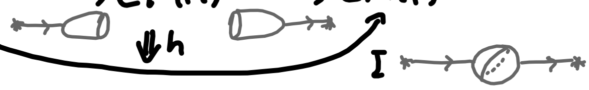
Application of abstract regularization techniques

at the example of

a "non-equivariant" proof of Arnold Conjecture

Piunikhin-Salamon-Schwarz moduli spaces | via fiber products

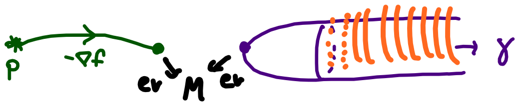
$$
\begin{tikzcd}
CM(f) \arrow[r, "PSS"] \arrow[rr, bend right, "I"'] & CF(H) \arrow[r, "SSP"] & CM(f)
\end{tikzcd}
\qquad \Downarrow h
$$



PSS: $$\mathcal{M}(p,\gamma,J) := \Big\{u:\mathbb{C}\to M \;\Big|\; \bar\partial_J u = \Theta^*(\beta\cdot J X_H)^{0,1}(u),\ u(0)\in W^u_p,\ E(u)<\infty,\ \lim_{s\to\infty} u|_{\Theta^{-1}(s,\cdot)}=\gamma\Big\}$$

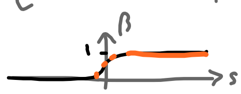
$$\simeq \mathcal{M}(p,M) \times \mathcal{M}_+(\gamma)$$

$ev \searrow\ M \swarrow ev$



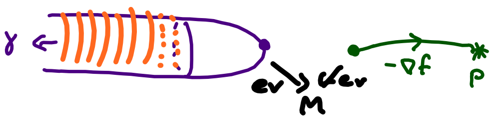
$\Theta:\mathbb{C}\setminus\{0\} \xrightarrow{\sim} \mathbb{R}\times S^1$

$e^{2\pi(s+it)} \longmapsto (s,t)$



SSP: $$\mathcal{M}(\gamma,p,J) := \Big\{u:\mathbb{C}\to M \;\Big|\; \bar\partial_J u = \hat\Theta^*((1-\beta) J X_H)^{0,1}(u),\ u(0)\in W^s_p,\ E(u)<\infty,\ \lim_{s\to-\infty} u|_{\hat\Theta^{-1}(s,\cdot)}=\gamma\Big\}$$

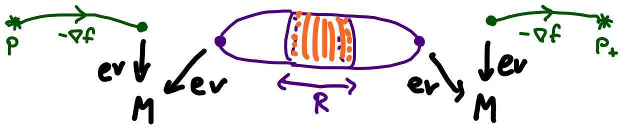
$$\simeq \mathcal{M}_-(\gamma) \underset{M}{\times} \mathcal{M}(M,p)$$



$\hat\Theta:\mathbb{C}\setminus\{0\} \xrightarrow{\sim} \mathbb{R}\times S^1$

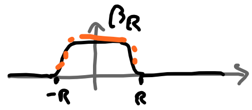
$e^{-2\pi(s+it)} \longmapsto (s,t)$

h: $$\mathcal{M}(p_-,p_+,J) := \mathcal{M}(p_-,M) \underset{M}{\times} \bigcup_{R\in[0,\infty]} \mathcal{M}_R \underset{M}{\times} \mathcal{M}(M,p_+)$$

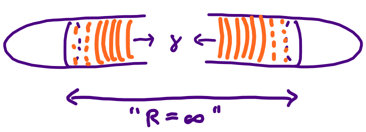


$\Theta: \mathbb{CP}^1\setminus\{0,\infty\} \to \mathbb{R}\times S^1$

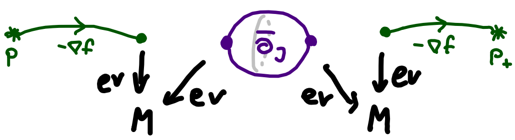
$e^{2\pi(s+it)} \longmapsto (s,t)$



$$\mathcal{M}_R := \begin{cases} \{u:\mathbb{CP}^1\to M \mid \bar\partial_J u = \Theta^*(\beta_R J X_H)(u),\ E(u)<\infty\} & ;\ 0\le R<\infty \\ \bigcup_{\gamma\in\mathcal{P}_H} \mathcal{M}_+(\gamma)\times\mathcal{M}_-(\gamma) & ;\ R=\infty \end{cases}$$

