Moduli spaces of pseudoholomorphic curves

general form $\overline{\mathcal{M}} = \frac{\bar\partial_J^{-1}(0)}{\mathrm{Aut}} \underset{\text{glue}}{\cup} \{\text{broken/nodal curves}\}$

$\bar\partial_J \left(\begin{smallmatrix} \mathcal{E} \\ \downarrow \\ \mathcal{B} \end{smallmatrix} \right.$ $= \{ \underset{\text{+ marked pts}}{u:(\Sigma,j)} \to (M,J) \mid [u]=A,\ u(\partial\Sigma)\subset L,\ (\Sigma,j)\in \mathcal{DM},\ (M,J)\in\mathcal{D} \}$

Aut $\circlearrowright \mathcal{B}$ by $u \mapsto u\circ\varphi$ for $\varphi:(\Sigma,j)\xrightarrow{\sim}(\Sigma',j')$

Language:

$(\Sigma, j, \underline{z}$ marked points$)$ is stable domain if $\mathrm{Stab}(\Sigma,j,\underline{z}) = \{\varphi \circlearrowright (\Sigma,j) \mid \varphi(z_i)=z_i\}$ finite

$u:\Sigma\to M$ is stable map if $\mathrm{Stab}(u)=\{\varphi \circlearrowright \mid u\circ\varphi = u\}$ finite

$([u]\in \mathcal{B}/\mathrm{Aut}$ stable curve$)$

Rmk: $[u]=A\neq 0$ guarantees stable maps

Aut is almost never discrete

	(Σ, j)	Aut	curves added in "compactification"
genus zero Gromov-Witten	$(\mathbb{P}^1, i)$ + marked points	$G(\mathbb{P}^1, i)$	
Gromov-Witten	Σ fixed + marked points j can vary	(Σ, j) $\xrightarrow{\sim} (\Sigma, j')$	nodes
Hamiltonian Floer	$(\mathbb{R}\times S^1, i)$ $\subset \mathbb{C}/\mathbb{Z}$ "energy"	$\mathbb{R}$	breaking
Lagrangian Floer	$(\mathbb{R}\times [0,1], i)$ $\subset \mathbb{C}$	$\mathbb{R}$	
Fukaya A_∞-algebra	(D, i) + marked points on ∂D $\cup$ disk in $\mathbb{C}$	$\frown (D, i)$	
Fukaya A_∞-category	$(D \setminus \{z_0, \dots, z_k\}, i)$ $z_0, \dots, z_k \in \partial D$	$G(D, i)$	
contact homology	k pos. punctures genus 0 1 neg. puncture output input	M $=$ $\mathbb{R}\times Y$	"buildings" & "nodes" max princ
Symplectic Field Theory	punctured Riemann surfaces	$\mathbb{R}^+\times Y^+$ IT $\mathbb{R}^-\times Y^-$	buildings & nodes & sphere bubbles
relative SFT	punctured Riemann surfaces with boundary $u(\partial\Sigma)\subset L$	$\mathbb{R}^+\times Y^+$ IT $\mathbb{R}^-\times Y^-$ L	buildings & interior/boundary nodes & sphere/disk bubbles

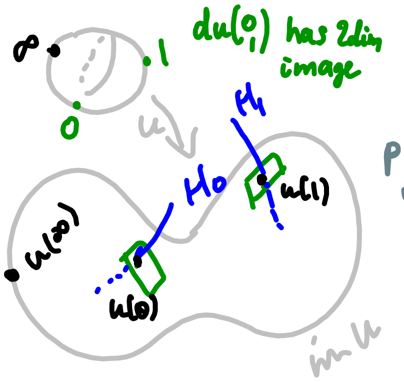
	Regularization	Algebra
Gromov-Witten	$[\overline{\mathcal{M}}] \in H_d(\overline{\mathcal{M}})$ $\quad ev_i : \overline{\mathcal{M}} \to M$	$\alpha_1 \wedge .. \alpha_k \in H^d(M) \rightsquigarrow \int_{[\overline{\mathcal{M}}]} ev_1^*\alpha_1 \wedge .. ev_k^* \alpha_k$ " = # curves through $PD(\alpha_i)$
Floer Theories	0/1-manifolds/cobordism $\partial \overline{\mathcal{M}}^1 \simeq \overline{\mathcal{M}}^0 \underset{\text{matching ends}}{\times} \overline{\mathcal{M}}^0$	$\overline{\mathcal{M}}^0 \rightsquigarrow \partial : CF \to CF$ $\overline{\mathcal{M}}^1 \rightsquigarrow \partial \circ \partial = 0$
contact homology, SFT A_∞-category	0/1-manifolds, $\partial \overline{\mathcal{M}}^1 \simeq \overline{\mathcal{M}}^0 \times \overline{\mathcal{M}}^0$ discrete set of limit orbits	$\overline{\mathcal{M}}^0 \rightsquigarrow$ operation $\hat{m}$ $\overline{\mathcal{M}}^1 \rightsquigarrow$ relations " $\hat{m} \circ \hat{m} = 0$ "
Fukaya A_∞-algebra	$ev_0, .., ev_k : \overline{\mathcal{M}}_k \to L$ $\partial \overline{\mathcal{M}}_k \simeq \bigcup_\ell \overline{\mathcal{M}}_{k-\ell} \times \overline{\mathcal{M}}_\ell$ 	$\mu_k : \otimes^k C_*L \to C_*L$ $(C_1, \dots, C_k) \mapsto ev_0 : (\underbrace{ev_1^* C_1 \cap \dots \cap ev_k^* C_k}_{\text{on } \overline{\mathcal{M}}_k}) \to L$ " $(\sum_k \mu_k) \circ (\sum_k \mu_k) = 0$ "

