Polyfold-Fredholm theory

- overview
- scale calculus

next: - splicings/retracts

literature:
- Hofer-Wysocki-Zehnder
- Hofer - surveys
- Fabert-Fish-Golovko-Wehrheim: "Polyfolds - A first and second look"

General Form of Polyfold Regularization

$\overline{\mathcal{M}}$ compact metrizable moduli space

TODO: Construct polyfold bundle $\begin{smallmatrix}\mathcal{E}\\ \downarrow \\ \mathcal{B}\end{smallmatrix}$ (categories modeled on M-polyfolds)

and Fredholm section $\sigma : \mathcal{B} \to \mathcal{E}$ s.t. $\mathcal{M} \simeq |\sigma^{-1}(0)| = \frac{\sigma^{-1}(0) \subset \mathrm{Obj}_{\mathcal{B}}}{\mathrm{Mor}_{\mathcal{B}}}$

Thm [HWZ] $\exists\, \mathcal{P} = \{\nu \pitchfork 0\} \subset \{$multi-sections of $\mathcal{E} \to \mathcal{B}\}$:

- $\forall \nu \in \mathcal{P}$: $|(\sigma+\nu)^{-1}(0)|$ weighted branched manifold, $\partial^k |(\sigma+\nu)^{-1}(0)| = |(\sigma+\nu)^{-1}(0) \cap \partial^k \mathcal{B}|$ $\forall k \in \mathbb{N}$ corner index
- $\forall \nu \# \mu \in \mathcal{P}$: $|(\sigma+\mu)^{-1}(0)|$ cobordant
- if $\sigma|_{\mathcal{U}} \pitchfork 0$ for $\mathcal{U} \subset \mathrm{Obj}_{\mathcal{B}}$ open, $|\sigma^{-1}(0) \cap \mathcal{U}|$ compact then $\exists\, \nu \in \mathcal{P}$: $\nu|_{\mathcal{U}} \equiv 0$
- if $\nu^{\partial} \pitchfork 0$ then $\exists\, \nu \in \mathcal{P}$: $\nu|_{\partial\mathcal{B}} = \nu^{\partial}$
- $\sigma_1 + \nu_1 \pitchfork 0$, $\sigma_2 + \nu_2 \pitchfork 0$ $\Rightarrow$ $\sigma_1 \times \sigma_2 + \nu_1 \times \nu_2 \pitchfork 0$

TODO':
- choice of $(\mathcal{E}, \mathcal{B}, \sigma)$
- variation of J

$\Big\}$ yield $\underbrace{\text{equivalent}}_{\text{t.b.d.}}$ polyfold Fredholm sections

Examples : SFT Fredholm sections for PSS moduli spaces

Conj. [HWZ - in progress]: $\exists$ polyfolds $\mathcal{B}_{\pm} = \bigcup_{\gamma\in\mathcal{P}_H} \mathcal{B}_{\pm}(\gamma)$, $\mathcal{B}_0$, $\mathcal{B}_{stretch}$

p.bundles $\mathcal{E}_{\pm}$, $\mathcal{E}_0$, $\mathcal{E}_{stretch}$

p.Fredholm sections $\sigma_* : \mathcal{B}_* \to \mathcal{E}_*$

s.t. $ev_{0/\infty} : \mathcal{B}_* \to M$ p.-smooth , $\partial\mathcal{B}_{stretch} = \mathcal{B}_0^- \cup \mathcal{B}_+ \times_{\mathcal{P}_H} \mathcal{B}_-$,

$|\sigma_*^{-1}(0)| \simeq \overline{\mathcal{N}}_*$ SFT compactifications of

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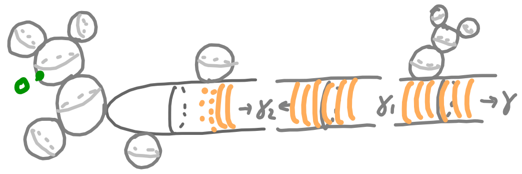
$$\mathcal{N}_{\pm}(\gamma) := \{ \hat{u} : \mathbb{C} \to \underset{\ni 0}{\mathbb{C}^{\pm}} \times M \mid \bar{\partial}_{\hat{J}_{\pm}} \hat{u} = 0 ,\ [\hat{u}] = [id]\times[u], E(u) < \infty,\ \hat{u}|_{nbhd(\infty)} \sim id_{\mathbb{C}} \times \gamma \} / Aut(\mathbb{C})$$

pos/neg puncture

$$\cap$$

$$\overline{\mathcal{N}}_{\pm} = \bigcup_{\gamma\in\mathcal{P}_H} \overline{\mathcal{N}}_{\pm}(\gamma)$$

