensional space and are sc*$^\infty$ *as map*

$$\pi : U \times \mathbb{E}' \to \mathbb{E}', \qquad (v,f) \mapsto \pi_v(f).$$

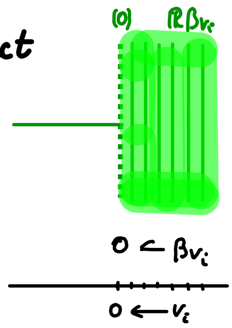
(not requiring continuity $U \to L(\mathbb{E}')$, $v \mapsto \pi_v$)

The **splicing core** *of a splicing* $(\pi_v)_{v \in U}$ *is the subset of* $\mathbb{R}^d \times \mathbb{E}'$ *given by the images of the projections,*

$$K^\pi := \{(v,e) \in U \times \mathbb{E}' \mid \pi_v e = e\} = \bigcup_{v \in U} \{v\} \times \operatorname{im} \pi_v \subset \mathbb{R}^d \times \mathbb{E}'.$$

$K^\pi = \varrho(U \times \mathbb{E}')$ sc-retract

$\operatorname{im}\pi_v = \begin{cases} \mathbb{R}\beta_v & , v>0 \\ \{0\} & ; v \le 0 \end{cases}$



$\forall f \in L^2 \; \langle f, \beta_{v_i} \rangle \to 0$

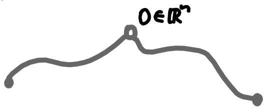
Exercise: $T_{(v,0)} K^\pi = \begin{cases} T_v \mathbb{R} \times \{0\} & ; v \le 0 \\ T_v \mathbb{R} \times \mathbb{R}\beta_v & ; v>0 \end{cases}$ $\qquad T_{(v,e)} K^\pi$ spanned by β_v and $\partial_t|_{t=0} (v+t, \pi_{v+t} e)$

Pregluing as M-polyfold chart

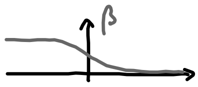
(ambient space for $\sigma^{-1}(0) = \mathcal{M}_{Morse}(\mathbb{R}^n, \mathbb{R}^n)$, $\mathrm{Crit}\, f = \{0\}$)

$$\mathfrak{X} = \bigcup_{L \geq 0} H^1([-L,L], \mathbb{R}^n) \;\underset{\text{preglue}}{\cup}\; H^1([0,\infty), \mathbb{R}^n) \times H^1((-\infty,0], \mathbb{R}^n)$$

$0 \in \mathbb{R}^n$

neighbourhood of $(\gamma_-^0, \gamma_+^0) \in \mathfrak{X}$: $\{\oplus_R(\gamma_-, \gamma_+) \mid \gamma_\pm \approx \gamma_\pm^0,\ R \in (R_0, \infty]\} = U$



$$\| \quad \begin{cases} \beta \cdot \gamma_-(\cdot + R) + (1-\beta)\gamma_+(\cdot - R) & ; R < \infty \\ (\gamma_-, \gamma_+) & ; R = \infty \end{cases}$$

$v \rightsquigarrow [0, v_0)$ "gluing profile" ($v = 0 \hat{=} R = \infty$ is a boundary of the M-polyfold)

$$[R_0, \infty] \times \underbrace{\mathbb{H}^1([0,\infty), \mathbb{R}^n) \times \mathbb{H}^1((-\infty,0], \mathbb{R}^n)}_{\mathbb{E}}$$

$$\downarrow (\pi_R)_{R > R_0}$$

$$\mathcal{O} \simeq \bigcup_{R > R_0} \{v(R)\} \times \mathrm{im}\, \pi^R$$

$U \subset \mathfrak{X}$

$\varphi(v(R), \gamma_-, \gamma_+) = \oplus_R(\gamma_-, \gamma_+)$

$$\pi^R = \begin{cases} \text{projection to complement of } \ker \oplus_R & ; R < \infty \\ \mathrm{id} & ; R = \infty \end{cases}$$

Anti-pregluing

$\ominus_R(\gamma_-, \gamma_+)$ = "data forgotten by $\oplus_R(\gamma_-, \gamma_+)$"

