Abstract approaches to regularizing moduli spaces of pseudoholomorphic curves

Approach #2: Regularization via finite dimensional reductions

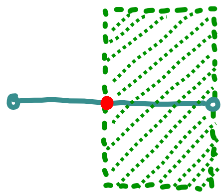
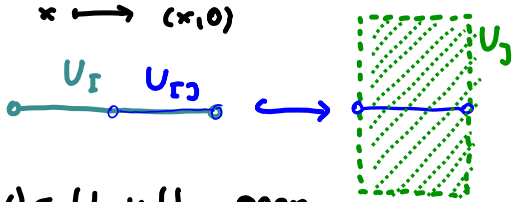
Fukaya-Ono 1999 -Oh-Ohta 2009 + 2012	Kuranishi structure (of germs) + categories
Joyce	—— " —— $\rightsquigarrow$ d-orbifold (2-category)
McDuff - W. 2012	Kuranishi atlas & categories
Pardon	—— " ——
⋮	

NIGHTMARES IN POINT SET TOPOLOGY I

typical example of coordinate change $U_I \supset U_{IJ} \hookrightarrow U_J$

$(-1,1) \supset (0,1) \longrightarrow (0,1)\times(-2,2)$

$x \longmapsto (x,0)$

$\mathcal{U} = \dfrac{U_I \cup U_J}{U_{IJ}}$  

with

quotient topology: $V \subset \mathcal{U}$ open $:\Leftrightarrow pr^{-1}(V) \subset U_I \cup U_J$ open

$pr_I \searrow \mathcal{U} \swarrow pr_J$ $\quad \Leftrightarrow pr_I^{-1}(V), pr_J^{-1}(V)$ open

is not

- **locally compact**: $[0] \in \mathcal{U}$ has no compact neighbourhood $K \subset \mathcal{U}$

$\rightarrow [0] \in V \subset K$, V open $\Rightarrow \exists \varepsilon > 0 : [-\varepsilon, \varepsilon] \subset pr_I^{-1}(V) \Rightarrow (\varepsilon, 0) \in pr_J^{-1}(V)$

$\Rightarrow \forall k \in \mathbb{N}\ \exists \delta_k > 0 : \{2^{-k}\varepsilon\} \times [-\delta_k, \delta_k] \subset pr_J^{-1}(V)$, $\delta_k \leq \varepsilon$

$\Rightarrow \forall n \in \mathbb{N}\ W_n := pr_J\Big(\Big(\bigcup_{k \geq n} (2^{-k-1}\varepsilon, 2^{-k+1}\varepsilon) \times (-2\varepsilon, 2\varepsilon) \setminus \{(2^{-k}\varepsilon, \delta_k)\}\Big) \cup (2^{-n}\varepsilon, 2\varepsilon) \times (-2\varepsilon, 2\varepsilon)\Big)$

$\cup\, pr_I((-2\varepsilon, 2\varepsilon))$ $\quad$ is open in $\mathcal{U}$ and contains $(2^{-k}\varepsilon, \delta_k) \in pr_J^{-1}(V)$ iff $n > k$

$\Rightarrow K \subset \mathcal{U} \setminus pr([-\varepsilon, \varepsilon] \cup (0, \varepsilon] \times [-\varepsilon, \varepsilon]) \cup \bigcup_{n \in \mathbb{N}} W_n$ has no finite subcover $\lightning$

more generally

every continuous map $f : (0,1) \to (0, \infty)$ defines an open neighbourhood

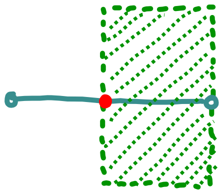
$V_f := pr_I(U_I) \cup pr_J(\{(x,y) \in U_J \mid |y| < f(x)\}) \subset \mathcal{U}$ of $[0]$

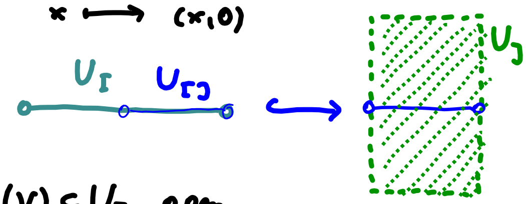
NIGHTMARES IN POINT SET TOPOLOGY (II)

typical example of coordinate change $U_I \supset U_{IJ} \hookrightarrow U_J$

$(-1,1) \supset (0,1) \longrightarrow (0,1)\times(-2,2)$
$\qquad\qquad\qquad x \longmapsto (x,0)$

$\mathcal{U} = \dfrac{U_I \cup U_J}{U_{IJ}}$





with

quotient topology: $V \subset \mathcal{U}$ open $:\Longleftrightarrow$ $pr_I^{-1}(V) \subset U_I$ open, $pr_J^{-1}(V) \subset U_J$ open

is not

- **metrizable** $\overset{\text{Nagata Smirnov}}{\Longleftrightarrow}$ regular and has countably locally finite basis $(\mathcal{W}_n)_{n\in\mathbb{N}}$

  (closed sets $C \subset \mathcal{U}$ and points $p \in \mathcal{U}$ can be separated: $C \subset V_C$ open, $p \in V_p$ open, $V_C \cap V_p = \emptyset$)

  ($\forall n$: $\mathcal{W}_n$ locally finite collection of open sets $W \subset \mathcal{U}$; $\forall O \subset \mathcal{U}$ open, $x \in O$ $\exists n, W \in \mathcal{W}_n$: $x \in W \subset O$)

  $\Downarrow\ W_n = B_{\frac{1}{n}}(x)$ $\qquad\qquad \Downarrow$

- **1^{st} countable** : $[0] \in \mathcal{U}$ has no countable neighbourhood base

