L9 – quilted Floer homology

Note Title 2/25/2008

<u>Construction of $HF(\underline{\mathcal{L}})$ for $\underline{\mathcal{L}}$ cyclic correspondence</u>: Choose regular Hamiltonians $\underline{H}$, strip widths $\underline{\delta}$, regular almost complex structures $\underline{J}$.

We defined $\quad CF(\underline{\mathcal{L}},\underline{H}) := \bigoplus_{\underline{p}\in\cap_{\underline{H}}\underline{\mathcal{L}}} \mathbb{Z}_2 \langle \underline{p} \rangle$

and will define ∂ from the moduli spaces $\mathcal{M}(\underline{\mathcal{L}},\underline{p}^-,\underline{p}^+,\underline{H},\underline{\delta},\underline{J})$

$$= \left\{ \underline{u} \in \bigoplus_{j=1}^{k} W^{1,p}_{loc}(\mathbb{R}\times[0,\delta_j],N_j) \;\middle|\; \begin{array}{l} \lim_{s\to\pm\infty} u_j(s,t) = \phi_{H_j}^{t/\delta_j}(p_j^\pm) \;,\; \mathcal{E}(\underline{u})<\infty \\ (u_{j-1}(s,\delta_{j-1}),u_j(s,0)) \in L_{(j-1)j} \\ \partial_s u_j + J_j(\partial_t u_j - \delta_j^{-1} X_{H_j}) = 0 \end{array} \right\}$$

with energy $\quad \mathcal{E}(\underline{u}) = \frac{1}{2}\sum_{j=1}^{k} \int_{\mathbb{R}\times[0,\delta_j]} |\partial_s u_j|^2 + |\partial_t u_j - \delta_j^{-1} X_{H_j}|^2$

<u>Remark</u>: Rescaling $w_j(s,\tau) := u_j(s,\delta_j\tau)$ identifies $\mathcal{M}$ with a moduli space of "$(\delta_j^{-1}J_j)$-holomorphic" quilts

$$\left\{ \underline{w} \in \bigoplus_{j=1}^{k} W^{1,p}_{loc}(\mathbb{R}\times[0,1],N_j) \;\middle|\; \begin{array}{l} \lim_{s\to\pm\infty} w_j(s,\tau) = \phi_{H_j}^{\tau}(p_j^\pm) \;,\; \mathcal{E}_{\underline{\delta}}(\underline{w})<\infty \\ (w_{j-1}(s,1),w_j(s,0)) \in L_{(j-1)j} \\ \partial_s w_j + \delta_j^{-1} J_j(\partial_\tau w_j - X_{H_j}) = 0 \end{array} \right\}$$

with energy $\quad \mathcal{E}_{\underline{\delta}}(\underline{w}) = \frac{1}{2}\sum_{j=1}^{k} \int_{\mathbb{R}\times[0,1]} \delta_j |\partial_s w_j|^2 + \delta_j^{-1} |\partial_\tau w_j - X_{H_j}|^2 \; ds\, d\tau$

For a single strip $\underline{u} = (u : \mathbb{R}\times[0,\delta]\to N)$ as in "classical" $HF(L,L')$ or $HF(\varphi)$ the moduli spaces for width δ and width 1 are identified by $w(s,t) := u(\delta s,\delta t)$.

$\boxed{\text{Energy \& Index}}$

Assume monotonicity (with $\tau \geq 0$) : $\int \underline{v}^*\underline{\omega} = \tau \cdot I_{Maslov}(\underline{v}) \quad \forall \underline{v} : S^1 \to \mathcal{P}$

Let $N_{\underline{L}} \in \mathbb{N}$ be the generator of $\{I_{Maslov}(\underline{v})\} \subset \mathbb{Z}$.

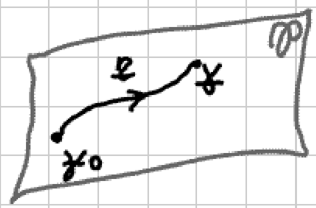
Fix a base point $\underline{\gamma}_0 \in \mathrm{Crit}\, \mathcal{A}_{\underline{H}} \cong \cap_{\underline{H}} \underline{L}$ in each connected component of

$\mathcal{P} = \{ \underline{\gamma} = (\gamma_j : [0,1] \to N_j) \mid L_{(j-1)j}\text{-conditions} \}$ that has a crit. pt.. Then we have

(I) S^1-valued action $\mathcal{A}_{\underline{H}} : \mathcal{P} \longrightarrow \mathbb{R}/_{\tau N_{\underline{L}} \mathbb{Z}}$, $\underline{\gamma} \mapsto -\int \underline{\eta}^* \underline{\omega} - \underline{H}(\underline{\gamma})$

- $\underline{H}(\underline{\gamma}) := \sum_{j=1}^{k} \int_0^1 H_j(\gamma_j(t))\, dt \quad \left(\text{if } \dot{\gamma}_j = X_{H_j} \text{ then } \frac{d}{dt} H_j(\gamma_j) = dH_j(X_j) = \omega_j(X_j, X_j) = 0 \right)$
- $\underline{\eta} : [0,1] \to \mathcal{P}$ path from $\underline{\gamma}_0$ to $\underline{\gamma}$

  i.e. $\underline{\eta} = (\eta_j : [0,1] \times [0,1] \to N_j)$



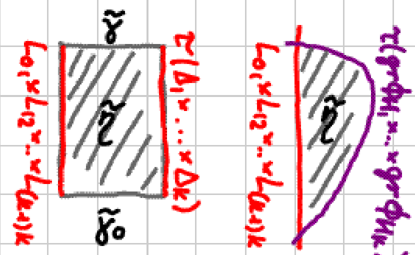
(II) $\mathbb{Z}_{N_{\underline{L}}}$-grading on $CF(\underline{L}, \underline{H})$

