Application of abstract regularization techniques

Proof of Arnold Conjecture

- Morse trajectory spaces
- PSS moduli spaces
- dream proof
- "nightmare" compact-cation

$\boxed{\text{Morse trajectory spaces}}$ $(f: M \to \mathbb{R},\ g \text{ metric on } M)$ "Morse-Smale"

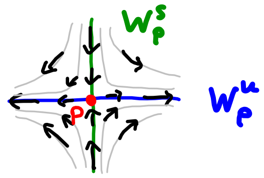
- $\forall p \in \mathrm{Crit} f = \{p \in M \mid \nabla f(p) = 0\}$ $\quad \nabla^2 f(p): T_pM \to T_p^*M$ bijective

  $\Rightarrow \forall v \in \mathbb{R} \setminus f(\mathrm{Crit} f)$: $f^{-1}(v) \subset M$ smooth submanifold "level set"

  $\Rightarrow \forall p \in \mathrm{Crit} f \ \exists \mathcal{C}^0\text{-chart } \mathrm{Nbhd}(p) \simeq \underbrace{\mathbb{R}^k}_{\text{negative}} \times \underbrace{\mathbb{R}^{\dim M - k}}_{\text{positive}}, \quad -\nabla f \simeq (x,y) \longmapsto (x,-y)$

  negative positive eigenspace of $\nabla^2 f(p)$

  [(f,g) "Euclidean" $\Rightarrow \mathcal{C}^\infty$
  otherwise $\mathcal{C}^\infty$ with eigenvalues $\neq \pm 1$]

  $\Rightarrow M \supset W_p^u = \{z \in M \mid (-\nabla f\ \mathrm{flow})_t(z) \xrightarrow[t\to-\infty]{} p\} \simeq \mathbb{R}^k \times \{0\}$ smooth

  $ M \supset W_p^s = \{z \in M \mid (-\nabla f\ \mathrm{flow})_t(z) \xrightarrow[t\to\infty]{} p\} \simeq \{0\} \times \mathbb{R}^{\dim M - k}$ submanifolds

  $|p| := k = \dim W_p^u$

- $\forall p, q \in \mathrm{Crit} f$: $W_p^u \pitchfork W_q^s \iff (W_p^u \cap f^{-1}(v)) \pitchfork (W_q^s \cap f^{-1}(v)) \subset f^{-1}(v)$

  $\forall v \in (f(q), f(p)) \setminus f(\mathrm{Crit} f)$

For $p \neq q \in \mathrm{Crit} f$ define manifolds of Morse trajectories

$W_p^u \cap W_q^s \cap f^{-1}(v)$ $\simeq$ $\quad x(\mathbb{R}) \cap f^{-1}(v)$ $\quad$ has compatible smooth structures $\forall v \in (f(p), f(q)) \setminus f(\mathrm{Crit} f)$ $\quad \dim = |p| - |q| - 1$

$$\mathcal{M}(p,q) := \{x : \mathbb{R} \to M \mid \dot{x} + \nabla f(x) = 0,\ x(s) \xrightarrow[s\to-\infty]{} p,\ x(s) \xrightarrow[s\to\infty]{} q\} / \mathbb{R}$$

$$\mathcal{M}(p,M) := \{x : (-\infty, 0] \to M \mid \text{—"—},\ \text{—"—}\} \xrightarrow{\sim} W_p^u, \quad x \longmapsto x(0)$$

$$\mathcal{M}(M,q) := \{x : [0,\infty) \to M \mid \text{—"—},\ \text{—"—}\} \xrightarrow{\sim} W_q^s, \quad x \longmapsto x(0)$$

$$\mathcal{M}(M,M) := \{x : [0,L] \to M \mid \text{—"—},\ L \geq 0\} \xrightarrow{\sim} M \times [0,\infty), \quad \longmapsto (x(0), L)$$

$\boxed{\text{Morse trajectory spaces}}$ $(f: M \to \mathbb{R},\ g \text{ metric on } M)$ "Morse-Smale"

$\mathcal{M}(p,q) := \{x : \mathbb{R} \to M \mid \dot{x} + \nabla f(x) = 0,\ x(s) \xrightarrow[s\to-\infty]{} p,\ x(s) \xrightarrow[s\to\infty]{} q\} / \mathbb{R}$ $\quad \mathcal{M}(p,p) := \emptyset$