  $\big([0] \in W_n \subset \mathcal{U} \text{ open}\big)_{n\in\mathbb{N}}$: $\forall [0] \in V \subset \mathcal{U}$ open $\exists n$: $W_n \subset V$

since
- $pr^{-1}(W_n) \subset U_I$ open $\Rightarrow \exists \varepsilon_n > 0$: $pr_I([-\varepsilon_n, \varepsilon_n]) \subset W_n$

  $\Rightarrow x \in (0,\varepsilon_n]$: $[(x,0)] \in pr_J^{-1}(W_n)$ $\Rightarrow \exists h_n(x) = h > 0$: $\{x\}\times[-h,h] \subset pr_J^{-1}(W_n)$
- inductively define $\delta_n > 0$ s.t. $0 < \delta_n \le \min\{\frac{1}{2}\delta_{n-1}, \varepsilon_1, \dots, \varepsilon_n\}$
- ________ $f : (0,1) \to (0,\infty)$ s.t. $f(\delta_n) = h_n(\delta_n)$ $\Rightarrow$ $[(\delta_n, f(\delta_n))] \in W_n \setminus V_f$

  $V_f = pr_J^{-1}\{|y| < f(x)\} \cup \dots$

  e.g. by $f_n(x\delta_n + (1-x)\delta_{n+1}) = x h_n(\delta_n) + (1-x) h_{n+1}(\delta_{n+1})$ for $x \in [0,1]$

GOOD NEWS IN POINT SET TOPOLOGY

(for $X = |(s+r)^{-1}(0)|$)

Lemma: X compact, locally homeomorphic to $\mathbb{R}^n$ $\Rightarrow$ second countable

$\Updownarrow$

$\exists\, (W_n \subset X \text{ open})_{n\in\mathbb{N}} : \forall O \subset X \text{ open}, x \in O\ \exists n : x \in W_n \subset O$

$(\Rightarrow O = \bigcup_{n_x} W_{n_x})$

Proof: cover X with finitely many charts $U_i \hookrightarrow \mathbb{R}^n$

take $\mathcal{W}_i$ countable basis of U_i $\Rightarrow (W_n)_{n\in\mathbb{N}} = \bigcup_i \mathcal{W}_i$ basis for X

"nested uniqueness of compact Hausdorff topologies"

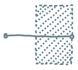
$\tau_1 \subset \tau_2$ compact Hausdorff topologies on X $\Rightarrow$ $\tau_1 = \tau_2$

$f = id : \underset{X_2 \text{ compact}}{(X,\tau_2)} \to \underset{X_1 \text{ Hausdorff}}{(X,\tau_1)}$ continuous, bijective $\Big\} \Rightarrow f^{-1}$ continuous

$\overset{\circledast}{\Rightarrow} f$ closed $\Rightarrow f$ open

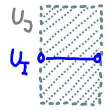
$\circledast$ $C \subset X_2$ closed $\Rightarrow C$ compact $\Rightarrow f(C)$ compact $\Rightarrow f(C)$ closed

MORE CHALLENGES & IDEAS

① avoid  by restricting to coordinate changes $U_I = U_{IJ} \hookrightarrow U_J$

$\to$ "Kuranishi structure" • chart $(U_p, \dots)$ $\forall p \in \overline{\mathcal{M}}$

"tame" • coord. change $(U_q \hookrightarrow U_p, \dots)$ $\forall q \in F_p$



⚠ this is stronger than the usual def^n of Kuranishi structure which actually does allow $U_q \supsetneq U_{qp} \hookrightarrow U_p$

• in practice obtained from atlas:

pick $K_i \subset F_i$ compact s.t. $\overline{\mathcal{M}} \subset \bigcup_{i=1}^{N} K_i$

for $p \in \overline{\mathcal{M}}$ let $I_p := \{ i \mid p \in K_i \} = \max \{ I \mid p \in \bigcap_{i \in I} K_i \}$ and

$$U_p := \Big(U_{I_p} \setminus \bigcup_{j \notin I_p} \psi_{I_p}^{-1}(K_j) \Big) \cap \bigcap_{\substack{J \supsetneq I_p \\ p \in F_J}} U_{I_p J}, \quad s_p := s_{I_p}|_{U_p}, \quad \psi_p := \psi_{I_p}|_{s_p^{-1}(0)}$$

$$\Rightarrow F_p = \operatorname{im} \psi_p = \Big(F_{I_p} \setminus \bigcup_{j \notin I_p} K_j \Big) \cap \bigcap_{\substack{J \supsetneq I_p \\ p \in F_J}} F_J = F_{J_p} \setminus \bigcup_{j \notin I_p} K_j \quad \text{with } J_p = \{ j \mid p \in F_j \}$$

Now for $q \in F_p$ have $I_q \subset I_p$ and $U_q \subset U_{I_q I_p}$ so that $\varphi_{pq} : U_q \to U_{I_p}$ is defined. We have $\varphi_{pq}(U_q) \subset U_p$ if the strong cocycle condition holds (e.g. after "taming").

• looses partial order $I \subset J$ of charts since $q \in F_p$, $p \in F_q$ is "generic"

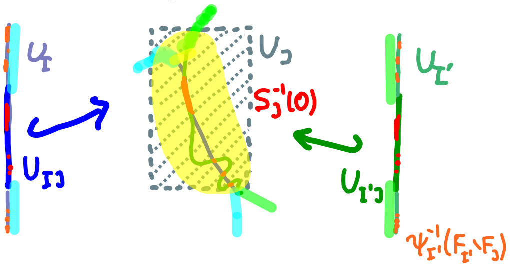
② construction of compatible perturbations requires overlap control

$$\begin{array}{ccccc} E_I & \longrightarrow & & & E_J \\ \uparrow \nu_I & & & & \uparrow \nu_J \\ U_I & \supset & U_{IJ} & \hookrightarrow & U_J \end{array}$$

"$U_I \cap U_{I'} \neq \emptyset$" $\Rightarrow$ $I \subset I'$ or $I' \subset I$

