Polyfold - Fredholm theory

- implicit function theorem for nonlinear sc-Fredholm maps
- Gromov-Witten application near smooth curves
- towards Fredholm description near nodal curves

literature:
- Hofer-Wysocki-Zehnder
- Hofer - surveys
- Fabert-Fish-Golovko-Wehrheim: "Polyfolds - A first and second look"
- Wehrheim: "Fredholm notions in scale calculus and Hamiltonian Floer Theory"

Correction of L21 (video): $\tau : F_\infty \to E_\infty$ sc^0 extends to scale-differentiable $\mathbb{F} \to \mathbb{E}$

if $\forall x \in F_1 \ \exists D_x\tau : F_1 \to E_0$: $\forall k \geq 0$

* $$\frac{\| \tau(x+h) - \tau(x) - D_x\tau(h) \|_{E_k}}{\|h\|_{F_{k+1}}} \xrightarrow[\|h\|_{F_{k+1}} \to 0]{} 0 \qquad \left(\Leftrightarrow \tau|_{F_{k+1}} : F_{k+1} \to E_k \text{ differentiable } \forall k \right)$$

* $$\forall x \in F_{k+1} \quad \sup_{h \neq 0} \frac{\| D_x\tau(h) \|_{E_k}}{\|h\|_{F_k}} < \infty \qquad \left(\Leftrightarrow D\tau : \underbrace{\mathbb{F}^1 \times \mathbb{F}}_{(F_{k+1} \times F_k)_{k \in \mathbb{N}_0}} \to \mathbb{E} \ , \ (f_1, e) \mapsto D_f\tau \cdot e \right)$$

Implicit Function Theorem:

$\mathbb{E}, \mathbb{F}$ sc-Banach spaces

$f: \mathbb{E} \to \mathbb{F}$ sc^∞, sc-Fredholm

$f \pitchfork 0$ ($\forall e \in f^{-1}(0)$: $D_e f : \mathbb{E} \to \mathbb{F}$ surjective)

$\Rightarrow f^{-1}(0) \subset E_\infty$ submanifold of (local) dimension ind Df

Proof : Contraction Mapping Principle

Def^n: $f: \mathbb{E} \to \mathbb{F}$ is sc-Fredholm

- regularizing: $f(e) \in F_k \Rightarrow e \in E_k$
- contraction in local coordinates near each $e \in f^{-1}(0)$ is $(0,0)$

$$\mathbb{R}^n \times \mathbb{E}^c \longrightarrow \mathbb{R}^m \times \mathbb{F}^c$$
$$(v, w) \longmapsto (A(v,w), w - B(v,w))$$

$\forall k \in \mathbb{N}_0, \theta > 0 \ \exists \varepsilon > 0 : \ \forall \|v\|, \|w_1\|_{E_k}, \|w_2\|_{E_k} < \varepsilon :$

$$\|B(v,w_1) - B(v,w_2)\|_{F_k} \leq \theta \|w_1 - w_2\|_{E_k}$$

Rmk: Implicit Function Theorem only requires $\theta < 1$, but Fredholm stability Thm uses small $\theta > 0$.
For $f \in \mathcal{C}^1$ have $\theta \sim \sup_{\|w\| < \varepsilon} \|d_w f - d_0 f\|_{L(E_k, F_k)} \xrightarrow[\varepsilon \to 0]{} 0$

Lemma [W.]: $f: \mathbb{E} \to \mathbb{F}$ sc^∞, regularizing

$$\left.\begin{array}{l} \bullet \text{ uniformly } \mathcal{C}^1 \text{ up to finite dimensions} \\ \bullet \text{ uniformly linearized Fredholm near } e = 0 \end{array}\right\} \Rightarrow \text{contraction in local coordinates near } e = 0$$

$$f : \mathbb{E} = \mathbb{R}^d \times \tilde{\mathbb{E}} \longrightarrow \mathbb{F}$$
$$(r, e) \longmapsto f_r(e)$$