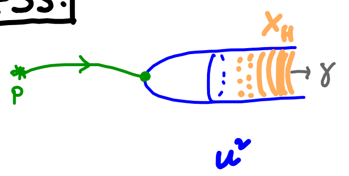


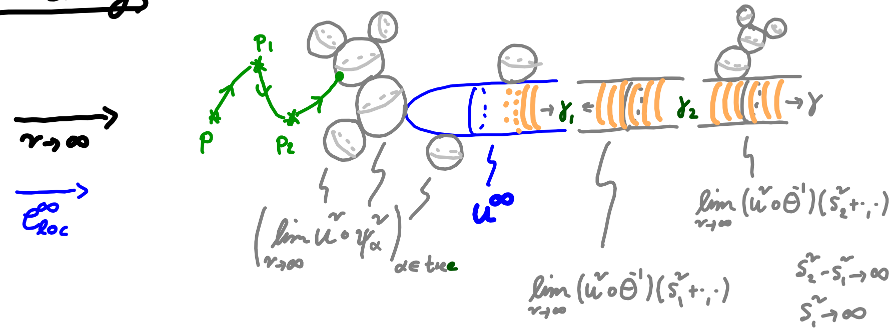
I: $$\mathcal{M}_{R=0}(p_-,p_+,J) := \mathcal{M}(p_-,M) \underset{M}{\times} \mathcal{M}_{R=0} \underset{M}{\times} \mathcal{M}(M,p_+)$$



Breaking & Bubbling

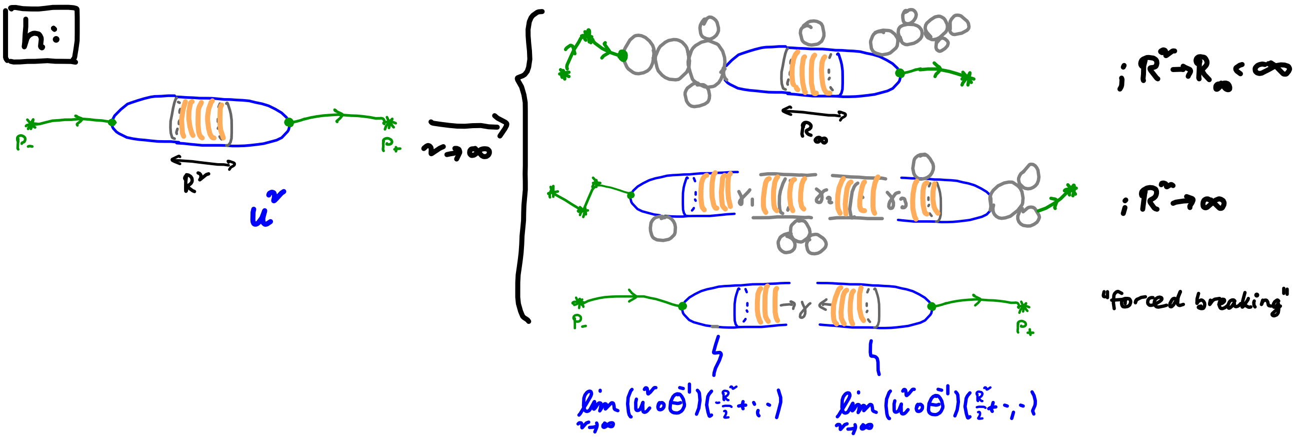
PSS:



