# Abstract approaches to regularizing moduli spaces of pseudoholomorphic curves

**Approach #2**: Regularization via finite dimensional reductions

| | |
|---|---|
| Fukaya-Ono-Oh-Ohta 1999 | Kuranishi structure (of germs) |
| Joyce | —— " —— ⇝ d-orbifold |
| McDuff-W. | Kuranishi atlas |
| Pardon | —— " —— |
| ⋮ | |

**Guiding Questions** for studying regularization approaches via abstract perturbations / "virtual" fundamental class

$\overline{\mathcal{M}} = s^{-1}(0) \rightsquigarrow [(s+\nu)^{-1}(0)] =: [\overline{\mathcal{M}}]$

→ what is the abstract form of section s ?

$$\begin{array}{c} \mathcal{E} \\ \downarrow \uparrow s \\ \mathcal{U} \end{array}$$

section of "etale" category bundle ⇝ • object spaces $Obj_{\mathcal{E}} = \coprod_i U_i \times E_i$, $Obj_{\mathcal{U}} = \coprod_i U_i$ are smooth manifolds

($\dim U_i, \dim E_i$ vary with $i$ but $\dim U_i - \dim E_i = \text{virdim}$ constant)

• functor is smooth on objects $s = \coprod s_i : Obj_{\mathcal{U}} \to Obj_{\mathcal{E}}$

$\overline{\mathcal{M}} \simeq |s^{-1}(0)| = \coprod_i s_i^{-1}(0) / \text{morphisms}$

→ morphisms ?

→ how is s constructed for pseudoholomorphic curve moduli spaces ?

by finite dimensional reduction of local Fredholm descriptions

and stabilized gluing

→ morphisms = transition data ?

## WHAT WE WANT

(smooth, trivial isotropy)
**regularization theorem**

$\exists\, \mathcal{P} \subset \{\text{sections } \nu : \mathcal{U} \to \mathcal{E}\}$ : 2nd countability ⊛ (see later)

$\forall \nu \in \mathcal{P}$ : $|(s+\nu)^{-1}(0)|$ compact manifold

$\forall \nu_0 \neq \nu_1 \in \mathcal{P}$ $\; |(s+\nu_0)^{-1}(0)| \underset{\text{cobord}}{\sim} |(s+\nu_1)^{-1}(0)|$

$\Rightarrow [\overline{\mathcal{M}}] := [\,|(s+\nu)^{-1}(0)|\,]$ well defined

$\in H_*(\overline{\mathcal{M}})$ $\quad \check{H}_*(\overline{\mathcal{M}}) \ni \lim\limits_{\overleftarrow{\delta \to 0}} [|(s+\nu_\delta)^{-1}(0)|]$, $\quad \in \check{H}_*(\delta\text{-Nbhd}(\overline{\mathcal{M}}))$

## WHAT WE DREAM

$|\mathcal{E}|$
$\downarrow \uparrow |s| \quad \rightsquigarrow \quad |s+\nu| \pitchfork 0$
$|\mathcal{U}|$

$|\mathcal{U}|$ locally compact, Hausdorff

$\|\nu\|$ small $\Rightarrow |(s+\nu)^{-1}(0)|$ compact $\subset \delta\text{-Nbhd}(\overline{\mathcal{M}})$

$|\mathcal{U}|$ metric
$\cup$
$\overline{\mathcal{M}} = \bigcap_{\delta > 0} \delta\text{-Nbhd}(\overline{\mathcal{M}})$

* $A \subset B \subset$ metric space induces $\check{H}_*(A) \xrightarrow{\iota_*} \check{H}_*(B)$
* Given $\delta > 0$ find $\nu_\delta$ s.t. $Z_\delta := |(s+\nu_\delta)^{-1}(0)| \subset \delta$-nbhd defines a cycle $[Z_\delta] \in \check{H}_*(\delta\text{-nbhd})$
* For $\delta' > \delta$, cobordism $Z_\delta \sim Z_{\delta'}$ implies $\iota_*[Z_\delta] = [Z_{\delta'}] \in \check{H}_*(\delta'\text{nbhd})$
* inverse limit property of (rational) Čech homology for $\delta_i \to 0$ implies existence & uniqueness of $C_\infty$ s.t. $\iota_* C_\infty = [Z_\delta]$ $\forall \delta > 0$