[FO]: glue germs to "good coordinate system"

[MW]: reduction $V_I \subset U_I$ e.g. $V_I, V_{I'}, V_J$

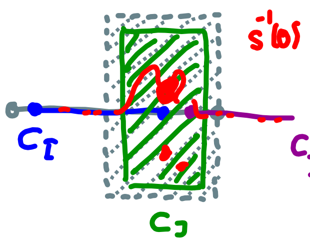


③ preserve compactness of $|(s+\nu)^{-1}(0)| = \bigsqcup (s_I + \nu_I)^{-1}(0) \big/ \varphi_{IJ}$ for small ν

by precompact shrinking $C_I \sqsubset U_I$ (i.e. $\overline{C}_I$ compact)

with $\|\nu\| < \delta$ $\Rightarrow$ $|(s+\nu)^{-1}(0)| \subset \bigsqcup C_I \big/ \varphi_{IJ}$

(! not automatic near $C_I \cap \partial \overline{C}_J$, but can use ν_J taking values in E_I near C_I, then perturbed zeros near $C_I \hookrightarrow U_J$ lie in $s_J^{-1}(E_I) = \operatorname{im} \varphi_{IJ}$ (tameness))



$\to$ also need to control supports

$\to$ need compatible metrics

MORE CHALLENGES & IDEAS

④ quotient by ambiguities (as in algebraic geometry)

$(U_p, \dots) \sim (U'_p, \dots)$ if

$$\begin{tikzcd} E & F \arrow[l, "\sim"'] \arrow[r, "\sim"] & E' \\ U \arrow[u, "s"] & V \arrow[l, hook'] \arrow[r, hook] \arrow[u, "t"'] & U' \arrow[u, "s'"'] \\ s^{-1}(0) \arrow[u, phantom, "\subset" rotate=90] & t^{-1}(0) \arrow[l, hook'] \arrow[r, hook] \arrow[u, phantom, "\subset" rotate=90] \arrow[d] & s'^{-1}(0) \arrow[u, phantom, "\subset" rotate=90] \\ & \mathrm{nbhd}(p) \subset \overline{\mathcal{M}} & \end{tikzcd}$$

e.g.

$$\begin{tikzcd} E \arrow[r, "\sim"] & E' \\ U \arrow[u, "s"] \arrow[r, "I"] & U' \arrow[u, "s'"'] \\ s^{-1}(0) \arrow[u, phantom, "\subset" rotate=90] \arrow[r, "id"'] & s'^{-1}(0) \arrow[u, phantom, "\subset" rotate=90] \end{tikzcd}$$

↳ coordinate changes between germs of charts are conjugacy classes $[\varphi_{pq}] = [I_p^{-1} \varphi'_{pq} I_q] = [\varphi'_{pq}] : [(U_q \dots)] \to [(U_p \dots)]$

↳ cocycle condition $\varphi_{pq} \circ \varphi_{qr} = \varphi_{pr}$ problematic

in [FO-1999] , [J-Kuranishi]

$\Downarrow$ $I = id$ [FOOO-2009]

$\Downarrow$ restrict (I_p, I_q) to get 2-category [J- d-orbifolds]

$$\begin{tikzcd} U_q \arrow[r, hook, "\varphi_{pq}"] \arrow[d, "I_q"'] & U_p \arrow[r, hook, "\varphi_{rp}"] \arrow[d, bend right, "I_p"'] \arrow[d, bend left, "\tilde{I}_p"] & U_r \arrow[d, "I_r"] \\ U'_q \arrow[r, hook, "\varphi'_{pq}"] & U'_p \arrow[r, hook, "\varphi'_{rp}"] & U'_r \end{tikzcd}$$

Def$^{\underline{n}}$: additive weak **Kuranishi atlas** for compact metrizable $\overline{\mathcal{M}}$ consists of
(with isotropy)

- **Kuranishi charts** $\left(\underset{\text{finite}}{\Gamma_i} \circlearrowright \begin{matrix} E_i \ \text{vector space} \\ \uparrow s_i \\ U_i \ \text{manifold} \end{matrix} \qquad s_i^{-1}(0)/\Gamma_i \ \underset{\text{homeo}}{\overset{\psi_i}{\cong}} \ \overset{\text{"footprint"}}{F_i} \underset{\text{open}}{\subset} \overline{\mathcal{M}} \right)_{i=1..N}$ covering $\overline{\mathcal{M}} = \bigcup_i F_i$

- **Transition data** consisting of
  - additive **sum charts** for $I \subset \{1 \ldots N\}$ s.t. $\bigcap_{i\in I} F_i \neq \emptyset$

    $\overset{\prod \Gamma_i}{\overset{\shortparallel}{\Gamma_I}} \circlearrowright \begin{matrix} E_I = \bigoplus_{i\in I} E_i \\ \downarrow \uparrow s_I \\ U_I \end{matrix} \qquad s_I^{-1}(0)/\Gamma_I \simeq \bigcap_{i\in I} F_i \subset \overline{\mathcal{M}}$

  - **coordinate changes** for $I \subsetneq J$

$$\begin{array}{ccc} E_I & \longrightarrow & E_J \\ \uparrow s_I & & \uparrow s_J \\ U_I \supset U_{IJ} & \xhookrightarrow{\varphi_{IJ}} & U_J \end{array} \qquad \left(\begin{array}{ccc} \Gamma_J \circlearrowright & \tilde{U}_{IJ} \hookrightarrow U_J & \\ & \downarrow \text{covering} & \\ U_I \supset & U_{IJ} \circlearrowleft \Gamma_I & \end{array} \right)$$

satisfying the index condition: $\ker ds_I \overset{d\varphi_{IJ}}{\cong} \ker ds_J$, $\quad E_I/\mathrm{im}\, ds_I \simeq E_J/\mathrm{im}\, ds_J$