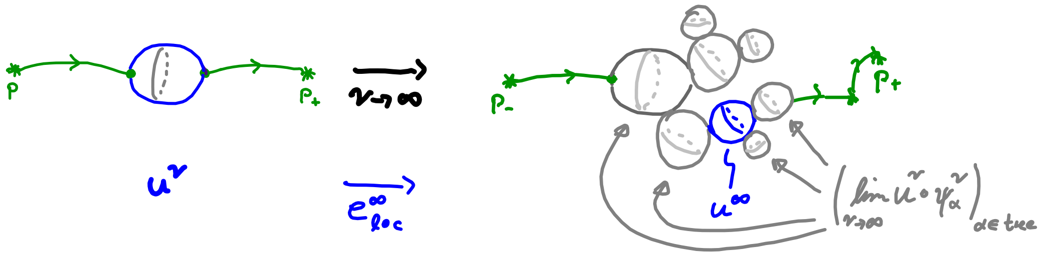


SSP: similar

h:

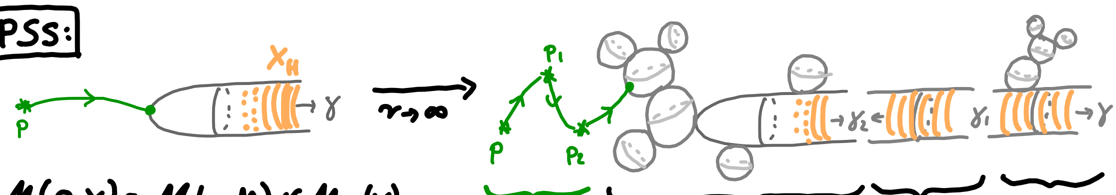


I:



Breaking & Bubbling ⇝ "Compact-cation"

PSS:



$$\mathcal{M}(p,\gamma) = \mathcal{M}(p,M) \times_M \mathcal{M}_+(\gamma)$$

$$\overline{\mathcal{M}}(p,M) \xrightarrow{ev} M \xleftarrow{ev} \overline{\mathcal{M}}^{S^2}_+(\gamma_2) \qquad \overline{\mathcal{M}}^{S^2}(\gamma_2,\gamma_1) \qquad \overline{\mathcal{M}}^{S^2}(\gamma_1,\gamma)$$

$$\overline{\mathcal{M}}(p,\gamma) := \bigcup_{k=0}^{\infty} \bigcup_{\gamma_1\ldots\gamma_k\in\mathcal{P}_H} \underbrace{\overline{\mathcal{M}}(p,M)}_{\substack{\text{compactified}\\ \text{Morse trajectory}\\ \text{space}}} \times_M \underbrace{\overline{\mathcal{M}}^{S^2}_+(\gamma_k)}_{\substack{\text{PSS +}\\ \text{sphere bubbles}}} \times \underbrace{\overline{\mathcal{M}}^{S^2}(\gamma_k,\gamma_{k-1})}_{\substack{\text{Floer trajectories}\\ \text{+ sphere bubbles}}} \ldots \times \underbrace{\overline{\mathcal{M}}^{S^2}(\gamma_1,\gamma)}_{\substack{\text{SSP +}\\ \text{sphere bubbles}}}$$

Floer trajectories

$$\mathcal{M}(\gamma_-,\gamma_+) := \{u:\mathbb{R}\times S^1\to M \mid \bar\partial_J u = JX_H(u),\ E(u):=\int|\partial_s u|^2<\infty,\ \lim_{s\to\pm\infty} u(s,\cdot)=\gamma_\pm\}/\mathbb{R}$$

sphere bubble trees

$$\mathcal{M}_k(J) := \{(u:\mathbb{P}^1\to M,\ \underline{z}\in(\mathbb{P}^1)^k\setminus diag) \mid \bar\partial_J u = 0,\ E(u):=\tfrac12\int|du|^2<\infty,\ k\geq 3 \text{ or } E(u)>0\}/Aut(\mathbb{P}^1)$$

$$\mathcal{M}_k(J) \xrightarrow{ev} M^k$$

$$\overline{\mathcal{M}}_1(J) := \bigcup_{T\text{ tree}} \mathcal{M}_{k_{\alpha_0}+1} \times \prod_{\alpha\in T\setminus\{\alpha_0\}} \mathcal{M}_{k_\alpha} \times \Delta_M^E$$

$$\overline{\mathcal{M}}_1(J) \xrightarrow{ev_0} M, \qquad \mathcal{M}_{k_{\alpha_0}+1} \xrightarrow{ev_0} M, \qquad \mathcal{M}_{k_{\alpha_0}+1} \times \prod \mathcal{M}_{k_\alpha} \xrightarrow{ev} (M\times M)^{2E} \leftarrow \Delta_M^E$$