$$\begin{array}{ccccccccc}
\check{H}_*(\overline{\mathcal{M}}) & \xrightarrow{\iota_*} & \dots \to \check{H}_*(\delta_{n+1}\text{-nbhd}) & \xrightarrow{\iota_*} & \check{H}_*(\delta_n\text{-nbhd}) & \xrightarrow{\iota_*} & \dots & \xrightarrow{\iota_*} & \check{H}_*(\delta_0\text{-nbhd}) \\
\cup & & \cup & & \cup & & & & \cup \\
C_\infty & & [(s+\nu_{n+1})^{-1}(0)] & \mapsto & [(s+\nu_n)^{-1}(0)] & \mapsto & \dots & \mapsto & [(s+\nu_0)^{-1}(0)] \\
 & & \| & & \| & & & & \| \\
 & & [Z_{\delta_{n+1}}] & & [Z_{\delta_n}] & & & & [Z_{\delta_0}]
\end{array}$$

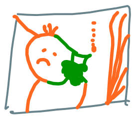
## WHAT WE HAVE

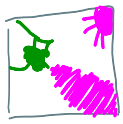
- Def$^{n}$: Kuranishi chart : $\Gamma \circlearrowright \hat{E} \downarrow \uparrow s$ over $U$, $\quad s^{-1}(0)/\Gamma \overset{\psi}{\simeq} F \subset \overline{\mathcal{M}}$

  (with isotropy) $\quad \Gamma$ finite $\quad \hat{E}$ vector bundle (trivial) $= U \times E \quad U$ manifold $\quad \psi$ homeo $\quad F$ open, "footprint"

- analytic construction of Kuranishi charts (up to smoothness questions for stabilized gluing) covering $\overline{\mathcal{M}} = \bigcup_{i=1}^{N} F_i$

- charts induce transition maps on footprints





$$\begin{tikzcd}
E_i \arrow[d] & & & & & & E_j \arrow[d] \\
U_i \arrow[drrr, bend right] & \supset s_i^{-1}(0) \supset & \psi_i^{-1}(F_j) \arrow[r, "\psi_j^{-1}\circ\psi_i"] \arrow[dr] & \psi_j^{-1}(F_i) \arrow[dl] & \subset s_j^{-1}(0) \subset & & U_j \arrow[dlll, bend left] \\
& & & F_i \cap F_j \subset \overline{\mathcal{M}} \subset \mathcal{B} & & &
\end{tikzcd}$$

$\psi_i : s_i^{-1}(0) \to \overline{\mathcal{M}}, \quad \psi_j : s_j^{-1}(0) \to \overline{\mathcal{M}}$

## WHAT WE DREAM

(smooth) transition maps between open subsets

$$U_i \supset U_{ij} \xrightarrow{\sim} U_{ji} \subset U_j$$

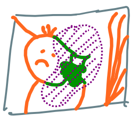
$$s_i^{-1}(0) \cap U_{ij} = \psi_i^{-1}(F_j) \xrightarrow{\psi_j^{-1}\circ\psi_i} \psi_j^{-1}(F_i) = U_{ji} \cap s_j^{-1}(0)$$

induced by charts / ambient space $\mathcal{B}$ / ....

## WHAT WE HAVE

• **Kuranishi charts** $\left( \underset{\text{finite}}{\Gamma_i} \circlearrowright \begin{matrix} E_i^{\text{ vector bundle}} \\ \downarrow \uparrow s_i \\ U_i \text{ manifold} \end{matrix} \quad s_i^{-1}(0)/\Gamma_i \underset{\text{homeo}}{\overset{\psi_i}{\simeq}} F_i \underset{\text{open}}{\subset} \overline{\mathcal{M}} \right)_{i=1..N}$ "footprint" covering $\overline{\mathcal{M}} = \bigcup_i F_i$