$\Updownarrow$

[J]/[FO^3]-"tangent bundle condition" $\qquad$ ($\Rightarrow$ index ds_I = index ds_J = "virdim")

- satisfying the weak cocycle condition $\boxed{\varphi_{JK} \circ \varphi_{IJ} = \varphi_{IK} \quad \forall I \subset J \subset K}$
  where both are defined $(\varphi_{IJ}^{-1}(U_{JK}) \cap U_{IK})$

Obj: $\amalg U_I \times E_I \xrightarrow{pr} \amalg U_I$ (assuming cocycle condition $\varphi_{IJ}^{-1}(U_{JK}) \subset U_{IK}$)

Lemma 6.1.9. *The functor* $\mathrm{pr}_{\mathcal{K}} : \mathbf{E}_{\mathcal{K}} \to \mathbf{B}_{\mathcal{K}}$ *induces a continuous map*

$$|\mathrm{pr}_{\mathcal{K}}| : |\mathbf{E}_{\mathcal{K}}| \to |\mathcal{K}| = |\mathcal{U}|$$

which we call the **obstruction bundle** *of* $\mathcal{K}$*, although its fibers generally do not have the structure of a vector space.*[18] *However, it has a continuous zero section*

$$|0_{\mathcal{K}}| : |\mathcal{K}| \to |\mathbf{E}_{\mathcal{K}}|, \quad [I, x] \mapsto [I, x, 0].$$

Moreover, the section $s_{\mathcal{K}} : \mathbf{B}_{\mathcal{K}} \to \mathbf{E}_{\mathcal{K}}$ *descends to a continuous section*

$$|s_{\mathcal{K}}| : |\mathcal{K}| \to |\mathbf{E}_{\mathcal{K}}|.$$

These maps are sections in the sense that $|\mathrm{pr}_{\mathcal{K}}| \circ |s_{\mathcal{K}}| = |\mathrm{pr}_{\mathcal{K}}| \circ |0_{\mathcal{K}}| = \mathrm{id}_{|\mathcal{K}|}$*. Moreover, there is a natural homeomorphism from the realization of the subcategory* $s_{\mathcal{K}}^{-1}(0)$ *to the zero set of the section, with the relative topology induced from* $|\mathcal{K}|$,

$$|s_{\mathcal{K}}^{-1}(0)| = {s_{\mathcal{K}}^{-1}(0)} \big/ {\sim_{s_{\mathcal{K}}^{-1}(0)}} \xrightarrow{\cong} |s_{\mathcal{K}}|^{-1}(0) := \{[I, x] \mid s_I(x) = 0\} \subset |\mathcal{K}|.$$

$$\begin{array}{c} \amalg U_I \times E_I / \sim \\ |pr| \downarrow \quad \uparrow |s|, |0| \\ \amalg U_I / \sim \\ \cup \\ |s|^{-1}(0) \simeq \amalg s_I^{-1}(0) / \sim \\ \wr\!\wr \; |\psi| \\ \bar{M} \end{array}$$

zero set of a continuous map $|s|$

Lemma 6.1.10. *The footprint functor* $\psi_{\mathcal{K}} : s_{\mathcal{K}}^{-1}(0) \to \mathbf{X}$ $= \bar{M}$ *descends to a homeomorphism* $|\psi_{\mathcal{K}}| : |s_{\mathcal{K}}|^{-1}(0) \to X$*. Its inverse is given by*

$$\iota_{\mathcal{K}} := |\psi_{\mathcal{K}}|^{-1} : X \longrightarrow |s_{\mathcal{K}}|^{-1}(0) \subset |\mathcal{K}|, \qquad p \mapsto [(I, \psi_I^{-1}(p))],$$

where $[(I, \psi_I^{-1}(p))]$ *is independent of* $I \in \mathcal{I}_{\mathcal{K}}$ *with* $p \in F_I$*.*

Theorem 6.2.6. *Let $\mathcal{K}$ be an additive weak Kuranishi atlas on a compact metrizable space X. Then an appropriate shrinking of $\mathcal{K}$ provides a metrizable tame Kuranishi atlas $\mathcal{K}'$ with domains $(U_I' \subset U_I)_{I \in \mathcal{I}_{\mathcal{K}'}}$ such that the realizations $|\mathcal{K}'|$ and $|\mathbf{E}_{\mathcal{K}'}|$ are Hausdorff in the quotient topology. In addition, for each $I \in \mathcal{I}_{\mathcal{K}'} = \mathcal{I}_{\mathcal{K}}$ the projection maps $\pi_{\mathcal{K}'} : U_I' \to |\mathcal{K}'|$ and $\pi_{\mathcal{K}'} : U_I' \times E_I \to |\mathbf{E}_{\mathcal{K}'}|$ are homeomorphisms onto their images and fit into a commutative diagram*

$$\begin{array}{ccc} U_I' \times E_I & \xrightarrow{\pi_{\mathcal{K}'}} & |\mathbf{E}_{\mathcal{K}'}| = \amalg U_I' \times E_I /\sim \\ \downarrow & & \downarrow |\mathrm{pr}_{\mathcal{K}'}| \\ U_I' & \xrightarrow{\pi_{\mathcal{K}'}} & |\mathcal{K}'| = \amalg U_I' /\sim \end{array}$$

where the horizontal maps intertwine the vector space structure on E_I with a vector space structure on the fibers of $|\mathrm{pr}_{\mathcal{K}'}|$.