($\alpha_0\in T$ root
$k_\alpha = \#$edges at vertex α)

$\overline{\mathcal{M}}_2(J) :=$ similar

$$M \xleftarrow{ev_0} \overline{\mathcal{M}}_2(J) \xrightarrow{ev_1} M$$

sphere compactified moduli spaces

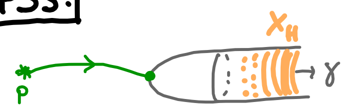
$$\overline{\mathcal{M}}^{S^2}_\cdot(\ldots) := \bigcup_{n\geq 0} \{(u\in\mathcal{M}_\cdot(\ldots),\ (y_1..y_n)\in domain(u)) \mid y_i \text{ disjoint}\} \times \overbrace{\overline{\mathcal{M}}_1(J)\times\ldots\times\overline{\mathcal{M}}_1(J)}^{n}$$

$$\{\ldots\} \xrightarrow{ev} M^n \xleftarrow{ev} \overline{\mathcal{M}}_1(J)\times\ldots\times\overline{\mathcal{M}}_1(J)$$

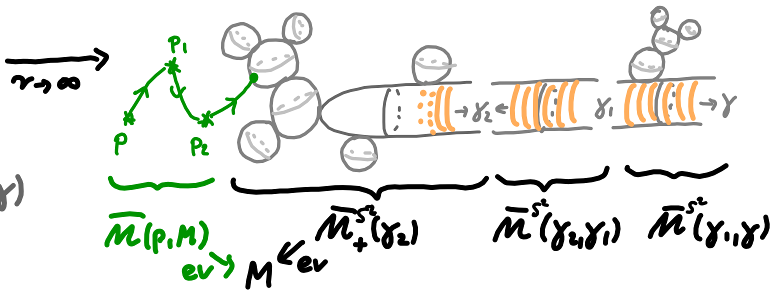
⚠ If one of the fixed marked points on the domain coincides with a y_i (i.e. $y_i=0$ for PSS, SSP, $\mathcal{I}$, h or $y_i=\infty$ for $\mathcal{I}$, h) then one (or two) $\overline{\mathcal{M}}_1(J)$ are replaced by $\overline{\mathcal{M}}_2(J)$ which has 2 marked points
- the node
- the marked point of dom(u) "bubbled" to a sphere

Breaking & Bubbling ⇝ "Compact-cation"

PSS:

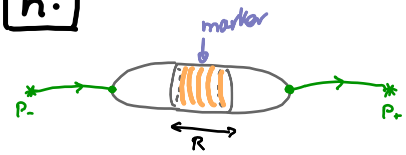


$$\mathcal{M}(p,\gamma) = \mathcal{M}(p,M) \times_M \mathcal{M}_+(\gamma)$$

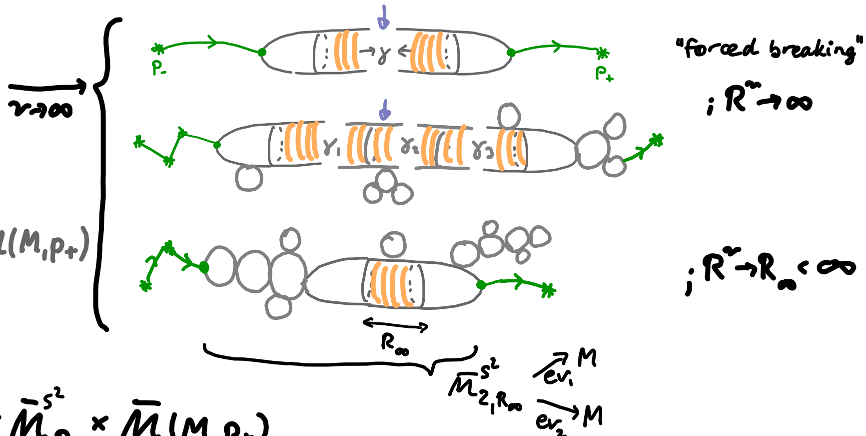


$$\overline{\mathcal{M}}(p,\gamma) := \bigcup_{k=0}^{\infty} \bigcup_{\gamma_1..\gamma_k\in\mathcal{P}_H} \overline{\mathcal{M}}(p,M) \times_M \overline{\mathcal{M}}_+^{S^2}(\gamma_k) \times \overline{\mathcal{M}}^{S^2}(\gamma_k,\gamma_{k-1}) \dots \times \overline{\mathcal{M}}^{S^2}(\gamma_1,\gamma)$$

h:



$$\mathcal{M}(p_-,p_+) = \mathcal{M}(p_-,M) \times_M \bigcup_R \mathcal{M}_R \times_M \mathcal{M}(M,p_+)$$



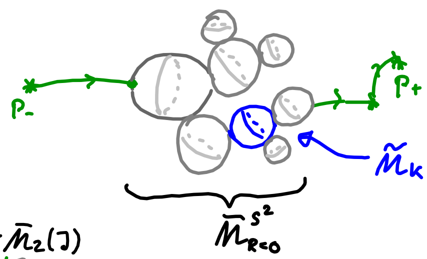
$$\overline{\mathcal{M}}(p_-,p_+) = \bigcup_{0\le R<\infty} \overline{\mathcal{M}}(p_-,M) \times_M \overline{\mathcal{M}}^{S^2}_R \times_M \overline{\mathcal{M}}(M,p_+)$$

$$\cup \bigcup_{k=0}^{\infty} \bigcup_{\gamma_1..\gamma_k\in\mathcal{P}_H} \underbrace{\overline{\mathcal{M}}(p_-,M) \times_M \overline{\mathcal{M}}_+^{S^2}(\gamma_1) \times \overline{\mathcal{M}}^{S^2}(\gamma_1,\gamma_2) \dots}_{\in\,\overline{\mathcal{M}}(p_-,\gamma_\ell)} \underset{\text{marker}}{\uparrow} \underbrace{\times\, \overline{\mathcal{M}}^{S^2}(\gamma_{k-1},\gamma_k) \times \overline{\mathcal{M}}_-^{S^2}(\gamma_k) \times_M \overline{\mathcal{M}}(M,p_+)}_{\in\,\overline{\mathcal{M}}(\gamma_\ell,p_+)}$$

I:

$$\overline{\mathcal{M}}^I(p_-,p_+) = \overline{\mathcal{M}}(p_-,p_+) \cap \{R=0\}$$

$$= \overline{\mathcal{M}}(p_-,M) \times_M \overline{\mathcal{M}}^{S^2}_{R=0} \times_M \overline{\mathcal{M}}(M,p_+)$$



$$\overline{\mathcal{M}}^{S^2}_{R=0} = \{(u,0,\infty) \mid \bar\partial_J u = 0\} \cup \bigcup_{n\ge1} \tilde{\mathcal{M}}_n \times \overline{\mathcal{M}}_2(J) \times \overline{\mathcal{M}}_1(J) \times .. \times \overline{\mathcal{M}}_1(J) \times \overline{\mathcal{M}}_2(J)$$

ev_0, ev_∞: M, M; ev_0; ev_∞

$$\cup \bigcup_{n\ge1} \tilde{\mathcal{M}}_{1+n} \times \overline{\mathcal{M}}_1(J) \times .. \times \overline{\mathcal{M}}_1(J) \times \overline{\mathcal{M}}_2(J) \quad \cup \bigcup_{n\ge1} \tilde{\mathcal{M}}_{1+n} \times \overline{\mathcal{M}}_2(J) \times .. \times \overline{\mathcal{M}}_1(J) \times \overline{\mathcal{M}}_1(J) \quad \cup \bigcup_{n\ge1} \tilde{\mathcal{M}}_{2+n} \times \overline{\mathcal{M}}_1(J) \times .. \times \overline{\mathcal{M}}_1(J) \times \overline{\mathcal{M}}_1(J)$$

