Regularization of moduli spaces of pseudoholomorphic curves

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0 Conventions

(M,ω) symplectic manifold $\to M$ 2n-dim (closed) smooth manifold oriented

$\hookrightarrow \Omega^2(M)$ $d\omega = 0$, $\underbrace{\omega\wedge\dots\wedge\omega}_{n}$ volume form

Ex: $M = \mathbb{CP}^1 \times T^{2n-2}$, $\omega = (1+\varepsilon)\,dvol_{\mathbb{CP}^1} \times K\left[\sum_{l=1}^{n-1} dx_l \wedge dy_l\right]_{\mathbb{C}^{n-1}/\mathbb{Z}^{2n-2}} \ni (x_l + i y_l)$

$\varepsilon, K > 0$

$\mathcal{J}(M,\omega) \ni J$ compatible almost complex structure

$J: TM \to TM$, $J^2 = -id$, $\omega(\cdot, J\cdot) = g_J$ metric

(both TM over M)

Ex: $J_0 = i_{\mathbb{P}^1} \times [i]_{\mathbb{C}^{n-1}}$

Fact: $\mathcal{J}(M,\omega)$ nonempty, contractible

$\Rightarrow$ Chern classes $c_l(TM, J) \in H^{2l}(M)$ J-independent

$\underline{(\Sigma, j) \text{ Riemann surface}}$

$\to \Sigma$ oriented 2-mfd

$\to j : T\Sigma \to T\Sigma$, $j^2 = -id$

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$\underline{\text{Ex}}$: $\Sigma = \mathbb{CP}^1$

$j = i$

$\underline{\text{Fact}}$: locally $(\Sigma, j) \simeq (\mathbb{C}, i)$

$\underline{\text{Cauchy Riemann operator}}$

$\bar{\partial}_J : Map(\Sigma, M) \to \Omega^{0,1}(\ldots)$

$u \mapsto \frac{1}{2}(du + J \circ du \circ j)$

$\underline{\text{Fact } 0}$

$\bar{\partial}_J u = 0$

$\Updownarrow$

$du \circ j = J \circ du$

$\underline{\text{Fact}}$: locally $2\bar{\partial}_J u = (\partial_s u + J \partial_t u) ds - J(\partial_s u + J \partial_t u) dt$

① pseudoholomorphic curves

pseudoholomorphic <u>map</u>: $u:(\Sigma,j)\to(M,\omega,J)$

$\bar\partial_J u = 0$

<u>Fact</u>: $\bar\partial_J u=0 \Rightarrow \bar\partial_J(u\circ\psi)=0$
$\forall \psi\in Aut(\Sigma,j)$

<u>Ex</u>: $Aut(\mathbb{CP}^1,i)\simeq PSL(2,\mathbb{C})$
$\simeq\left\{z\mapsto\frac{az+b}{cz+d}\,\middle|\,ad-bc\neq 0\right\}$
6-dim, noncompact

pseudoholomorphic <u>curve</u>:
$[u]\in Map(\Sigma,M)/Aut(\Sigma,j)$

$\forall v\in[u]: \bar\partial_J v=0$

compare algebraic geometry
"curve" $C\subset(M,J)$
"$u(\Sigma)$"

② moduli spaces of (Gromov-Witten) pseudoholom. curves

fix $A \in H_2(M)$

$\widetilde{\mathcal{M}}(A,J) := \{ u : \Sigma \to M \mid \bar{\partial}_J u = 0,\ u_*[\Sigma] = A \}$

space of pseudoholomorphic maps

$\mathcal{M}(A,J) := \widetilde{\mathcal{M}}(A,J) / \mathrm{Aut}(\Sigma, j)$ moduli space of pseudoholomorphic curves

$\overline{\mathcal{M}}(A,J) := \mathcal{M}(A,J) \cup \{\text{bubble trees}\}$ "Gromov compactification"

(in which $\mathcal{M}(A,J)$ may not be dense)

<u>Ex</u>: $A = [\mathbb{CP}^1 \times pt] \in H_2(\mathbb{CP}^1 \times T^{2n-2})$

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<u>Ex</u>: $A = [\mathbb{CP}^1 \times pt] \in H_2(\mathbb{CP}^1 \times T^{2n-2})$

$$\mathcal{M}(A,J) = \{u : \mathbb{CP}^1 \to \mathbb{CP}^1 \times T \mid \bar\partial_J u = 0,\ u_*[\mathbb{CP}^1] = A\} / Aut(\mathbb{CP}^1, i)$$

<u>MATH 278</u> proved <u>Gromov nonsqueezing</u>

$$\left(\begin{tikzcd} B^{2n}(R) \arrow[r, hook, "symp", "\varphi"'] \arrow[dr, "\tilde\varphi"'] & B^2(r)\times\mathbb{R}^{2n-2} \arrow[d, hook] \\ & \mathbb{CP}^1 \times T^{2n-2} \end{tikzcd} \quad \Rightarrow R \le r \right)$$

by finding J_1 , $[u] \in \mathcal{M}(A, J_1)$ s.t. $\tilde\varphi^* J_1 = i$ on $B^{2n}(R) \subset \mathbb{C}^n$