Moreover, any two such shrinkings are cobordant by a metrizable tame Kuranishi cobordism whose realization also has the above Hausdorff, homeomorphism, and linearity properties.

weak K.atlas

unique up to cobordism

metrizable tame K.atlas

Definition 6.2.4. *A Kuranishi atlas $\mathcal{K}$ is said to be* **metrizable** *if there is a bounded metric d on the set $|\mathcal{K}|$ such that for each $I \in \mathcal{I}_{\mathcal{K}}$ the pullback metric $d_I := (\pi_{\mathcal{K}}|_{U_I})^* d$ on U_I induces the given topology on the manifold U_I. In this situation we call d an* **admissible metric** *on $|\mathcal{K}|$. A* **metric Kuranishi atlas** *is a pair $(\mathcal{K}, d)$ consisting of a metrizable Kuranishi atlas together with a choice of admissible metric d. For a metric*

Definition 6.2.7. *A weak Kuranishi atlas is* **tame** *if it is additive, and for all $I, J, K \in \mathcal{I}_{\mathcal{K}}$ we have*

$$(6.2.4) \qquad U_{IJ} \cap U_{IK} = U_{I(J \cup K)} \qquad \forall I \subset J, K;$$

$$(6.2.5) \qquad \phi_{IJ}(U_{IK}) = U_{JK} \cap s_J^{-1}\big(\widehat{\phi}_{IJ}(E_I)\big) \qquad \forall I \subset J \subset K.$$

Here we allow equalities, using the notation $U_{II} := U_I$ and $\phi_{II} := \mathrm{Id}_{U_I}$. Further, to allow for the possibility that $J \cup K \notin \mathcal{I}_{\mathcal{K}}$, we define $U_{IL} := \emptyset$ for $L \subset \{1, \dots, N\}$ with $L \notin \mathcal{I}_{\mathcal{K}}$. Therefore (6.2.4) includes the condition

$$U_{IJ} \cap U_{IK} \neq \emptyset \implies F_J \cap F_K \neq \emptyset \quad (\iff J \cup K \in \mathcal{I}_{\mathcal{K}}).$$

just to show that these are rather technical notions

Lemma 6.2.9. *If $\mathcal{K}$ is a tame Kuranishi atlas, then for any $H, I, J \in \mathcal{I}_{\mathcal{K}}$ with $H \cap I \neq \emptyset$ and $H \cup I \subset J$ the two submanifolds $\operatorname{im} \phi_{HJ}$ and $\operatorname{im} \phi_{IJ}$ of $\operatorname{im} \phi_{(H \cup I)J}$ intersect transversally in $\operatorname{im} \phi_{(H \cap I)J}$.*

one (noncentral) use of tameness

<u>WHAT WE WANT</u> : Analogue of

<u>regularization theorem</u> for $\begin{matrix}\mathcal{E}\\ \downarrow\uparrow s\\ \mathcal{U}\end{matrix}$ $\mathcal{C}^\infty$ section of finite dim. bundle
$s^{-1}(0)$ compact

$\exists\, \mathcal{P} \subset \{\text{sections } \nu : \mathcal{U} \to \mathcal{E}\}$:

$\forall \nu \in \mathcal{P}$: $|(s+\nu)^{-1}(0)|$ compact manifold

$\forall \nu_0 \neq \nu_1 \in \mathcal{P}$ $\quad |(s+\nu_0)^{-1}(0)| \underset{\text{cobord}}{\sim} |(s+\nu_1)^{-1}(0)|$

Proposition 7.2.7. *Let $\mathcal{K}$ be a tame d-dimensional Kuranishi atlas with a reduction $\mathcal{V} \sqsubset \mathcal{K}$, and suppose that $\nu : \mathbf{B}_{\mathcal{K}}|_{\mathcal{V}} \to \mathbf{E}_{\mathcal{K}}|_{\mathcal{V}}$ is a precompact transverse perturbation. Then $|\mathbf{Z}_\nu| = |(s+\nu)^{-1}(0)|$ is a smooth closed d-dimensional manifold. Moreover, its quotient topology agrees with the subspace topology on $\|(s+\nu)^{-1}(0)\| \subset |\mathcal{K}|$.*

by nested uniqueness

The "implicit function theorem" holds for certain perturbations ν...

Proposition 7.3.5. *Let $(\mathcal{K}, d)$ be metric tame Kuranishi atlas with nested reduction $\mathcal{C} \sqsubset \mathcal{V}$. Then for any $0 < \delta < \delta_{\mathcal{V}}$ and $0 < \sigma \le \sigma(\delta, \mathcal{V}, \mathcal{C})$ there exists a $(\mathcal{V}, \mathcal{C}, \delta, \sigma)$-adapted perturbation ν of $s_{\mathcal{K}}|_{\mathcal{V}}$. In particular, ν is admissible, precompact, and transverse, and its perturbed zero set $|\mathbf{Z}_\nu| = |(s+\nu)^{-1}(0)|$ is compact with $\pi_{\mathcal{K}}\big((s+\nu)^{-1}(0)\big)$ contained in $\pi_{\mathcal{K}}(\mathcal{C})$.*

defined on last page

...which exist, just depend on a lot of data...

Proposition 7.3.7. *Let $(\mathcal{K}, d)$ be a metric tame Kuranishi cobordism with nested cobordism reduction $\mathcal{C} \sqsubset \mathcal{V}$, let $0 < \delta < \min\{\varepsilon, \delta_{\mathcal{V}}\}$, where ε is the collar width of $(\mathcal{K}, d)$ and the reductions $\mathcal{C}, \mathcal{V}$. Then we have $\sigma_{\rm rel}(\delta, \mathcal{V}, \mathcal{C}) > 0$ and the following holds.*

(i) *Given any $0 < \sigma \le \sigma_{\rm rel}(\delta, \mathcal{V}, \mathcal{C})$, there exists an admissible, precompact, transverse cobordism perturbation ν of $s_{\mathcal{K}}|_{\mathcal{V}}$ with $\pi_{\mathcal{K}}\big((s+\nu)^{-1}(0)\big) \subset \pi_{\mathcal{K}}(\mathcal{C})$, whose restrictions $\nu|_{\partial^\alpha\mathcal{V}}$ for $\alpha = 0, 1$ are $(\partial^\alpha\mathcal{V}, \partial^\alpha\mathcal{C}, \delta, \sigma)$-adapted perturbations of $s_{\partial^\alpha\mathcal{K}}|_{\partial^\alpha\mathcal{V}}$.*