ev_0 ↘ M^n ← ; ev_∞ ; ev_0 ↘ M^n ← ; ev_∞ ; ev_0, ev_∞ ↘ M^n ←

② Construction of PSS, SSP, I, h

$$\langle * \rangle \longmapsto \sum_{\bullet} \sum_{u \in \mathcal{M}(*,\bullet)_0} a_u q^{E(u)} \langle \bullet \rangle = \sum_{\bullet} \sum_{i=0}^{\infty} m_{E_i}(*,\bullet)\, q^{E_i} \langle \bullet \rangle$$

$$m_{E_i}(*,\bullet) = \#\{u \in \mathcal{M}(*,\cdot) \mid \operatorname{ind} D_u = 0,\ E(u) = E_i\}$$

Gromov topology on $\overline{\mathcal{M}}.(\ldots)$

↳ $\operatorname{ind} \underline{u} = \sum_{u \in \underline{u}} (\operatorname{ind} D_u - \dim \operatorname{Aut})$, $E(\underline{u}) = \sum_u E(u)$ locally constant

↳ $\overline{\mathcal{M}}.(\ldots) \cap \{E(\underline{u}) \le C\}$ compact

$\Rightarrow \overline{\mathcal{M}}.(\ldots) \cap \{\operatorname{ind} \underline{u} = k\} = \bigcup_{i=0}^{\infty} \underbrace{\overline{\mathcal{M}}.(\ldots)_{k,E_i}}_{\text{compact}}$; $E_0 < E_1 < \ldots E_i \xrightarrow[i\to\infty]{} \infty$

Claim: $m_{E_i}(*,\cdot) := \langle [\overline{\mathcal{M}}.(*,\cdot)_{0,E_i}], 1 \rangle \in \mathbb{Q}$ is well defined

tbd

unique up to [redacted]-cobordism

Proof:

TODO • $\overline{\mathcal{M}}.(*,\cdot)_{0,E_i} = \sigma^{-1}(0)$ is the (compact) zero set of a [redacted]-section of index 0 (with boundary & corners)

• [redacted]-sections of index 0 with compact zero set have regularizations as
- Čech homology class $\in \check{H}_0(\sigma^{-1}(0), \mathbb{Q})$
- branched weighted 0-manifold $(\sigma + \underline{\nu})^{-1}(0)$

quotable abstract theory

that are unique up to "algebra induced by boundary strata"
and invariant under [redacted]-cobordism

E.g.

Siebert: [redacted] = "section of Banach orbibundle with Fredholm structure and global stabilization" $\mathcal{E} \underset{\downarrow}{} \circlearrowright s$, $\mathcal{B}$

(!no good notion of boundary & corners on topological orbifold $\mathcal{B}$!)

FO, ..., MW: [redacted] = "Kuranishi-section" $\bigsqcup_I U_I \times E_I \xrightarrow{} \bigsqcup_I U_I$ with sections s_I, $\circlearrowleft \Gamma_I$

$\sigma^{-1}(0) = \bigcup_I s_I^{-1}(0) / \Gamma_I$ / transition data

U_I manifolds with boundary & corners

HWZ: [redacted] = "polyfold Fredholm section" $\hat{\mathcal{E}} \downarrow \uparrow \sigma$, $\hat{\mathcal{B}}$ – polyfold with boundary & corners

"non-equivariant" proof of Arnold Conjecture

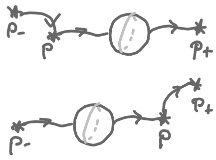
① Morse complex with Novikov coefficients

② $\overline{\mathcal{M}}(\ldots)_{0,E_i} = \sigma^{-1}(0)$ for "proper" index 0 [redacted]-sections unique up to [redacted] cobordism

$\Rightarrow$ PSS, SSP, h, I : $\langle * \rangle \longmapsto \sum_{\bullet,i} \langle [\overline{\mathcal{M}}(*,\cdot)_{0,E_i}], 1 \rangle \, q^{E_i} \langle \cdot \rangle$ well defined