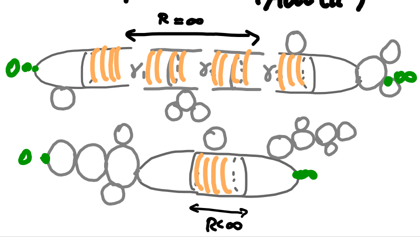


$$\mathcal{N}_{stretch} := \bigcup_{R\geq 0} \{ \hat{u} : \mathbb{P}^1 \to \underset{\ni 0,\infty}{\mathbb{P}^1} \times M \mid \bar{\partial}_{\hat{J}_R} \hat{u} = 0, [\hat{u}] = [id]\times[u], E(u) < \infty \} / Aut(\mathbb{P}^1)$$

$$\cap$$

$$\overline{\mathcal{N}}_{stretch}$$

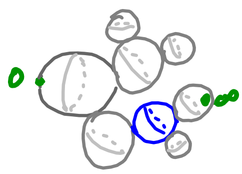
$$(\mathbb{P}^1\times M, \hat{J}_R) \underset{R\to\infty}{\longrightarrow} (\mathbb{C}^+\times M, \hat{J}_+) \cup (\mathbb{C}^-\times M, \hat{J}_-)$$



$$\mathcal{N}_0 := \{ \hat{u} : \mathbb{P}^1 \to \underset{\ni 0,\infty}{\mathbb{P}^1} \times M \mid \bar{\partial}_{\hat{J}_0} \hat{u} = 0, [\hat{u}] = [id]\times[u], E(u) < \infty \} / Aut(\mathbb{P}^1)$$

$$\cap$$

$$\overline{\mathcal{N}}_0$$



Examples : SFT Fredholm sections for PSS moduli spaces

Conj.: polyfolds $\mathcal{B}_\pm, \mathcal{B}_0, \mathcal{B}_{stretch}$
p.bundles $\mathcal{E}_\pm, \mathcal{E}_0, \mathcal{E}_{stretch}$
p.Fredholm sections $\sigma_* : \mathcal{B}_* \to \mathcal{E}_*$

$\Big\}$ are up to equivalence tbd uniquely determined by

• polyfolds : $|\mathcal{B}_0| \underset{dense}{\supset} \{ \hat{u} : \mathbb{P}^1 \xrightarrow{\mathcal{C}^\infty} \mathbb{P}^1 \times M \mid [\hat{u}] = [id] \times [u],\ E(u) < \infty \} / Aut(\mathbb{P}^1)$

$\frac{Obj\, \mathcal{B}_\pm}{Mor\, \mathcal{B}_\pm} = |\mathcal{B}_\pm| \underset{dense}{\supset} \{ \hat{u} : \mathbb{C} \xrightarrow{\mathcal{C}^\infty} \mathbb{C} \times M \mid [\hat{u}] = [id] \times [u],\ E(u) < \infty,\ \hat{u}|_{nbhd(\infty)} \sim id_{\mathbb{C}} \times \gamma \} / Aut(\mathbb{C})$

$\mathcal{B}_{stretch} = [0,\infty) \times \mathcal{B}_0 \underset{pregluing}{\cup} \mathcal{B}_+ \underset{\mathcal{P}_H}{\times} \mathcal{B}_-$

• polyfold bundles : $|\mathcal{E}_0^{\hat{J}}| \underset{dense}{\supset} \bigcup_{\hat{u} \in \mathcal{B}_0} \Omega^{0,1}(\mathbb{P}^1, \hat{u}^* T(\mathbb{P}^1 \times M))$ wrt j on $\mathbb{P}^1$, $\hat{J}$ on $\mathbb{P}^1 \times M$

$|\mathcal{E}_\pm^{\hat{J}_\pm}| \underset{dense}{\supset} \dots$

$\mathcal{E}_{stretch} = \bigcup_{R \geq 0} \mathcal{E}_0^{\hat{J}_R} \underset{pregluing}{\cup} \mathcal{E}_+^{\hat{J}_+} \underset{\mathcal{P}_H}{\times} \mathcal{E}_-^{\hat{J}_-}$

• Fredholm sections : $\sigma|_{\text{dense subset}} = \bar{\partial}_{\hat{J}_0}$ on $\mathcal{B}_0$

$\bar{\partial}_{\hat{J}_\pm}$ on $\mathcal{B}_\pm$

$(R, \hat{u}) \longmapsto \bar{\partial}_{\hat{J}_R} \hat{u}$
$(\hat{u}_+, \hat{u}_-) \longmapsto (\bar{\partial}_{\hat{J}_+} \hat{u}_+, \bar{\partial}_{\hat{J}_-} \hat{u}_-)$
on $\mathcal{B}_{stretch}$