- $\forall k, r$ small: $f_r : \tilde{E}_k \to F_k$ classically $\mathcal{C}^1$
- continuity of $Df_r : \tilde{E}_k \to L(\tilde{E}_k, F_k)$ uniform for $(r,e) \approx (0,0)$
  ($\forall \delta > 0, k \in \mathbb{N}_0 \ \exists \varepsilon > 0 : \ |r|, \|e\|_k, \|e - e'\|_k < \varepsilon \Rightarrow \|D_e f_r - D_{e'} f_r\|_{L(\tilde{E}_k, F_k)} < \delta$)
- $r_i \to 0$, $\|e_i\|_{E_k} \leq 1$, $\|D_0 f_{r_i}(e_i)\|_{F_k} \to 0 \Rightarrow \|D_0 f_0(e_i)\|_{F_k} \to 0$
- $D_0 f_r : \tilde{\mathbb{E}} \to \mathbb{F}$ sc-Fredholm operator $\forall r$ small with index independent of r

Thm [HWZ]: $\exists$ polyfold $\mathcal{B}$

p.Fredholm section $\sigma_J : \mathcal{B} \to \mathcal{E}^J$ $\quad \forall J \in \mathcal{J}(M,\omega)$

s.t. $|\sigma_J^{-1}(0)| \simeq \bigcup_{A\neq 0} \overline{\mathcal{M}}(A,J)$

Gromov compactification of $\{u:\mathbb{P}^1\to M \mid \bar\partial_J u = 0,\ u_*[\mathbb{P}^1]=A\}/\mathrm{Aut}\,\mathbb{P}^1$

Main
Steps of Proof

• object level: cover $\overline{\mathcal{M}}$ with local Fredholm descriptions

(a) near smooth curve $[u]\in\overline{\mathcal{M}}$

√L2
L21 scale Banach bundle $\mathcal{E}|_{\mathcal{U}} = \overline{\bigcup_{v\in\mathcal{U}} \Omega^{0,1}(\mathbb{P}^1, v^*TM)}^{H^2}$ $\quad \uparrow \bar\partial_J = \sigma_{\mathcal{U}}$ $\quad$ scale smooth Fredholm section

scale Banach manifold $\mathcal{U} = \{v\in H^3(\mathbb{P}^1,M) \mid d(v,u)<\delta,\ v(z)\in H_z \text{ for } z=0,1,\infty\}$

$\sigma^{-1}(0)/_{\Gamma=\mathrm{Stab}(u)} \hookrightarrow F_{\mathcal{U}} \underset{\text{open}}{\subset} \overline{\mathcal{M}}$ $\quad$ homeomorphism

CHECK: $\mathbb{E} = \{\xi\in H^{3+k}(\mathbb{P}^1,u^*TM) \mid \xi(z)\in T_{u(z)}H_z \text{ for } z=0,1,\infty\}_{k\ge0} \xrightarrow{f} \mathbb{F} = (H^{2+k}(\mathbb{P}^1,\Lambda^{0,1}u^*TM))_{k\ge 0}$

is sc-Fredholm $\quad \xi \longmapsto \bar\partial_J \exp_u \xi$

• regularizing: $f(e)\in F_k \Rightarrow e\in E_k$ $\quad$ i.e. $\bar\partial_J \exp_u(\xi)\in H^{2+k} \Rightarrow \xi\in H^{3+k}$ ✓ if $u\in\mathcal{C}^\infty$

• $\forall k, r$ ~~small~~: $f_r : \underset{}{\overset{=E_k}{\tilde E_k}} \to F_k$ classically $\mathcal{C}^1$

• continuity of $Df_r : \tilde E_k \to L(\tilde E_k, F_k)$ uniform for $(r,e)\approx(0,0)$