(ii) *Given any perturbations ν^α of $s_{\partial^\alpha\mathcal{K}}|_{\partial^\alpha\mathcal{V}}$ for $\alpha = 0, 1$ that are $(\partial^\alpha\mathcal{V}, \partial^\alpha\mathcal{C}, \delta, \sigma)$-adapted with $\sigma \le \sigma_{\rm rel}(\delta, \mathcal{V}, \mathcal{C})$, the perturbation ν of $s_{\mathcal{K}}|_{\mathcal{V}}$ in (i) can be constructed to have boundary values $\nu|_{\partial^\alpha\mathcal{V}} = \nu^\alpha$ for $\alpha = 0, 1$.*

(iii) *In the case of a product cobordism $\mathcal{K} \times [0,1]$ with product metric and product reductions $\mathcal{C} \times [0,1] \sqsubset \mathcal{V} \times [0,1]$, both (i) and (ii) hold without requiring δ to be bounded in terms of the collar width.*

...but are all "cobordant".

Definition 7.1.2. *A* **reduction** *of a tame Kuranishi atlas $\mathcal{K}$ is an open subset $\mathcal{V} = \bigcup_{I\in\mathcal{I}_\mathcal{K}} V_I \subset \mathrm{Obj}_{\mathbf{B}_\mathcal{K}}$ i.e. a tuple of (possibly empty) open subsets $V_I \subset U_I$, satisfying the following conditions:*

(i) *$V_I \sqsubset U_I$ for all $I \in \mathcal{I}_\mathcal{K}$, and if $V_I \neq \emptyset$ then $V_I \cap s_I^{-1}(0) \neq \emptyset$;*
(ii) *if $\pi_\mathcal{K}(\overline{V_I}) \cap \pi_\mathcal{K}(\overline{V_J}) \neq \emptyset$ then $I \subset J$ or $J \subset I$;*
(iii) *the zero set $\iota_\mathcal{K}(X) = |s_\mathcal{K}|^{-1}(0)$ is contained in $\pi_\mathcal{K}(\mathcal{V}) = \bigcup_{I\in\mathcal{I}_\mathcal{K}} \pi_\mathcal{K}(V_I)$.*

Given a reduction $\mathcal{V}$, we define the **reduced domain category** $\mathbf{B}_\mathcal{K}|_\mathcal{V}$ *and the* **reduced obstruction category** $\mathbf{E}_\mathcal{K}|_\mathcal{V}$ *to be the full subcategories of $\mathbf{B}_\mathcal{K}$ and $\mathbf{E}_\mathcal{K}$ with objects $\bigcup_{I\in\mathcal{I}_\mathcal{K}} V_I$ resp. $\bigcup_{I\in\mathcal{I}_\mathcal{K}} V_I \times E_I$, and denote by $s|_\mathcal{V} : \mathbf{B}_\mathcal{K}|_\mathcal{V} \to \mathbf{E}_\mathcal{K}|_\mathcal{V}$ the section given by restriction of $s_\mathcal{K}$.*

A reduction
$(V_I \subset U_I)_{I\subset\{1..N\}}$
- organizes overlaps
- still covers $\overline{M}$.

Definition 6.2.16. *For any subset $\mathcal{A} \subset \mathrm{Obj}_{\mathbf{B}_\mathcal{K}}$ of the union of domains of a Kuranishi atlas $\mathcal{K}$, we denote by*

$$\|\mathcal{A}\| := \pi_\mathcal{K}(\mathcal{A}) \subset |\mathcal{K}|, \qquad |\mathcal{A}| := \pi_\mathcal{K}(\mathcal{A}) \cong \mathcal{A}/_\sim$$

the set $\pi_\mathcal{K}(\mathcal{A})$ equipped with its subspace topology induced from the inclusion $\pi_\mathcal{K}(\mathcal{A}) \subset |\mathcal{K}|$ resp. its quotient topology induced from the inclusion $\mathcal{A} \subset \mathrm{Obj}_{\mathbf{B}_\mathcal{K}}$ and the equivalence relation $\sim$ on $\mathrm{Obj}_{\mathbf{B}_\mathcal{K}}$ (which is generated by all morphisms in $\mathbf{B}_\mathcal{K}$, not just those between elements of $\mathcal{A}$).

⚠ The reductions $\mathcal{A} = \cup V_I$ carry different topologies..

Lemma:
(i) *For any subset $\mathcal{A} \subset \mathrm{Obj}_{\mathbf{B}_\mathcal{K}}$ the identity map $\mathrm{id}_{\pi_\mathcal{K}(\mathcal{A})} : |\mathcal{A}| \to \|\mathcal{A}\|$ is continuous.*
(ii) *If $\mathcal{A} \sqsubset \mathrm{Obj}_{\mathbf{B}_\mathcal{K}}$ is precompact, then both $|\overline{\mathcal{A}}|$ and $\|\overline{\mathcal{A}}\|$ are compact. In fact, the quotient and subspace topologies on $\pi_\mathcal{K}(\overline{\mathcal{A}})$ coincide, that is $|\overline{\mathcal{A}}| = \|\overline{\mathcal{A}}\|$ as topological spaces.*
(iii) *If $\mathcal{A} \sqsubset \mathcal{A}' \subset \mathrm{Obj}_{\mathbf{B}_\mathcal{K}}$, then $\pi_\mathcal{K}(\overline{\mathcal{A}}) = \overline{\pi_\mathcal{K}(\mathcal{A})}$ and $\pi_\mathcal{K}(\mathcal{A}) \sqsubset \pi_\mathcal{K}(\mathcal{A}')$ in the topological space $|\mathcal{K}|$.*
(iv) *If $\mathcal{A} \sqsubset \mathrm{Obj}_{\mathbf{B}_\mathcal{K}}$ is precompact, then $\|\overline{\mathcal{A}}\| = |\overline{\mathcal{A}}|$ is metrizable; in particular this implies that $\|\mathcal{A}\|$ is metrizable.*