Thm [HWZ]: $\exists$ polyfold $\mathcal{B}$

p.Fredholm section $\sigma_J : \mathcal{B} \to \mathcal{E}^J \qquad \forall J \in \mathcal{J}(M,\omega)$

s.t. $|\sigma_J^{-1}(0)| \simeq \bigcup_{A \neq 0} \overline{\mathcal{M}}(A,J)$

Gromov compactcation of $\{u : \mathbb{P}^1 \to M \mid \bar\partial_J u = 0,\ u_*[\mathbb{P}^1] = A\}/_{\mathrm{Aut}\,\mathbb{P}^1}$

Main **Steps of Proof**

- **object level**: cover $\overline{\mathcal{M}}$ with local Fredholm descriptions

(a) near smooth curve $[u] \in \overline{\mathcal{M}}$ pick representative u s.t. $d_z u$ injective for $z = 0,1,\infty$

$\checkmark$ L2   submanifolds $H_z \subset M$, $H_z \pitchfork u \ni u(z)$ —"—

Banach bundle $\mathcal{E}|_{\mathcal{U}} = \bigcup_{v \in \mathcal{U}} \overline{\Omega^{0,1}(\mathbb{P}^1, v^*TM)}^{H^2}$ $\uparrow \bar\partial_J = \sigma_{\mathcal{U}}$ Fredholm section

Banach manifold $\mathcal{U} = \{ v \in H^3(\mathbb{P}^1, M) \mid d(v,u) < \delta,\ v(z) \in H_z \text{ for } z = 0,1,\infty \}$

$\sigma^{-1}(0)/_{\Gamma = \mathrm{Stab}(u)} \hookrightarrow F_{\mathcal{U}} \underset{\text{open}}{\subset} \overline{\mathcal{M}}$ homeomorphism

(b) near nodal curve

TODO $\rightsquigarrow$ $\mathcal{U}$ M-polyfold

$\rightsquigarrow$ $\mathrm{Obj}\,\mathcal{B} = \bigsqcup_{\mathcal{U}} \mathcal{U}$, $\mathrm{Obj}\,\mathcal{E} = \bigsqcup_{\mathcal{U}} \mathcal{E}|_{\mathcal{U}}$, $\sigma|_{\mathcal{U}} = \sigma_{\mathcal{U}}$ with $\bigcup_{\mathcal{U}} F_{\mathcal{U}} = \overline{\mathcal{M}}$

- **morphism level**:

(a) $\leftrightarrow$ (a) $\mathrm{Mor}\,\mathcal{B} \supset (s \times t)^{-1}(\mathcal{U}, \mathcal{U}') = \{ (v, \varphi) \in \mathcal{U} \times \mathrm{Aut}\,\mathbb{P}^1 \mid v \circ \varphi \in \mathcal{U}' \}$

scale Banach manifold (locally $\simeq \mathcal{U}$)

- **structure maps**

$$\left.\begin{aligned}
id &: \mathrm{Obj}\,\mathcal{B} \to \mathrm{Mor}\,\mathcal{B},\quad v \mapsto (v, id)\\
s &: \mathrm{Mor}\,\mathcal{B} \to \mathrm{Obj}\,\mathcal{B},\quad (v,\varphi) \mapsto v\\
t &: \mathrm{Mor}\,\mathcal{B} \to \mathrm{Obj}\,\mathcal{B},\quad (v,\varphi) \mapsto v \circ \varphi \quad \text{⚠ not classically differentiable}\\
\circ &: \mathrm{Mor}\,\mathcal{B} \times \mathrm{Mor}\,\mathcal{B} \to \mathrm{Mor}\,\mathcal{B},\quad ((v,\varphi),(v\circ\varphi,\psi)) \mapsto (v, \psi\circ\varphi)
\end{aligned}\right\}\ \text{scale smooth}$$

(a) $\leftrightarrow$ (b)
(b) $\leftrightarrow$ (b)   TODO $\rightsquigarrow$ scale smooth maps between M-polyfolds

SCALE CALCULUS

Guiding Example: $\tau : S^1 \times \mathcal{C}^\infty(S^1) \longrightarrow \mathcal{C}^\infty(S^1)$ $\qquad S^1 = \mathbb{R}/\mathbb{Z}$

$$(s,\gamma) \longmapsto \gamma(s+\cdot)$$

Goal: Notion of smooth structure on $\mathcal{C}^\infty(S^1)$ s.t.