• $r_i\to 0$, $\|e_i\|_{E_k}\le 1$, $\|D_0 f_{r_i}(e_i)\|_{F_k}\to 0 \Rightarrow \|D_0 f_0(e_i)\|_{F_k}\to 0$

• $D_0 f_r : \tilde{\mathbb{E}} \to \mathbb{F}$ sc-Fredholm operator $\forall r$ ~~small~~
with index ~~independent~~ of r

$D_0 f = D_u\bar\partial_J$
· sc^0 $\rightsquigarrow D_u\bar\partial_J : H^{3+k}\to H^{2+k}$ bounded $\forall k$
· regularizing $\rightsquigarrow D_u\bar\partial_J^{-1}(H^{2+k}) \subset H^{3+k}$ $\forall k$
· $E_0\to F_0$ Fredholm $\to D_u\bar\partial_J : H^3\to H^2$

- **object level**: ✓ (a) near smooth curve $[u] \in \overline{\mathcal{M}}$ (b) near nodal curve

$$\rightsquigarrow \quad \mathrm{Obj}\,\mathcal{B} = \bigsqcup_{\mathcal{U}} \mathcal{U}, \quad \mathrm{Obj}\,\mathcal{E} = \bigsqcup_{\mathcal{U}} \mathcal{E}|_{\mathcal{U}}, \quad \sigma|_{\mathcal{U}} = \sigma_{\mathcal{U}} \qquad \text{with } \bigcup_{\mathcal{U}} F_{\mathcal{U}} = \overline{\mathcal{M}}$$

- **morphism level**:

(a) ⇔ (a) $\quad \mathrm{Mor}\,\mathcal{B} \supset (s \times t)^{-1}(\mathcal{U}, \mathcal{U}') = \{ (v, \varphi) \in \mathcal{U} \times \mathrm{Aut}\,\mathbb{P}^1 \mid v \circ \varphi \in \mathcal{U}' \}$

scale Banach manifold (locally $\simeq \mathcal{U}$)

$$\left(\begin{array}{l} \text{For } u \in \mathcal{U},\ \varphi_0 \in \mathrm{Aut}(\mathbb{P}^1) \text{ s.t. } u \circ \varphi_0 \in \mathcal{U}' \\ \text{have } \ \mathrm{nbhd}(u) \simeq (s \times t)^{-1}(\mathrm{nbhd}(u), \mathrm{nbhd}(u \circ \varphi_0)) \ \text{ via } \ v \longmapsto (v, \varphi) \\ \text{with } \ \varphi \approx \varphi_0 \ \text{ determined by } \ \varphi(z) = v^{-1}(H'_z) \cap \mathrm{nbhd}(\varphi_0(z)) \ \text{ for } z = 0, 1, \infty \end{array} \right)$$

correction of L21 video

- **structure maps** ✓L21

$$\left. \begin{array}{l} id : \mathrm{Obj}\,\mathcal{B} \to \mathrm{Mor}\,\mathcal{B}, \quad v \longmapsto (v, id) \\ s : \mathrm{Mor}\,\mathcal{B} \to \mathrm{Obj}\,\mathcal{B}, \quad (v, \varphi) \longmapsto v \\ t : \mathrm{Mor}\,\mathcal{B} \to \mathrm{Obj}\,\mathcal{B}, \quad (v, \varphi) \longmapsto v \circ \varphi \ \ (*) \\ \circ : \mathrm{Mor}\,\mathcal{B} \times \mathrm{Mor}\,\mathcal{B} \to \mathrm{Mor}\,\mathcal{B}, \quad ((v, \varphi), (v \circ \varphi, \psi)) \longmapsto (v, \psi \circ \varphi) \end{array} \right\} \text{scale smooth}$$