$\oplus_R \times \ominus_R : \mathbb{E} \xrightarrow{\sim} \mathbb{H}^1([-R,R], \mathbb{R}^n) \times \mathbb{H}^1_*((-\infty,\infty), \mathbb{R}^n)$

$\Rightarrow \mathbb{E} = \ker \oplus_R \oplus \ker \ominus_R$

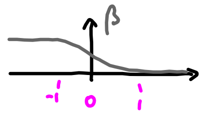
$$\pi_{e^{-1/R}} := \text{projection to } \ker \ominus_R \text{ along } \ker \oplus_R \qquad \begin{cases} \oplus_R(\eta_-, \eta_+) = \oplus_R(\gamma_-, \gamma_+) \\ \ominus_R(\eta_-, \eta_+) = 0 \end{cases}$$

$$(\gamma_-, \gamma_+) \longmapsto (\eta_-, \eta_+)$$

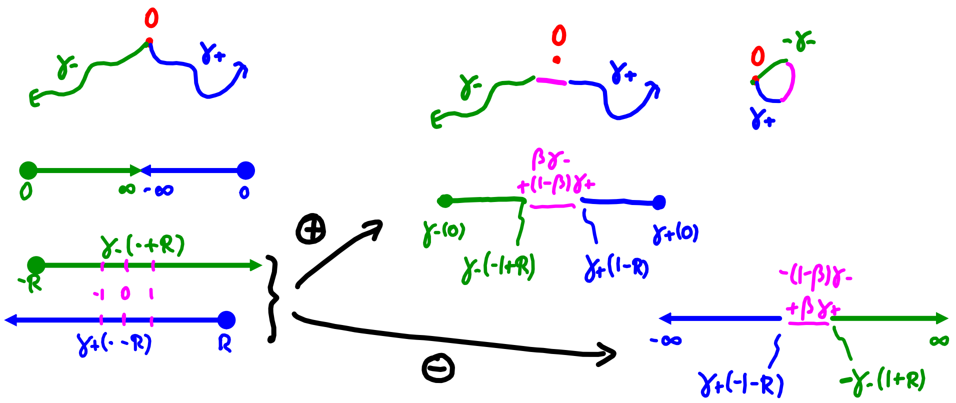
The Anti-pre-gluing splicing

$$\mathbb{E} = \mathbb{H}^1([0,\infty),\mathbb{R}^n) \times \mathbb{H}^1((-\infty,0],\mathbb{R}^n)$$

$$\oplus_R \times \ominus_R : \mathbb{E} \xrightarrow{\sim} \mathbb{H}^1([-R,R],\mathbb{R}^n) \times \mathbb{H}^1_*((-\infty,\infty),\mathbb{R}^n)$$



$$(\gamma_-,\gamma_+) \mapsto \left(\begin{array}{c|c} \beta\cdot\gamma_-(\cdot+R) & (\beta-1)\gamma_-(\cdot+R) \\ +(1-\beta)\gamma_+(\cdot-R) & +\beta\,\gamma_+(\cdot-R) \end{array} \right)$$



$$\Pi_R := \text{projection to } \ker \ominus_R \text{ along } \ker \oplus_R \qquad \begin{cases} \oplus_R(\eta_-,\eta_+) = \oplus_R(\gamma_-,\gamma_+) \\ \ominus_R(\eta_-,\eta_+) = 0 \end{cases}$$

$$= \begin{pmatrix} \tau_{-R} & 0 \\ 0 & \tau_R \end{pmatrix} \begin{pmatrix} \beta & 1-\beta \\ \beta-1 & \beta \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \beta & 1-\beta \\ \beta-1 & \beta \end{pmatrix} \begin{pmatrix} \tau_R & 0 \\ 0 & \tau_{-R} \end{pmatrix}$$

$$(\gamma_-,\gamma_+) \longmapsto \left(\begin{array}{l} \eta_- = \frac{\beta^2}{\beta^2+(1-\beta)^2}(\cdot-R)\,\gamma_- + \frac{\beta(1-\beta)}{\beta^2+(1-\beta)^2}(\cdot-R)\,\gamma_+(\cdot-2R) \\[2ex] \eta_+ = \frac{\beta(1-\beta)}{\beta^2+(1-\beta)^2}(\cdot+R)\,\gamma_-(\cdot+2R) + \frac{(1-\beta)^2}{\beta^2+(1-\beta)^2}(\cdot+R)\,\gamma_+ \end{array} \right)$$