...but when $V_I \sqsubset U_I$ are precompact, than (on closures) they agree, and we get
- compactness
- metrizability

Proposition 7.1.11. (a) *Every tame Kuranishi atlas $\mathcal{K}$ has a reduction $\mathcal{V}$.*
(b) *Every tame Kuranishi cobordism $\mathcal{K}^{[0,1]}$ has a cobordism reduction $\mathcal{V}^{[0,1]}$.*
(c) *Let $\mathcal{V}^0, \mathcal{V}^1$ be reductions of a tame Kuranishi atlas $\mathcal{K}$. Then there exists a cobordism reduction $\mathcal{V}$ of $\mathcal{K} \times [0,1]$ such that $\partial^\alpha \mathcal{V} = \mathcal{V}^\alpha$ for $\alpha = 0, 1$.*

Reductions exist and are unique up to cobordism.

Definition 7.1.12. *Let $\mathcal{K}$ be a Kuranishi atlas (or cobordism). Then we call a pair of subsets $\mathcal{C}, \mathcal{V} \subset \mathrm{Obj}_{\mathbf{B}_\mathcal{K}}$ a* **nested (cobordism) reduction** *if both are (cobordism) reductions of $\mathcal{K}$ and $\mathcal{C} \sqsubset \mathcal{V}$.*

We work with pairs of reductions, with
$\underbrace{C_I \sqsubset}_{\text{precompact}} V_I \subset U_I$

Definition 7.2.1. *A* **reduced section** *of $\mathcal{K}$ is a smooth functor $\nu : \mathbf{B}_{\mathcal{K}}|_{\mathcal{V}} \to \mathbf{E}_{\mathcal{K}}|_{\mathcal{V}}$ between the reduced domain and obstruction categories of some reduction $\mathcal{V}$ of $\mathcal{K}$, such that $\mathrm{pr}_{\mathcal{K}} \circ \nu$ is the identity functor. That is, $\nu = (\nu_I)_{I \in \mathcal{I}_{\mathcal{K}}}$ is given by a family of smooth maps $\nu_I : V_I \to E_I$ such that for each $I \subsetneq J$ we have a commuting diagram*

$$\begin{array}{ccc} V_I \cap \phi_{IJ}^{-1}(V_J) & \xrightarrow{\nu_I} & E_I \\ \downarrow{\scriptstyle \phi_{IJ}} & & \downarrow{\scriptstyle \widehat{\phi}_{IJ}} \\ V_J & \xrightarrow{\nu_J} & E_J. \end{array} \tag{7.2.1}$$

We say that a reduced section ν is an **admissible perturbation** *of $s_{\mathcal{K}}|_{\mathcal{V}}$ if*

$$\mathrm{d}_y \nu_J (\mathrm{T}_y V_J) \subset \operatorname{im} \widehat{\phi}_{IJ} \qquad \forall\, I \subsetneq J,\ y \in V_J \cap \phi_{IJ}(V_I \cap U_{IJ}). \tag{7.2.2}$$

"admissibility" transfers $s_I + \nu_I \pitchfork 0$ to $s_J + \nu_J|_{\operatorname{im} \varphi_{IJ}} \pitchfork 0$

Lemma 7.2.3. *Let $\mathcal{V}$ be a reduction of $\mathcal{K}$, and ν an admissible perturbation of $s_{\mathcal{K}}|_{\mathcal{V}}$. If $z \in V_I$ and $w \in V_J$ map to the same point in the virtual neighbourhood $\pi_{\mathcal{K}}(z) = \pi_{\mathcal{K}}(w) \in |\mathcal{K}|$, then z is a transverse zero of $s_I|_{V_I} + \nu_I$ if and only if w is a transverse zero of $s_J|_{V_J} + \nu_J$.*

Definition 7.2.4. *A* **transverse perturbation** *of $s_{\mathcal{K}}|_{\mathcal{V}}$ is a reduced section $\nu : \mathbf{B}_{\mathcal{K}}|_{\mathcal{V}} \to \mathbf{E}_{\mathcal{K}}|_{\mathcal{V}}$ whose sum with the canonical section $s|_{\mathcal{V}}$ is transverse to the zero section $0_{\mathcal{V}}$, that is $s_I|_{V_I} + \nu_I \pitchfork 0$ for all $I \in \mathcal{I}_{\mathcal{K}}$.*

Given a transverse perturbation ν, we define the **perturbed zero set** *$|\mathbf{Z}_\nu|$ to be the realization of the full subcategory $\mathbf{Z}_\nu$ of $\mathbf{B}_{\mathcal{K}}$ with object space*

$$(s + \nu)^{-1}(0) := \textstyle\bigcup_{I \in \mathcal{I}_{\mathcal{K}}} (s_I|_{V_I} + \nu_I)^{-1}(0) \subset \operatorname{Obj}_{\mathbf{B}_{\mathcal{K}}}.$$