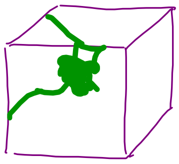
induced by
stabilizations $E_i, E_j$
slicing conditions $G, H$
domains $\mathcal{U}_i, \mathcal{U}_j \subset W^{k,p}$



$$\left\{ \underset{\mathcal{U}_i}{(u}, \underset{E_i}{e)} \,\middle|\, \begin{matrix} \bar\partial_J u = e \\ u(p) \in G_0 \end{matrix} \right\} = \underset{E_i \downarrow}{U_i} \supset s_i^{-1}(0) \cap \psi_i^{-1}(F_j) \xrightarrow{\psi_j^{-1}\circ\psi_i} \psi_j^{-1}(F_i) \cap s_j^{-1}(0) \subset \underset{E_j \downarrow}{U_j} = \left\{ \underset{\mathcal{U}_j}{(u}, \underset{E_j}{f)} \,\middle|\, \begin{matrix} \bar\partial_J u = f \\ u(p) \in H_0 \end{matrix} \right\}$$

$U_{iI} \underset{\text{open}}{\subset} U_i$ $\qquad$ $U_{jI} \underset{\text{open}}{\subset} U_j$

$\varphi_{iI}$ smooth embedding $U_{iI} \to U_{I=\{i,j\}}$ $\qquad$ $\varphi_{jI}$ smooth embedding $U_{jI} \to U_{I=\{i,j\}}$



$(u, e, 0, (0,1,u^{-1}(H_i)))$ $\qquad$ $(u, 0, f, u^{-1}(G_i), 0, 1)$

$$\{(u,e,f,(0,1,h_0,h_1))\} \simeq \left\{ (u,e,f,(g_0,g_1,h_0,h_1)) \,\middle|\, \begin{matrix} \bar\partial_J u = \text{"}e+f\text{"} \\ u(g_i)\in G_i,\ u(h_i) \in H_i \end{matrix} \right\}_{/\sim} \simeq \{(u,e,f,(g_0,g_1,0,1))\}$$

$$\wr$$

$$(u\circ\varphi, \vdots, \vdots, \varphi^{-1}(g_0,g_1,h_0,h_1))$$

• **transition data**

induced by
stabilizations $E_i, E_j$ that "sum well" under $\Phi_{ij} : u \mapsto u\circ\varphi_u$ so that $E_i \times \Phi_{ij}^* E_j$ stabilizes $\bar\partial_J|_{\mathcal{U}_{iI}}$
slicing conditions $G, H$ $\qquad$ $\Phi_{ij*} E_i \times E_j$ — " — $\bar\partial_J|_{\mathcal{U}_{jI}}$
domains $\mathcal{U}_i, \mathcal{U}_j \subset W^{k,p}$ and choices of $\mathcal{U}_{iI} \subset \mathcal{U}_i \cap \{u \pitchfork H_i\}$
$\mathcal{U}_{jI} \subset \mathcal{U}_j \cap \{u \pitchfork G_i\}$ so that $\mathcal{U}_{\cdot I} \cap \{\bar\partial_J u = 0\} \simeq F_i \cap F_j \in \overline{\mathcal{M}}$

## WHAT WE HAVE

• **Kuranishi charts** $\left( \underset{\text{finite}}{\Gamma_i} \circlearrowright \begin{matrix} E_i \\ \downarrow \uparrow s_i \\ U_i \end{matrix} \quad s_i^{-1}(0)/\Gamma_i \underset{\text{homeo}}{\overset{\psi_i}{\cong}} F_i \underset{\text{open}}{\subset} \overline{\mathcal{M}} \right)_{i=1..N}$ covering $\overline{\mathcal{M}} = \bigcup_i F_i$

trivial $U_i \times E_i$ vector bundle; $U_i$ manifold; "footprint"

• **Transition data** consisting of

$$\begin{tikzcd}
E_i \arrow[rr, bend left] & & E_I = E_i \times E_j & & E_j \arrow[ll, bend right] \\
U_i \arrow[u, "s_i"] & & & & U_j \arrow[u, "s_j"'] \\
U_{iI} \arrow[rr, "\varphi_{iI}"'] & & U_{I=\{i,j\}} \arrow[uu, "s_I"] & & U_{jI} \arrow[ll, "\varphi_{jI}"]
\end{tikzcd}$$