Prop^n: $(v,\gamma_-,\gamma_+) \longmapsto (R,\eta_-,\eta_+)$ is sc^∞ for $R(v) = e^{1/v} - e$

$\left(\Leftrightarrow v(R) = 1/\ln(R-e)\right)$

Thm [HWZ]: $\exists$ polyfold $\mathcal{B}$

p.Fredholm section $\sigma_J : \mathcal{B} \to \mathcal{E}^J \quad \forall J \in \mathcal{J}(M,\omega)$

s.t. $|\sigma_J^{-1}(0)| \simeq \bigcup_{A \neq 0} \overline{\mathcal{M}}(A,J)$

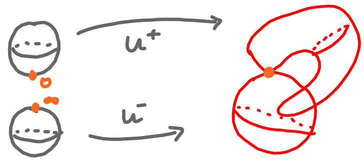
Gromov compactification of $\{u : \mathbb{P}^1 \to M \mid \bar\partial_J u = 0,\ u_*[\mathbb{P}^1] = A\} / \mathrm{Aut}\,\mathbb{P}^1$

Main Steps of Proof

- object level: cover $\overline{\mathcal{M}}$ with local Fredholm descriptions

✓ (a) near smooth curve $[u] \in \overline{\mathcal{M}}$

(b) near nodal curve e.g. $([u^-],[u^+])$



$$\mathcal{E} \downarrow \uparrow \sigma : (a, v_-, v_+) \longmapsto \begin{cases} \bar\partial_J \#_a(v_-,v_+) & ; a \neq 0 \\ (\bar\partial_J v_-, \bar\partial_J v_+) & ; a = 0 \end{cases}$$

$$\mathcal{O} \subset \bigcup_{|a|<\varepsilon} \{a\} \times \operatorname{im} \pi_a$$

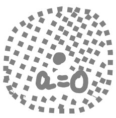
$$\cup$$

$$\sigma^{-1}(0) \longleftrightarrow \overline{\mathcal{M}}(A,J)$$

$$(a, v_-, v_+) \longmapsto \begin{cases} [\#_a(v_-,v_+)] \\ ([v_-],[v_+]) \end{cases}$$

$$\mathbb{E} = \left\{ (\xi_-,\xi_+) \in H^k(\mathbb{P}^1, u_-^*TM) \times H^k(\mathbb{P}^1, u_+^*TM) \;\; \begin{matrix} \xi_-(z) \in TH_z^- \text{ for } z=0,1,\ \xi_+(z) \in TH_z^+ \text{ for } z=1,\infty \\ \xi_-(\infty) = \xi_+(0) \end{matrix} \right\}_{k \geq 3}$$

$\circlearrowleft$ π_a = "projection to $\ker \ominus_a$ along $\ker \oplus_a$"



$$\mathbb{E} \times D \xrightarrow{\#} \mathcal{U}, \qquad (e,a) \longmapsto \#_a(e)$$

$$(e,a) \mapsto (\pi_a e, a) \qquad \downarrow \ sc^\infty \text{ surj.}$$

$$\bigcup_{a \in D} \operatorname{im} \pi_a \times \{a\} = K^\pi \xrightarrow{\#|_{K^\pi}} \mathcal{U} \quad \text{homeomorphism}$$

M-polyfold chart

$\operatorname{im} \pi_a$ = "complement to $\ker \#_a$"

$a=0$: $\operatorname{im} \pi_a = \mathbb{E} \approx \mathcal{C}^\infty(\mathbb{P}^1) \times \mathcal{C}^\infty(\mathbb{P}^1)$

$a \neq 0$: $\operatorname{im} \pi_a \subset \mathbb{E}$ ∞-dim, ∞-codim

$\cong \mathcal{C}^\infty(\mathbb{P}^1 \setminus B_a(\infty)) \times \mathcal{C}^\infty(\mathbb{P}^1 \setminus B_a(0))$

TODO: bundle structure on $\mathcal{E}$

Fredholm property of σ

morphisms

functoriality of σ