Analytic features of moduli spaces

(1) $\mathrm{Aut} \times \mathcal{B} \to \mathcal{B}$ nowhere differentiable for any known
$(\varphi, u) \mapsto u \circ \varphi$ Banach spaces $\mathcal{B} = W^{k,p}, \mathcal{C}^{\ell}, \dots$
<u>except</u> for $\mathcal{B} \subset \mathcal{C}^\infty$ finite dimensional

(2) local slices for $\mathcal{B}/\mathrm{Aut}$

$$\supset \mathcal{U} \underset{\text{homeo}}{\simeq} \frac{\text{Banach space}}{\Gamma \text{ finite}}$$



$$\mathcal{U} = \left\{ \mathcal{V} \in \mathcal{B}/_{\mathrm{Aut}} \;\middle|\; \exists v \in \mathcal{V} : d_{\mathcal{C}^1}(u,v) < \varepsilon \right\} \Rightarrow H_i \pitchfork \mathrm{im}\, v$$

$v^{-1}(H_i) \ni z_i$ for $i = 0, 1$

pick representative $[u]$ s.t. $du(i)$ is injective, then find codim 2 submfds

$H_0, H_1 \subset M \quad H_i \pitchfork \mathrm{im}\, u \ni u(i)$

$\Rightarrow \exists! \varphi \in \mathrm{Aut}, \varphi(i) = z_i$

$\Rightarrow \mathcal{V} = [v \circ \varphi]$

$$\Rightarrow \mathcal{U} \simeq \left\{ v' \in \mathrm{nbhd}(u) \subset \mathcal{B} \;\middle|\; v'(0) \in H_0, v'(1) \in H_1 \right\}$$

$$\begin{array}{c} \cup \\ \mathcal{V} = [v] \end{array} \longmapsto v' = v \circ \varphi$$

unique up to $v \circ \varphi = v$
$\rightsquigarrow \varphi \in \Gamma = \mathrm{Stab}_u$

Analytic features of moduli spaces

(1) $\mathrm{Aut} \times \mathcal{B} \to \mathcal{B}$ nowhere differentiable

(2) local slices for $\mathcal{B}/\mathrm{Aut}$

(2') transition maps rarely differentiable

$$\left\{ \begin{matrix} v(0) \in H_0 \\ v(1) \in H_1 \end{matrix} \right\} \simeq \mathcal{U} \subset \mathcal{B}/\mathrm{Aut} \supset \tilde{\mathcal{U}} \simeq \left\{ \begin{matrix} w(0) \in \tilde{H}_0 \\ w(1) \in \tilde{H}_1 \end{matrix} \right\}$$

$$v \mapsto [v] = [w] \mapsfrom w = v \circ \varphi_v$$

$\varphi_v \in \mathrm{Aut}(\mathbb{CP}^1, i, \infty)$ given by

$\varphi_v(i) = z_i$ for $v(z_i) \in \tilde{H}_i$; $i = 0, 1$

$$\mathcal{B} \xrightarrow{\mathcal{C}^\ell \text{ if } \mathcal{B} \hookrightarrow \mathcal{C}^\ell} \mathcal{B} \times \mathbb{CP}^1 \times \mathbb{CP}^1 \longrightarrow \mathcal{B} \times \mathrm{Aut} \xrightarrow{\text{not diff}} \mathcal{B}$$

$$v \longmapsto (v, z_0, z_1) \longmapsto (v, \varphi_v) \longmapsto v \circ \varphi_v$$

Analytic features of moduli spaces

(1) $Aut \times \mathcal{B} \to \mathcal{B}$ nowhere differentiable

(2) local slices for $\mathcal{B}/Aut$

(3) local Fredholm description of $\bar{\partial}_J^{-1}(0)/Aut$

$$\begin{array}{l} \mathcal{E} \\ \downarrow \uparrow s = \bar{\partial}_J|_{\mathcal{U}} \\ \mathcal{U} = \left\{ u \in \mathcal{B} \;\middle|\; \begin{matrix} u(0) \in H_0 \\ u(1) \in H_1 \end{matrix} \right\} \end{array} \qquad \frac{s^{-1}(0)}{\Gamma = Stab(u)} \xrightarrow{\text{homeo}} F \hookleftarrow \text{open nbhd of } [u]$$

(3) allows to turn analytic problems (1) & (2) into topological problems

ASIDE / PREVIEW

Geometric Regularization (varying J, ...)

→ uses global Fred description $\bar{\partial}_J : \mathcal{B} \to \mathcal{E}$

→ achieves Aut-equivariant transversality - using geometric control of curves

Abstract Regularization

→ uses local Fredholm description $\bar{\partial}_J|_{\text{local slice}}$

→ cannot (generally) achieve equivariant transversality

Analytic features of moduli spaces

(1) $\mathrm{Aut} \times \mathcal{B} \to \mathcal{B}$ nowhere differentiable

(2) local slices for $\mathcal{B}/\mathrm{Aut}$

(3) local Fredholm description of $\bar{\partial}_J^{-1}(0)/\mathrm{Aut}$

(4) ———— " ———— of {broken/nodal curves}

(5) compact metric topology on $\overline{\mathcal{M}}$ via "gluing"