$\mathcal{M}(p,M) := \{x : (-\infty,0] \to M \mid \text{—"—},\ \text{—"—}\}$

$\mathcal{M}(M,q) := \{x : [0,\infty) \to M \mid \text{—"—},\ \text{—"—}\}$

$\mathcal{M}(M,M) := \{x : [0,L] \to M \mid \text{—"—},\ L \geq 0\}$

<u>Compactification</u>:

$\overline{\mathcal{M}}(*,*) = \mathcal{M}(*,*) \cup \bigcup_p \mathcal{M}(*,p) \times \mathcal{M}(p,*) \cup \bigcup_{p_1,p_2} \mathcal{M}(*,p_1) \times \mathcal{M}(p_1,p_2) \times \mathcal{M}(p_2,*)$

$\cup \bigcup_{k\geq 3} \bigcup_{p_1 \ldots p_k} \mathcal{M}(*,p_1) \times \ldots \times \mathcal{M}(p_k,*)$

with Gromov-Hausdorff metric on images $\subset M$ (+ length difference for $\mathcal{M}(M,M)$) are compact metric spaces.

⊛ canonical for (f,g) Euclidean

<u>Folk Thm</u>: [... W'12] $\overline{\mathcal{M}}(\cdot,\cdot)$ can be given smooth⊛ manifold structure with "associative gluing maps" $\Rightarrow \partial\overline{\mathcal{M}} \setminus \text{corners} = \bigcup_p \mathcal{M}(\cdot,p) \times \mathcal{M}(p,\cdot)$

$\Rightarrow$ "corner degeneracy" $= k$

s.t. $ev : \overline{\mathcal{M}}(M,p_+) \to M$, $(x_0, x_1 \ldots x_k) \mapsto x_0(0)$, $\quad ev : \overline{\mathcal{M}}(p_-,M) \to M$, $(x_0, x_1, \ldots, x_k) \mapsto x_k(0)$ are smooth.

"minimized" PSS proof of Arnold Conjecture

(M,ω) closed

(f,g) Morse-Smale, $H: S^1\times M \to \mathbb{R}$ nondegenerate

Thm: $\# \mathcal{P}_H = \{\gamma : S^1 \to M \mid \dot\gamma = X_H(\gamma)\} \geq \sum_{i=0}^{\dim M} H_i(M;\mathbb{Q})$

Proof:

① $CM(f) := \bigoplus_{p\in \mathrm{Crit} f} \Lambda \langle p\rangle \circlearrowleft \partial : \langle p_-\rangle \longmapsto \sum_{\substack{p_+\in \mathrm{Crit} f \\ |p_+| = |p_-|-1}} \# \mathcal{M}(p_-,p_+)\langle p_+\rangle$

$$\begin{tikzcd} CM(f) \arrow[r, "PSS"] \arrow[rr, bend right, "I"'] & CF(H) \arrow[r, "SSP"] \arrow[d, phantom, "\Downarrow h"] & CM(f) \\ & {} & \end{tikzcd}$$

$\simeq$ on $HM(f) \cong H_*(M)$

② $PSS, SSP, I, h : \langle * \rangle \longmapsto \sum_{\bullet} \sum_{u\in \mathcal{M}(*,\cdot,J)_0} a_u q^{E(u)} \langle \cdot \rangle$

for $J \in \mathcal{J}(M,\omega)$

$\boxed{PSS:}$ $\mathcal{M}(p,\gamma,J) := \{ u : \mathbb{C} \to M \mid \bar\partial_J u = \hat\Theta^*(\hat\beta\cdot J X_H)^{0,1}(u),\ u(0)\in W^u_p,\ E(u)<\infty,\ u\circ\hat\Theta^{-1}(s,\cdot) \xrightarrow[s\to\infty]{} \gamma \}$