$U_{iI} \subset U_i$ open, $U_{jI} \subset U_j$ open

• additive **sum charts** for $I \subset \{1 \dots N\}$ s.t. $\bigcap_{i \in I} F_i \neq \emptyset$

$$E_I = \bigoplus_{i \in I} E_i, \quad U_I \xrightarrow{s_I} E_I, \quad s_I^{-1}(0)/\Gamma_I \cong \bigcap_{i \in I} F_i \subset \overline{\mathcal{M}}$$

• **coordinate changes** for $I \subsetneq J$

$$\begin{tikzcd}
E_I \arrow[rr] & & E_J \\
U_I \supset U_{IJ} \arrow[u, "s_I"] \arrow[rr, hook, "\varphi_{IJ}"] & & U_J \arrow[u, "s_J"']
\end{tikzcd}$$

• satisfying the **weak cocycle condition** $\boxed{\varphi_{JK} \circ \varphi_{IJ} = \varphi_{IK} \quad \forall I \subset J \subset K}$

where both are defined $(\varphi_{IJ}^{-1}(U_{JK}) \cap U_{IK})$

## WHAT WE WANT: A CATEGORY

$$\mathrm{Obj} = \bigsqcup_I U_I \qquad \mathrm{Mor} = \bigsqcup_{I \subset J} U_{IJ}$$

$\mathrm{Obj} \xrightarrow{id} \mathrm{Mor} \supset \bigsqcup_I U_{II}$, $\quad U_I = U_{II}$

$\mathrm{Mor} \overset{s}{\underset{t}{\rightrightarrows}} \mathrm{Obj}$, $\quad U_I \xleftarrow{s} U_{IJ} \xrightarrow{\varphi_{IJ}} U_J$

**Composition** $\mathrm{Mor} \times \mathrm{Mor} \to \mathrm{Mor}$

$U_{IJ} \hookrightarrow U_J \supset U_{JK} \hookrightarrow U_K$ $\qquad U_{IK} \hookrightarrow U_K$

requires **cocycle condition** $\boxed{\quad}$ on $\varphi_{IJ}^{-1}(U_{JK}) \subset U_{IK}$

## WHAT WE REALLY WANT

Hausdorff realization $\dfrac{\mathrm{Obj}}{\mathrm{Mor}} = \dfrac{\sqcup U_I}{\varphi_{IJ}}$ requires "tameness"

$\Longrightarrow$ **strong cocycle condition** $\boxed{\quad}$ on $\varphi_{IJ}^{-1}(U_{JK}) = U_{IK}$

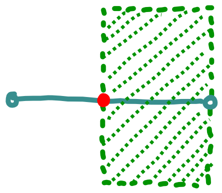
## WHAT WE SHOULD NOT DREAM OF

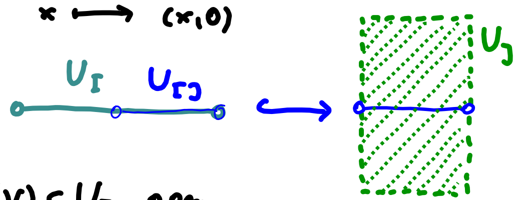
NIGHTMARES IN POINT SET TOPOLOGY

**typical example of coordinate change** $U_I \supset U_{IJ} \hookrightarrow U_J$

$$(-1,1) \supset (0,1) \longrightarrow (0,1)\times(-2,2)$$
$$x \longmapsto (x,0)$$

$$\mathcal{U} = \frac{U_I \cup U_J}{U_{IJ}}$$






with

**quotient topology**: $V \subset \mathcal{U}$ open $:\Longleftrightarrow$ $pr_I^{-1}(V) \subset U_I$ open, $pr_J^{-1}(V) \subset U_J$ open

is not

- **locally compact**: $[0] \in \mathcal{U}$ has no compact neighbourhood $K \subset \mathcal{U}$
  $(\exists V \subset \mathcal{U} \text{ open} : [0] \in V \subset K)$
- **1st countable**: $[0] \in \mathcal{U}$ has no countable neighbourhood base
  $\left([0] \in W_n \subset \mathcal{U} \text{ open}\right)_{n\in\mathbb{N}} : \forall\, [0] \in V \subset \mathcal{U} \text{ open } \exists n : W_n \subset V$
- **metrizable**: There is no metric on $\mathcal{U}$ that induces the quotient topology.