(*)

$$\begin{tikzcd}
\mathrm{Mor}\,\mathcal{B} \arrow[r] & \mathrm{Obj}\,\mathcal{B} \\
\mathcal{V}_{\mathcal{U}'} \subset \mathcal{U} \arrow[u, phantom, "\cup"] \arrow[d] & \mathcal{U}' \arrow[u, phantom, "\cup"] \\
\mathcal{V}_{\mathcal{U}'} \times (\mathbb{P}^1)^3 \arrow[r] & \mathcal{V}_{\mathcal{U}'} \times \mathrm{Aut}\,\mathbb{P}^1 \arrow[u]
\end{tikzcd}$$

$v \longmapsto \left(v, \left(\begin{array}{c} v^{-1}(H'_z) \cap \\ \mathrm{nbhd}(\varphi_0(z)) \end{array} \right)_{z = 0, 1, \infty} \right) \xrightarrow{\ \mathrm{pso}\ } (v, \varphi) \longmapsto v \circ \varphi$

$= \varphi(z)$ for $z = 0, 1, \infty$

$sc^\infty \Leftarrow \left\{ \begin{array}{l} \text{el on} \\ \mathcal{U} \cap H^{2+\ell}_{\mathrm{ref}\,\ell} \end{array} \right.$

sc^∞ analogous to $\tau : \mathrm{Maps} \times S^1 \to \mathrm{Maps}$

- **functoriality**

$$\begin{tikzcd}
\mathrm{Obj}\,\mathcal{E} \ni \eta \arrow[r, leftrightarrow, "\mathrm{Mor}\,\mathcal{E}"] & \eta \circ d\varphi \in \mathrm{Obj}\,\mathcal{E} \\
\mathrm{Obj}\,\mathcal{B} \ni \xi \arrow[u, "\sigma"] \arrow[r, leftrightarrow, "\mathrm{Mor}\,\mathcal{B}"] & \xi' \in \mathrm{Obj}\,\mathcal{B} \arrow[u, "\sigma"']
\end{tikzcd}$$

$\eta = \bar\partial_J u_\xi$, $\quad \eta \circ d\varphi = \bar\partial_J u_{\xi'}$

$\xi \leftrightarrow u_\xi = \exp_u(\xi)$, $\quad \xi' \leftrightarrow u_\xi \circ \varphi$

$$\boxed{\bar\partial_J (u \circ \varphi) = \bar\partial_J u \circ d\varphi}$$

Thm [HWZ]: $\exists$ polyfold $\mathcal{B}$

p.Fredholm section $\sigma_J : \mathcal{B} \to \mathcal{E}^J \qquad \forall J \in \mathcal{J}(M,\omega)$

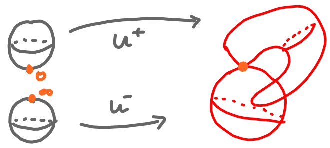
$$\text{s.t. } |\sigma_J^{-1}(0)| \simeq \bigcup_{A\neq 0} \underbrace{\overline{\mathcal{M}}(A,J)}_{\text{Gromov compactification of}} \quad \{u:\mathbb{P}^1\to M \mid \bar\partial_J u = 0,\ u_*[\mathbb{P}^1]=A\}/_{\mathrm{Aut}\,\mathbb{P}^1}$$

Main
Steps of Proof

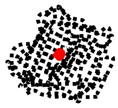
• object level: cover $\overline{\mathcal{M}}$ with local Fredholm descriptions

√(a) near smooth curve $[u]\in\overline{\mathcal{M}}$

(b) near nodal curve e.g. $([u^-],[u^+])$



Goal: $\begin{array}{l}\mathcal{E}_{\mathcal{U}} \\ \downarrow \uparrow \sigma_{\mathcal{U}} \\ \mathcal{U}\end{array}$ s.t. $\sigma_{\mathcal{U}}^{-1}(0) \simeq \text{nbhd}(([u^-],[u^+])) \subset \overline{\mathcal{M}}$