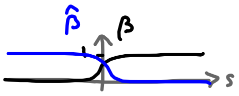
SSP $\quad \gamma \quad p$

$\hat\Theta : \mathbb{C}\setminus\{0\} \xrightarrow{\cong} \mathbb{R}\times S^1$

$e^{2\pi(s+it)} \longmapsto (s,t)$

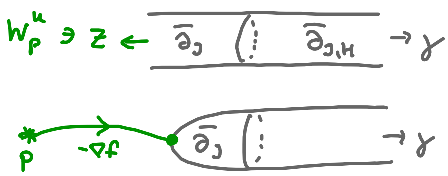
$\Updownarrow v = u \circ \hat\Theta^{-1}$

$\partial_s v + J(v)(\partial_t v - \hat\beta \cdot X_H(v)) = 0$



$E(u) := \int u^*\omega \pm \int_{S^1} H\circ\gamma$

$= \int |\partial_s v|^2 + \underbrace{\int \partial_s \hat\beta \cdot H\circ v}_{\text{uniformly bounded}}$



$\Downarrow$

$E(u) < \infty \Leftrightarrow \int |\partial_s v|^2 < \infty$

$E(u) \geq E_0$ fixed constant < 0

$\hookrightarrow \Lambda := \{ \sum_{i=1}^{\infty} a_i q^{E_i} \mid a_i \in \mathbb{Q},\ E_1 < E_2 < \dots,\ E_i \xrightarrow[i\to\infty]{} \infty \}$

Novikov field

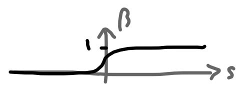
$$\mathrm{CM}(f) \xrightarrow{\mathrm{PSS}} \mathrm{CF}(H) \xrightarrow{\mathrm{SSP}} \mathrm{CM}(f) \qquad \Downarrow h \qquad I \qquad \simeq \text{ on } HM(f) \simeq H_*(M)$$

$\boxed{\text{PSS:}}$ $$\mathcal{M}(p,\gamma,J) := \left\{ u:\mathbb{C}\to M \;\middle|\; \bar\partial_J u = \Theta^*(\beta\cdot J X_H)^{0,1}(u),\ u(0)\in W^u_p,\ \begin{array}{l} E(u)<\infty \\ u\circ\Theta^{-1}(s,\cdot)\xrightarrow[s\to\infty]{}\gamma \end{array} \right\}$$

$\Theta : \mathbb{C}\setminus\{0\} \xrightarrow{\sim} \mathbb{R}\times S^1$, $e^{2\pi(s+it)} \mapsto (s,t)$

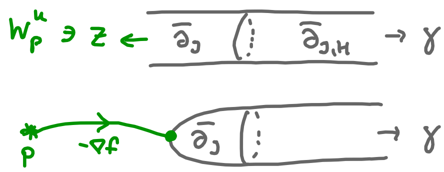
$\Updownarrow v = u\circ\Theta^{-1}$

$\partial_s v + J(v)(\partial_t v - \beta\cdot X_H(v)) = 0$



$$E(u) := \int u^*\omega - \int_{S^1} H\circ\gamma$$

$$= \int|\partial_s v|^2 + \underbrace{\int \partial_s\beta\cdot H\circ v}_{\text{uniformly bounded}}$$



$\boxed{\text{SSP:}}$ $$\mathcal{M}(\gamma,p,J) := \left\{ u:\mathbb{C}\to M \;\middle|\; \bar\partial_J u = \hat\Theta^*((1-\beta)J X_H)^{0,1}(u),\ u(0)\in W^s_p,\ \begin{array}{l} E(u)<\infty \\ u\circ\hat\Theta^{-1}(s,\cdot)\xrightarrow[s\to-\infty]{}\gamma \end{array} \right\}$$

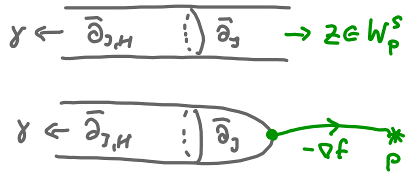
$\hat\Theta : \mathbb{C}\setminus\{0\} \xrightarrow{\sim} \mathbb{R}\times S^1$, $e^{-2\pi(s+it)} \mapsto (s,t)$