$\tilde\varphi(0) \in im(u)$

$\rightsquigarrow$ study moduli spaces with
- constraints

or
- evaluation maps

$\rightsquigarrow$ study moduli spaces with

- **constraints** or • **evaluation maps**

fix $z_0 \in \Sigma$, e.g. $0 \in \mathbb{CP}^1$ "marked point"
$Y \subset M$, e.g. $\tilde{\varphi}(0) \in \mathbb{CP}^1 \times T$ "constraint"
submanifold

$$\widetilde{\mathcal{M}}(A, J, Y) = \{ u \in \widetilde{\mathcal{M}}(A, J) \mid u(z_0) \in Y \}$$

$$\mathcal{M}(A, J, Y) = \widetilde{\mathcal{M}}(A, J, Y) \big/ \mathrm{Aut}(\Sigma, j, z_0)$$

$\{ \psi(z_0) = z_0 \}$

$$\widetilde{\mathcal{M}}_k(A, J) = \left\{ (u, \underline{z}) \;\middle|\; u \in \widetilde{\mathcal{M}}(A, J), \ \underline{z} = (z_1 .. z_k) \subset \Sigma \right\}$$

"marked points"

$$\mathcal{M}_k(A, J) := \widetilde{\mathcal{M}}_k(A, J) \big/ \mathrm{Aut}(\Sigma, j) \xrightarrow{Ev} M^k$$

$$[(u, \underline{z})] \mapsto (u(z_1) \dots u(z_k))$$

$$\psi^*(u, \underline{z}) = (u \circ \psi, \psi^{-1}(z_1), \dots, \psi^{-1}(z_k))$$

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<u>Fact</u>: $\widetilde{\mathcal{M}}_{(k)}(A, J)$ is the zero set of a Fredholm section $\begin{array}{c}\mathcal{E}\\ \downarrow \uparrow \bar\partial_J \\ \mathcal{B}\end{array}$ of Fredholm index

$$i = 2\langle c_1(TM, J), A\rangle + 2k + \chi(\Sigma)\, n$$

So if $\bar\partial_J$ is transverse to $\mathcal{O}$ and Aut acts freely, properly, then $\mathcal{M}_k(A, J)$ is a manifold of dimension $d = i - \dim \mathrm{Aut}$.

In the Gromov nonsqueezing example $\chi(\Sigma = \mathbb{CP}^1) = 2$, $k = 1$

$$d = 2\underbrace{\langle c_1(TM), A\rangle}_{\langle c_1(T\mathbb{CP}^1), [\mathbb{CP}^1]\rangle} + 2k + 2n - 6 = 2\cdot 2 + 2 + 2n - 6 = 2n$$

③ Regularization – slogans

" $\overline{\mathcal{M}}_k(A,J)$ is a d-manifold for generic J
(orbifold / weighted branched mfd)
and unique up to cobordism "

" virtual fundamental class :

$$\underbrace{(ev_1 \times \ldots \times ev_k)}_{Ev}{}_* \left[\overline{\mathcal{M}}_k(A,J)\right] \in H_d(M)$$

well defined and independent of J "

" fundamental class $[\overline{\mathcal{M}}_k(A,J)] \in H_d(\overline{\mathcal{M}}_k(A,J))$ "

" $\displaystyle\int_{\overline{\mathcal{M}}_k(A,J)} ev_1^*\alpha_1 \wedge \ldots \wedge ev_k^*\alpha_k$ = # J-holomorphic curves through $PD(\alpha_1), \ldots, PD(\alpha_k)$ "

$d = \sum \deg \alpha_i$

<u>MATH 278</u> proved (by Sard-Smale)

in <u>Ex</u> $M = \mathbb{CP}^1 \times T$, $A = [\mathbb{CP}^1 \times pt]$

$\exists\, \mathcal{J}^{reg} \subset \mathcal{J}(M,\omega)$ comeagre ($\Rightarrow$ dense) s.t.

$\overset{..}{=} \{ J$ regular i.e. $\boxed{\bar\partial_J \pitchfork 0} \}$ "transversality"

$$\left.\begin{array}{l} \bullet\ \forall J \in \mathcal{J}^{reg} \quad \mathcal{M}(A,J,\tilde\varphi(0)) \text{ is a compact 0-mfd} \\ \bullet\ \forall J_0, J_1 \in \mathcal{J}^{reg}\ \exists\, \mathcal{M}((J_t)) \text{ 1-cobordism with} \\ \qquad \partial \mathcal{M}((J_t)) = \mathcal{M}(J_0) \cup \mathcal{M}(J_1) \end{array}\right\} \text{"regularization"}$$

$\Longrightarrow \#\mathcal{M}(A,J,p_0) \in \mathbb{Z}_2$ well defined
independent of J and $p_0 \in M$

a (first example of a) <u>Gromov-Witten invariant</u>

<u>**end of nonsqueezing proof**</u>:

$$\left.\begin{array}{l} * \text{ special } J_0 \rightsquigarrow \#\mathcal{M}([\mathbb{CP}^1 \times pt], P, \tilde\varphi(0)) = 1 \quad \forall J \text{ regular} \\ * \text{ pick } J_1 \text{ with } \varphi^* J_1 = i \ \& \text{ find regular } J^\nu \xrightarrow[\nu\to\infty]{} J_1 \end{array}\right\}$$

$\Rightarrow \exists\, [u^\nu] \in \mathcal{M}(\dots)\ \forall \nu$

Gromov compactness (bubbling excluded since A has minimal energy)

$\Rightarrow [u^{\nu_i}] \xrightarrow[i\to\infty]{} [u^\infty] \in \mathcal{M}(A, J_1, \tilde\varphi(0))$