That is, we equip

$$|\mathbf{Z}_\nu| := \left|(s+\nu)^{-1}(0)\right| = {\textstyle\bigcup_{I \in \mathcal{I}_{\mathcal{K}}} (s_I|_{V_I} + \nu_I)^{-1}(0)} \Big/ {\sim}$$

with the quotient topology generated by the morphisms of $\mathbf{B}_{\mathcal{K}}|_{\mathcal{V}}$. By Lemma 7.1.5 this is equivalent to the quotient topology induced by $\pi_{\mathcal{K}}$, and the inclusion $(s+\nu)^{-1}(0) \subset \mathcal{V} = \operatorname{Obj}_{\mathbf{B}_{\mathcal{K}}|_{\mathcal{V}}}$ induces a continuous injection, which we denote by

$$i_\nu \,:\, |\mathbf{Z}_\nu| \longrightarrow |\mathcal{K}|. \tag{7.2.4}$$

Definition 7.2.5. *A reduced section $\nu : \mathbf{B}_{\mathcal{K}}|_{\mathcal{V}} \to \mathbf{E}_{\mathcal{K}}|_{\mathcal{V}}$ is said to be* **precompact** *if its domain is part of a nested reduction $\mathcal{C} \sqsubset \mathcal{V}$ such that $\pi_{\mathcal{K}}\big((s+\nu)^{-1}(0)\big) \subset \pi_{\mathcal{K}}(\mathcal{C})$.*

Definition 7.3.3. *Given a nested reduction $\mathcal{C} \sqsubset \mathcal{V}$ of a metric tame Kuranishi atlas $(\mathcal{K}, d)$ and constants $0 < \delta < \delta_{\mathcal{V}}$ and $0 < \sigma \le \sigma(\delta, \mathcal{V}, \mathcal{C})$, we say that a perturbation ν of $s_{\mathcal{K}}|_{\mathcal{V}}$ is* **$(\mathcal{V}, \mathcal{C}, \delta, \sigma)$-adapted** *if the sections $\nu_I : V_I \to E_I$ extend to sections over $V_I^{|I|}$ (also denoted ν_I) so that the following conditions hold for every $k = 1, \dots, M$ with*

$$M_{\mathcal{K}} := \max_{I \in \mathcal{I}_{\mathcal{K}}} |I|, \qquad \eta_k := 2^{-k}\eta_0 = 2^{-k}(1 - 2^{-\frac{1}{4}})\delta.$$

a) *The perturbations are compatible in the sense that the commuting diagrams in Definition 7.2.1 hold on $\bigcup_{|I| \le k} V_I^k$, that is*

$$\nu_I \circ \phi_{HI}|_{V_H^k \cap \phi_{HI}^{-1}(V_I^k)} = \widehat{\phi}_{HI} \circ \nu_H|_{V_H^k \cap \phi_{HI}^{-1}(V_I^k)} \qquad \text{for all } H \subsetneq I, |I| \le k.$$

b) *The perturbed sections are transverse, that is $(s_I|_{V_I^k} + \nu_I) \pitchfork 0$ for each $|I| \le k$.*

c) *The perturbations are strongly admissible with radius η_k, that is for all $H \subsetneq I$ and $|I| \le k$ we have*

$$\nu_I(B^I_{\eta_k}(N^k_{IH})) \subset \widehat{\phi}_{HI}(E_H) \qquad \text{with } N^k_{IH} = V_I^k \cap \phi_{HI}(V_H^k \cap U_{HI}).$$

In particular, the perturbations are admissible along the core N_I^k, that is we have $\operatorname{im} d_x \nu_I \subset \operatorname{im} \widehat{\phi}_{HI}$ at all $x \in N^k_{IH}$.

d) *The perturbed zero sets are contained in $\pi_{\mathcal{K}}^{-1}(\pi_{\mathcal{K}}(\mathcal{C}))$; more precisely*

$$(s_I|_{V_I^k} + \nu_I)^{-1}(0) \subset V_I^k \cap \pi_{\mathcal{K}}^{-1}(\pi_{\mathcal{K}}(\mathcal{C})) \qquad \forall |I| \le k,$$

or equivalently $s_I + \nu_I \ne 0$ on $V_I^k \setminus \pi_{\mathcal{K}}^{-1}(\pi_{\mathcal{K}}(\mathcal{C}))$.

e) *The perturbations are small, that is $\sup_{x \in V_I^k} \|\nu_I(x)\| < \sigma$ for $|I| \le k$.*

The iterative construction of perturbations uses a couple of stronger "adapted" conditions

Definition 7.3.1. *Given a reduction $\mathcal{V}$ of a metric Kuranishi atlas (or cobordism) $(\mathcal{K}, d)$, we set $\boxed{\delta_{\mathcal{V}} > 0}$ to be the maximal constant such that any $2\delta < 2\delta_{\mathcal{V}}$ satisfies the reduction properties of Lemma 7.1.14, that is*

$$B_{2\delta}(V_I) \sqsubset U_I \qquad \forall I \in \mathcal{I}_{\mathcal{K}}, \tag{7.3.11}$$

$$B_{2\delta}(\pi_{\mathcal{K}}(\overline{V_I})) \cap B_{2\delta}(\pi_{\mathcal{K}}(\overline{V_J})) \ne \emptyset \quad \Longrightarrow \quad I \subset J \text{ or } J \subset I. \tag{7.3.12}$$

Given a nested reduction $\mathcal{C} \sqsubset \mathcal{V}$ of a metric Kuranishi atlas $(\mathcal{K}, d)$ and $0 < \delta < \delta_{\mathcal{V}}$, we set $\eta_{|J|-\frac{1}{2}} := 2^{-|J|+\frac{1}{2}}\eta_0 = 2^{-|J|+\frac{1}{2}}(1 - 2^{-\frac{1}{4}})\delta$ and

$$\boxed{\sigma(\delta, \mathcal{V}, \mathcal{C})} := \min_{J \in \mathcal{I}_{\mathcal{K}}} \inf \left\{ \|s_J(x)\| \;\middle|\; x \in \overline{V_J^{|J|}} \setminus \left(\overline{C_J} \cup \textstyle\bigcup_{I \subsetneq J} B^J_{\eta_{|J|-\frac{1}{2}}}(N_{JI}^{|J|-\frac{1}{2}}) \right) \right\}.$$

The constants δ, σ govern

$\Big\}$ separation of domains

$\Big\}$ $\|s\| \geq \sigma > 0$ in areas where we need to avoid creating zeros