$\Updownarrow v = u\circ\Theta^{-1}$

$\partial_s v + J(v)(\partial_t v - (1-\beta)X_H(v)) = 0$

$$E(u) := \int u^*\omega + \int_{S^1} H\circ\gamma$$

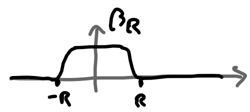
$$= \int|\partial_s v|^2 - \underbrace{\int \partial_s\beta\cdot H\circ v}_{\text{uniformly bounded}}$$



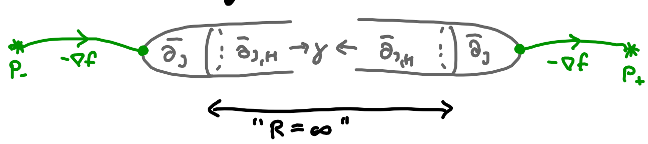
$\boxed{h:}$ $$\mathcal{M}(p_-,p_+,J) = \bigcup_{R\in[0,\infty]} \mathcal{M}_R(p_-,p_+,J)$$

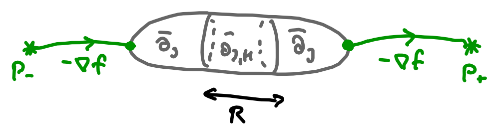
$$\mathcal{M}_R(p_-,p_+,J) \overset{R<\infty}{=} \left\{ u:\mathbb{CP}^1\to M \;\middle|\; \bar\partial_J u = \Theta^*(\beta_R J X_H)(u),\ u(0)\in W^u_{p_-},\ u(\infty)\in W^s_{p_+},\ E(u)<\infty \right\}$$

$\Theta : \mathbb{CP}^1\setminus\{0,\infty\} \to \mathbb{R}\times S^1$, $e^{2\pi(s+it)} \mapsto (s,t)$



$$\mathcal{M}_\infty(p_-,p_+,J) = \bigcup_\gamma \mathcal{M}(p_-,\gamma,J)\times\mathcal{M}(\gamma,p_+,J)$$



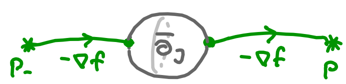


$$E(u) = \int u^*\omega = \int|\partial_s v|^2 - \int\partial_s\beta_R\cdot H\circ v$$

$$E(u_-,u_+) = \int u_-^*\omega + \int u_+^*\omega = E^{PSS}(u_-) + E^{SSP}(u_+)$$

$\boxed{I:}$ $$\mathcal{M}_{R=0}(p_-,p_+,J) = \left\{ u:\mathbb{CP}^1\to M \;\middle|\; \bar\partial_J u = 0,\ u(0)\in W^u_{p_-},\ u(\infty)\in W^s_{p_+},\ E(u)<\infty \right\}$$

$$E(u) = \int u^*\omega$$



② PSS, SSP, $\mathcal{I}$, h : $\langle * \rangle \mapsto \sum_{\cdot} \sum_{u \in \mathcal{M}(*,\cdot,J)_0} a_u q^{E(u)} \langle \cdot \rangle$

$\mathcal{M}(\dots)$ = zero set of Fredholm section $\sigma : u \mapsto \bar\partial_J u$ - 0^{th} order (e.g. $\Theta^*(\beta \cdot J X_H)^{0,1}(u)$)

$\mathcal{M}(\dots)_k := \{ u \in \mathcal{M}(\dots) \mid \operatorname{ind} D_u \sigma = k \}$

h: $\mathcal{M}(\dots) = \{(R,u) \mid \sigma_R(u) = 0\} \underset{\text{gluing}}{\cup} \mathcal{M}^{PSS} \times_{\mathcal{P}_H} \mathcal{M}^{SSP}$

$\mathcal{M}(\dots)_0 = \{(R,u) \in \mathcal{M} \mid \operatorname{ind} D_u \sigma_R = -1\}$ i.e. $\operatorname{ind} D_{(R,u)}((R,u) \mapsto \sigma_R(u)) = 0$

$\mathcal{M}(\dots)_1 = \{(R,u) \in \mathcal{M} \mid \operatorname{ind} D_u \sigma_R = 0\} \underset{\text{gluing}}{\cup} \mathcal{M}_0^{PSS} \times_{\mathcal{P}_H} \mathcal{M}_0^{SSP}$

(will need to be replaced by abstract regularization in general)

Dream Proof:

Find J s.t. all $\mathcal{M}(\dots)_{\leq 1}$ are cut out transversely (+gluing) and sphere bubbling is excluded (e.g. by "transversality + gluing" $\Rightarrow$ codim 2).