④ $\overline{\mathcal{M}}^{I}(p_-,p_+)_{1,E_i} = \sigma^{-1}(0)$ for "proper" index 1 [redacted]-sections

$$\begin{array}{c} \overline{\mathcal{M}}^{I}(p_-,p_+)_{1,E_i} \\ \cap \\ \overline{\mathcal{M}}(p_-,M) \times_M \overline{\mathcal{M}}^{S^2}_{R=0} \times_M \overline{\mathcal{M}}(M,p_+) \end{array}$$



with

PREVIEW

$$\sigma|_{\text{boundary}}^{-1}(0) = \underbrace{\partial\overline{\mathcal{M}}(p_-,M)}_{\bigcup_p \mathcal{M}(p_-,p)\times\overline{\mathcal{M}}(p,M)} \times_M \overline{\mathcal{M}}^{S^2}_{R=0} \times_M \mathcal{M}(M,p_+) \;\cup\; \mathcal{M}(p_-,M) \times_M \overline{\mathcal{M}}^{S^2}_{R=0} \times_M \underbrace{\partial\overline{\mathcal{M}}(M,p_+)}_{\bigcup_p \overline{\mathcal{M}}(M,p)\times\mathcal{M}(p,p_+)}$$

$$= \bigcup_p \mathcal{M}(p_-,p) \times \overline{\mathcal{M}}^{I}(p,p_+)_{E_i} \qquad \cup \qquad \bigcup_p \overline{\mathcal{M}}^{I}(p_-,p)_{E_i} \times \mathcal{M}(p,p_+)$$

$$\text{index } 1 = \underbrace{|p|-|p_-|}_{\geq 1} + \underbrace{k}_{\geq 0} \qquad\qquad \underbrace{k}_{\geq 0} + \underbrace{|p_+|-|p|}_{\geq 1}$$

$\emptyset$ unless ≥ 1 ... ≥ 1

regularized $\emptyset$ unless ≥ 0 ... ≥ 0

$\Rightarrow$ $\overline{\mathcal{M}}^{I}(p_-,p_+)_{1,E_i}$ is regularized to compact weighted branched 1-manifold

with boundary $\displaystyle\bigcup_{|p|=|p_-|+1} \mathcal{M}(p_-,p)\times\overline{\mathcal{M}}^{I}(p,p_+)^{reg}_{0,E_i} \cup \bigcup_{|p|=|p_+|-1} \overline{\mathcal{M}}^{I}(p_-,p)^{reg}_{0,E_i}\times\mathcal{M}(p,p_+)$

$\Rightarrow$ $0 = I\circ\partial + \partial\circ I$

③ $I - SSP\circ PSS = \partial\circ h - h\circ\partial$ from boundary of regularized $\overline{\mathcal{M}}^{h}(p_-,p_+)_{1,E_i}$

③&④ $\Rightarrow$ $I = SSP\circ PSS$ on $\ker\partial / \operatorname{im}\partial = HM(f)$

"non-equivariant" proof of Arnold Conjecture

1. $CM(f) := \bigoplus_{p \in \mathrm{Crit} f} \Lambda \langle p \rangle \circlearrowleft \partial \rightsquigarrow \ker\partial / \mathrm{im}\,\partial = HM(f) \simeq H_*(M;\Lambda)$

2. - 4.

$$\boxed{\begin{tikzcd} CM(f) \arrow[r, "PSS"] \arrow[rr, bend right, "I"'] & CF(H) \arrow[r, "SSP"] \arrow[d, phantom, "\Downarrow h"] & CM(f) \\ & {} & \end{tikzcd}}$$

5. $I : HM_* \to HM_*$ isomorphism because

$$\overline{\mathcal{M}}^I(p_-, p_+)_{0,E} = \emptyset \quad \text{for } E < 0$$
$$= \{u \equiv p_- = p_+\} \quad \text{for } E = 0$$

are regular and don't need to be perturbed for coherence with $E > 0$

$\Rightarrow I = id_{CM(f)} + \sum_{i=0}^{\infty} q^{E_i} I_i \qquad 0 < E_0 < E_1 < \ldots E_i \xrightarrow[i \to \infty]{} \infty$

$\Rightarrow \exists I^{-1} = id_{CM(f)} + (\text{determined iteratively})$