$\overline{\mathcal{M}} \ni ([u^-],[u^+])$

$$\mathcal{U}_\infty = \mathcal{C}^\infty(\mathbb{P}^1,M)/_{\mathrm{Aut}\,\mathbb{P}^1} \;\cup\; \frac{\mathcal{C}^\infty(\mathbb{P}^1,M)}{\mathrm{Aut}(\mathbb{P}^1,\infty)} \;{}_{ev_\infty}\!\times_{ev_0}\; \frac{\mathcal{C}^\infty(\mathbb{P}^1,M)}{\mathrm{Aut}(\mathbb{P}^1,0)} \;\cup\; \ldots$$

with pregluing topology

$$\begin{array}{ccc} \mathcal{U}_\infty & \ni [u] & \ni ([u_-],[u_+]) \\ \downarrow \sigma_{\mathcal{U}} & \downarrow & \downarrow \quad \ldots \\ \mathcal{E}|_{\mathcal{U}_\infty} & \ni [\bar\partial_J u] & \ni ([\bar\partial_J u_-],[\bar\partial_J u_+]) \end{array}$$

Need scale smooth structure near $([u^-],[u^+])$ in which $\sigma_{\mathcal{U}}$ is sc$^\infty$, Fredholm.

Idea: Generalize notion of Banach manifold so that pregluing is a chart.

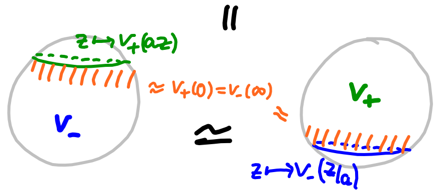
$$\frac{\mathcal{C}^\infty(\mathbb{P}^1,M)}{\mathrm{Aut}(\mathbb{P}^1,\infty)} \;{}_{ev_\infty}\!\times_{ev_0}\; \frac{\mathcal{C}^\infty(\mathbb{P}^1,M)}{\mathrm{Aut}(\mathbb{P}^1,0)} \times \{a = e^{-(R+i\theta)} \in \mathbb{C} \mid |a|<1\} \xrightarrow{\text{⚠ not injective}} \mathcal{U}_\infty$$

$$(v_-,v_+,a) \longmapsto \begin{cases} ([v_-],[v_+]) & ; a=0 \\ [\#_a(v_-,v_+)] & ; a\neq 0 \end{cases}$$

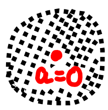
$\downarrow \mathcal{S}$ retraction $(\mathcal{S}^2=\mathcal{S})$

$\mathrm{im}\,\mathcal{S}$ $\xrightarrow{\#|_{\mathrm{im}\,\mathcal{S}}\ \text{homeom.}}$

$\|$



$$\begin{array}{ccc} \mathbb{E}\times D & \xrightarrow{\#} & \mathcal{U} \\ (e,a)\longmapsto \#_a(e) & & \\ (e,a) \downarrow (\pi_a e, a) & \downarrow \text{sc}^\infty \text{ surj.} & \nearrow \#|_{K^\pi} \text{ homeomorphism} \\ \bigcup_{a\in D} \mathrm{im}\,\pi_a \times \{a\} = K^\pi & & \end{array}$$



$\underbrace{\qquad\qquad\qquad}_{\text{M-polyfold chart}}$

$\mathrm{im}\,\pi_a$ = "complement to $\ker \#_a$"

$a=0$: $\mathrm{im}\,\pi_a = \mathbb{E} \simeq \mathcal{C}^\infty(\mathbb{P}^1)\times\mathcal{C}^\infty(\mathbb{P}^1)$

$a\neq 0$: $\mathrm{im}\,\pi_a \subset \mathbb{E}$ $\quad\infty$-dim, ∞-codim

$\quad\wr\!\wr$

$\mathcal{C}^\infty(\mathbb{P}^1\setminus B_a(\infty))\times\mathcal{C}^\infty(\mathbb{P}^1\setminus B_a(0))$