- Then $\mathcal{M}(\dots)_0$ are 0-manifolds with $\mathcal{M}(\dots)_0 \cap \{E(u) < E\}$ compact $\forall E > 0$.

$\Rightarrow$ " $\bigcup_{i=0}^{\infty} \{E(u) = E_i\}$ $\quad E_0 < E_1 < \dots E_i \xrightarrow[i\to\infty]{} \infty$

$\Rightarrow$ $\sum_{u \in \mathcal{M}(\dots)_0} a_u q^{E(u)} = \sum_{i \geq 0}^{\infty} \#\{E(u) = E_i\} q^{E_i} \in \Lambda$ well defined ($a_u = \pm 1$ from orientation of $\mathcal{M}(\dots)_0$)

$\Rightarrow$ PSS, SSP, h, $\mathcal{I}$ well defined

$\| \quad \int_{\mathcal{M}(\dots)_0} q^{E(u)} du = Q_*[\mathcal{M}(\dots)_0]$

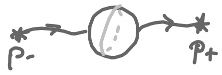
$Q : \mathcal{M}(\dots)_0 \to \Lambda,\ u \mapsto q^{E(u)}$

Dream Proof.

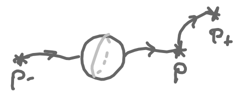
② $\mathcal{M}(\ldots)_0$ are 0-manifolds with $\mathcal{M}(\ldots)_0 \cap \{E(u)<E\}$ compact $\forall E>0$.

$\Rightarrow$ PSS, SSP, h, I : $\langle * \rangle \longmapsto \sum_{\cdot} \int_{\mathcal{M}(*,\cdot)_0} q^{E(u)}\, du \,\langle \cdot \rangle$ well defined

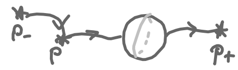
④ $\mathcal{M}^I(p_-,p_+,J)_1$ is 1-manifold with ends



$\bigcup_p \mathcal{M}^I(p_-,p,J)\times\mathcal{M}(p,p)$



$\cup \bigcup_p \mathcal{M}(p_-,p)\times\mathcal{M}^I(p,p_+,J)$

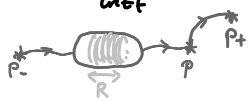


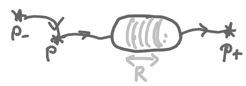
$\Rightarrow 0 = \partial I - I\partial$

③ $\mathcal{M}^h(p_-,p_+,J)_1 = \bigcup_{R\in[0,\infty]} \mathcal{M}_R(p_-,p_+,J)_0$ is 1-manifold with boundary @R=0,∞ and ends

$\mathcal{M}^I(p_-,p_+,J) \cup \mathcal{M}(p_-,\cdot,J)_0 \times_{\mathbb{P}_H} \mathcal{M}(\cdot,p_+,J)_0 \cup \mathcal{M}^h(p_-,\cdot,J) \times_{\mathrm{Crit} f} \mathcal{M}(\cdot,p_+,J) \cup \mathcal{M}(p_-,\cdot) \times_{\mathrm{Crit} f} \mathcal{M}^h(\cdot,p_+,J)$

R = 0   R = ∞





$\Rightarrow I - SSP \circ PSS = \partial h - h\partial$

③&④ $\Rightarrow$ $I = SSP \circ PSS$ on $\ker\partial / \mathrm{im}\,\partial = HM(f)$

"minimized" PSS proof of Arnold Conjecture

① $CM(f) := \bigoplus_{p \in \mathrm{Crit} f} \Lambda \langle p \rangle \circlearrowleft \partial \quad \rightsquigarrow \quad \ker\partial / \mathrm{im}\,\partial = HM(f) \simeq H_*(M;\Lambda)$

②-④

$$\boxed{\begin{tikzcd} CM(f) \arrow[r, "PSS"] \arrow[rr, bend right, "I"'] & CF(H) \arrow[r, "SSP"] \arrow[d, phantom, "\Downarrow h"] & CM(f) \\ & {} & \end{tikzcd}}$$