- τ is smooth
- chain rule
- implicit function theorem
  recovering classical smooth structure on finite dimensional submanifolds

Facts:

(0) $\tau : S^1 \times \mathcal{C}^k(S^1) \longrightarrow \mathcal{C}^k(S^1)$ continuous $\forall k \in \mathbb{N}_0$

(i) $\tau : S^1 \times \mathcal{C}^{k+1}(S^1) \longrightarrow \mathcal{C}^k(S^1)$ differentiable $\forall k$

(ii) $D\tau : S^1 \times \mathcal{C}^{k+1}(S^1) \longrightarrow L(\mathbb{R} \times \mathcal{C}^{k+1}(S^1), \mathcal{C}^k(S^1))$ continuous $\forall k$

$$(s,\gamma) \longmapsto D_{(s,\gamma)}\tau : (S,\Gamma) \longmapsto S \cdot \dot{\gamma}(s+\cdot) + \Gamma(s+\cdot)$$

to bounded linear operators

$D\tau : S^1 \times \mathcal{C}^k(S^1) \longrightarrow L(\mathbb{R} \times \mathcal{C}^k(S^1), \mathcal{C}^k(S^1))$ ill defined for $\gamma \in \mathcal{C}^k \setminus \mathcal{C}^{k+1}$

$D\tau : S^1 \times \mathcal{C}^\infty(S^1) \longrightarrow L(\mathbb{R} \times \mathcal{C}^k(S^1), \mathcal{C}^k(S^1))$ not continuous w.r.t. S^1 $\quad \sup_{\|\Gamma\|_{\mathcal{C}^0}=1} \|\Gamma(s+\cdot) - \Gamma\|_{\mathcal{C}^0} = 2 \quad \forall s \neq 0$

(iii) $D\tau : S^1 \times \mathcal{C}^{k+1}(S^1) \times \mathbb{R} \times \mathcal{C}^k(S^1) \longrightarrow \mathcal{C}^k(S^1)$ continuous $\forall k$

$$(s,\gamma,S,\Gamma) \longmapsto D_{(s,\gamma)}\tau(S,\Gamma) = S \cdot \underbrace{\dot{\gamma}(s+\cdot)}_{=\tau(s,\dot{\gamma})} + \underbrace{\Gamma(s+\cdot)}_{=\tau(s,\Gamma)}$$

SCALE CALCULUS

<u>Lemma</u>: $\tau : S^1 \times \mathbb{E} \longrightarrow \mathbb{E}$ is scale-smooth on sc-Banach space
$(s,\gamma) \longmapsto \gamma(s+\cdot)$ $\quad\quad \mathbb{E} := (\mathcal{C}^k(S^1))_{k\in\mathbb{N}_0}$

<u>Def^n</u>: $\mathbb{E} = (E_k)_{k\in\mathbb{N}_0}$ <u>scale-Banach space</u> consists of

- $(E_k, \|\cdot\|_k)$ Banach space $\forall k$ $\quad\quad (\mathcal{C}^k, \|\cdot\|_{\mathcal{C}^k})$
- $E_k \hookrightarrow E_j$ continuous, compact injection $\forall k>j$ $\quad\quad \mathcal{C}^k \hookrightarrow \mathcal{C}^j$
- $E_\infty := \bigcap_{j\in\mathbb{N}_0} E_j \subset E_k$ dense $\forall k$ $\quad\quad \mathcal{C}^\infty \subset \mathcal{C}^k$

$\mathbb{F} = (\cancel{S^1} \times \mathcal{C}^k)_{k\in\mathbb{N}_0}$
$\mathbb{R}$ v.space

<u>Def^n</u>: $\tau : \mathbb{F} \to \mathbb{E}$ is

(0) <u>scale-continuous</u> (sc^0) if $\tau|_{F_k} : F_k \to E_k$ $\mathcal{C}^0$ $\forall k$

(i) <u>scale-differentiable</u> if sc^0, $\tau|_{F_{k+1}} : F_{k+1} \to E_k$ differentiable $\forall k$
and derivative map $D\tau : \underbrace{\mathbb{F}^1 \times \mathbb{F}}_{(F_{k+1}\times F_k)_{k\geq 0}} \longrightarrow \mathbb{E}$, $(f,e) \mapsto D_f\tau\cdot e$ well defined

(ii) <u>ℓ-fold continuously scale-differentiable</u> (sc^ℓ) if (0), (i), $D\tau$ is $sc^{\ell-1}$

(iii) <u>scale-smooth</u> (sc^∞) if sc^ℓ $\forall \ell\in\mathbb{N}_0$