⑤ $I : HM_* \to HM_*$ isomorphism because

$$\mathcal{M}^I(p_-,p_+,J)_0 = \left\{ u : \mathbb{CP}^1 \to M \;\middle|\; \bar\partial_J u = 0,\ u(0) \in W^u_{p_-},\ u(\infty) \in W^s_{p_+},\ \begin{matrix} E(u) < \infty \\ \mathrm{ind}\, D_u \bar\partial = 0 \end{matrix} \right\}$$

$\Downarrow$

$E(u) = \frac{1}{2}\int |du|^2 \geq 0$

$E(u) = 0 \Rightarrow u = \mathrm{const} \in W^u_{p_-} \cap W^s_{p_+} \Rightarrow \mathrm{ind} \geq 1$ unless $p_- = p_+ \equiv u$

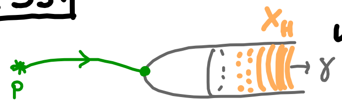
$$\Rightarrow \mathcal{M}^I(p_-,p_+,J)_0 = \{u \equiv p_- = p_+\} \cup \bigcup_{i=0}^{\infty} \{E(u) = E_i\} \qquad 0 < E_0 < E_1 \dots E_i \xrightarrow[i\to\infty]{} \infty$$

$$\Rightarrow I = id_{CM(f)} + \sum_{i=0}^{\infty} q^{E_i} I_i \qquad I_i : \langle p_- \rangle \mapsto \sum_{p_+} \# \begin{pmatrix} \mathcal{M}(p_-,p_+,J)_0 \\ \cap \{E(u) = E_i\} \end{pmatrix} \langle p_+ \rangle$$

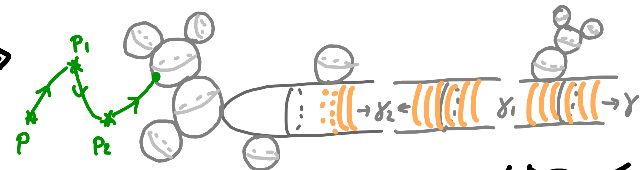
$$\Rightarrow \exists\, I^{-1} = id_{CM(f)} + (\text{determined iteratively})$$

PREVIEW: Breaking & Bubbling ⇝ "Compact-cation"

PSS:



$\xrightarrow[r\to\infty]{u^r}$



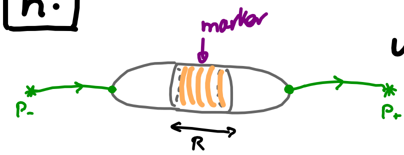
$$\mathcal{M}(p,\gamma)=\left\{\begin{matrix}(u,0)\in\mathcal{M}_1(\gamma):\\ ev(u)\in W^s_p\end{matrix}\right\}$$

$\overline{\mathcal{M}}(p,M)$ $\quad \overline{\mathcal{M}}_1^{S^2}(\gamma_2)$ $\quad \overline{\mathcal{M}}^{S^2}(\gamma_2,\gamma_1)$ $\quad \overline{\mathcal{M}}^{S^2}(\gamma_1,\gamma)$

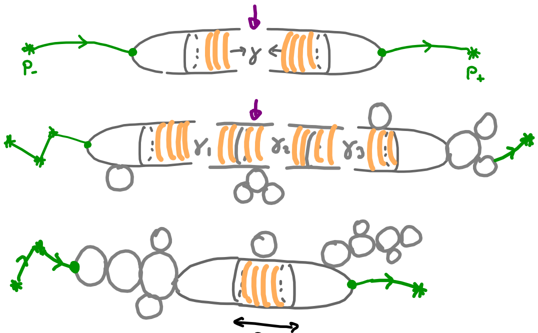
$\overline{\mathcal{M}}(p,M) \xrightarrow{ev} M \xleftarrow{ev} \overline{\mathcal{M}}_1^{S^2}(\gamma_2)$

$$\overline{\mathcal{M}}(p,\gamma) := \bigcup_{k=0}^{\infty}\ \bigcup_{\gamma_1\ldots\gamma_k\in\mathcal{P}_H} \overline{\mathcal{M}}(p,M)\times_M \overline{\mathcal{M}}_1^{S^2}(\gamma_k)\times\overline{\mathcal{M}}^{S^2}(\gamma_k,\gamma_{k-1})\ \ldots\times\overline{\mathcal{M}}^{S^2}(\gamma_1,\gamma)$$

h:



$\xrightarrow[r\to\infty]{u^r}$



"forced breaking" $;\ R^r\to\infty$

$;\ R^r\to R_\infty<\infty$

$$\bigcup_{0\le R<\infty}\mathcal{M}_R(p_-,p_+)$$

$\overline{\mathcal{M}}^{S^2}_{2,R_\infty}$ $\xrightarrow{ev_1} M$, $\xrightarrow{ev_2} M$

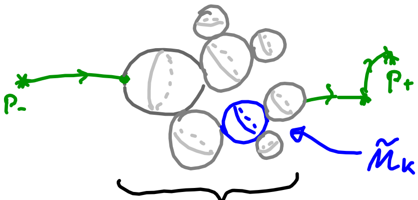
$$\overline{\mathcal{M}}(p_-,p_+)=\bigcup_{0\le R<\infty}\overline{\mathcal{M}}(p_-,M)\times_M\overline{\mathcal{M}}^{S^2}_{2,R}\times_M\overline{\mathcal{M}}(M,p_+)$$

$$\cup\bigcup_{k=0}^{\infty}\ \bigcup_{\gamma_1\ldots\gamma_k\in\mathcal{P}_H}\underbrace{\overline{\mathcal{M}}(p_-,M)\times_M\overline{\mathcal{M}}_{1,\,PSS}^{S^2}(\gamma_1)\times\overline{\mathcal{M}}(\gamma_1,\gamma_2)\ldots}_{\in\overline{\mathcal{M}}(p_-,\gamma_k)}\times\underbrace{\overline{\mathcal{M}}(\gamma_{k-1},\gamma_k)\times\overline{\mathcal{M}}_{1,\,SSP}^{S^2}(\gamma_k)\times_M\overline{\mathcal{M}}(M,p_+)}_{\in\overline{\mathcal{M}}(\gamma_k,p_+)}$$

marker

I: $\overline{\mathcal{M}}^I(p_-,p_+)=\overline{\mathcal{M}}^h(p_-,p_+)\cap\{R=0\}$

$$=\overline{\mathcal{M}}(p_-,M)\times_M\overline{\mathcal{M}}^{S^2}_{2,\,R=0}\times_M\overline{\mathcal{M}}(M,p_+)$$



$\overline{\mathcal{M}}_2^{S^2}$

$$\overline{\mathcal{M}}_2^{S^2}=\{(u,0,\infty)\mid\bar\partial_J u=0\}\cup\bigcup_{k\ge1}\widetilde{\mathcal{M}}_k\times\overline{\mathcal{M}}_2(J)\times\overline{\mathcal{M}}_1(J)\times\ldots\times\overline{\mathcal{M}}_1(J)\times\overline{\mathcal{M}}_2(J)$$

$ev_0\downarrow M$, $ev_\infty\downarrow M$; $\to M^k$; ev_0; ev_∞

$$\cup\bigcup_{k\ge1}\widetilde{\mathcal{M}}_{1+k}\times\overline{\mathcal{M}}_1(J)\times\ldots\times\overline{\mathcal{M}}_1(J)\times\overline{\mathcal{M}}_2(J)\quad\cup\bigcup_{k\ge1}\widetilde{\mathcal{M}}_{1+k}\times\overline{\mathcal{M}}_2(J)\times\ldots\times\overline{\mathcal{M}}_1(J)\times\overline{\mathcal{M}}_1(J)\quad\cup\bigcup_{k\ge1}\widetilde{\mathcal{M}}_{2+k}\times\overline{\mathcal{M}}_1(J)\times\ldots\times\overline{\mathcal{M}}_1(J)\times\overline{\mathcal{M}}_1(J)$$

ev_0, $\to M^k$, ev_∞; $\quad ev_\infty$, $\to M^k$, ev_0; $\quad ev_0$, ev_∞, $\to M^k$