$|\langle \underline{p} \rangle| := I_{Maslov}(\underline{\eta})$ for $\underline{\eta} : [0,1] \to \mathcal{P}$, $\underline{\eta}(0) = \underline{\gamma}_0$, $\underline{\eta}(1) = \underline{\gamma} \mathrel{\hat{=}} \underline{p} \in \cap_{\underline{H}} \underline{L}$

NOT IN LECTURE

Fold $\underline{\eta}$ to $\tilde{\eta} : [0,1] \times [0,\tfrac{1}{2}] \longrightarrow N_0 \times N_1 \times N_1 \times \ldots \times N_k$

then $\underline{\eta}(s,t) = (\phi^t_{H_k} \times Id_1 \times \phi^t_{H_1} \times Id_2 \times \ldots \times \phi^t_{H_{k-1}} \times Id_k)(\tilde{\eta}(s,t))$



induces 2 paths of Lagrangian subspaces that are transverse at the ends

$\underline{\eta}|^*_{t=0} T(L_{01} \times \ldots \times L_{(k-1)k})$, $\underline{\eta}|^*_{t=1} T\, \tau(gr\, \phi_{H_1} \ldots gr\, \phi_{H_k}) : [0,1] \longrightarrow \mathcal{L}ag(\underline{\eta}^* T(N_0 \times N_1 \times \ldots \times N_k))$

$\cong$ trivialize $\mathbb{C}^N$

These have a Maslov index [Robbin-Salamon].

(III) Energy and index identities

$\forall \underline{u} \in \mathcal{M}(\underline{p}^-, \underline{p}^+)$ $\quad \mathcal{E}(\underline{u}) = \mathcal{A}_{\underline{H}}(\underline{p}^-) - \mathcal{A}_{\underline{H}}(\underline{p}^+) \quad \text{mod } \tau N_{\underline{L}}$

$\text{index } D_{\underline{u}} = |\underline{p}^+| - |\underline{p}^-| \quad \text{mod } N_{\underline{L}}$

$\Rightarrow \mathcal{E}(\underline{u}) = \tau \cdot \text{index } D_{\underline{u}} + C_{\underline{p}^-, \underline{p}^+}$

$\Big(\text{"} \mathcal{A}_{\underline{H}}(\underline{p}^-) + \tau|\underline{p}^-| - \mathcal{A}_{\underline{H}}(\underline{p}^+) - \tau|\underline{p}^+|$ using same paths to $\underline{p}^-, \underline{p}^+$ in $\mathcal{P}$ $\Big)$

$\boxed{\mathbb{R}\text{-shift}}$ $\quad \underline{u} \mapsto \sigma * \underline{u} = (u_j(\sigma + \cdot, \cdot))$ maps $\mathcal{M}$ to $\mathcal{M}$ $\quad \forall \sigma \in \mathbb{R}$

$\Rightarrow (\partial_s u_j) \in \ker D_{\underline{u}} = T_{\underline{u}} \mathcal{M} \Rightarrow \text{index } D_{\underline{u}} \geq 1$ unless $\underline{u} = \underline{p}^- = \underline{p}^+$ $\mathbb{R}$-independent

$(\rightsquigarrow \mathcal{E}(\underline{u}) = 0, \ \text{index } D_{\underline{u}} = 0)$

⊛ $\mathbb{R}$ acts properly discontinuously on $\{\underline{u} \in \mathcal{M} \mid \text{index } D_{\underline{u}} \geq 1\}$

($\underline{u}$ cannot be periodic in $s \in \mathbb{R}$ since $0 < \mathcal{E}(\underline{u}) = \int_{\mathbb{R}} \sum \int |\partial_s u_j|^2 < \infty$)

$\Rightarrow$ For $k \geq 0$ $\quad \mathcal{M}^k(\underline{p}^-, \underline{p}^+) := \{\underline{u} \in \mathcal{M}(\underline{L}, \underline{p}^-, \underline{p}^+, \underline{H}, \underline{\delta}, \underline{J}) \mid \text{index } D_{\underline{u}} = k+1\} / \mathbb{R}$

is a smooth manifold of dimension k.

Thm (Gromov compactness, Gluing) (& orientations from "relative spin structure")

Assuming monotonicity and $N_{\underline{L}} \geq 2$ (e.g. $L_{(j-1)j}$ oriented)

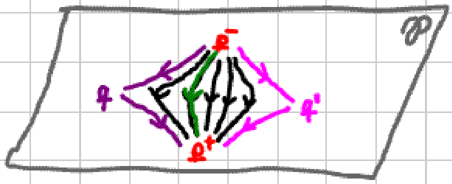
and for (ii) minimal Maslov $N_{L_{(j-1)j}} \geq 3$ for disks $(D, \partial D) \to (N_{j-1} \times N_j, L_{(j-1)j})$

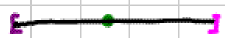
(i) $\mathcal{M}^0(\underline{p}^-, \underline{p}^+)$ is compact $\quad$ (& oriented $\mathcal{M}^0(\underline{p}^-, \underline{p}^+) \to \{\pm 1\}$)

(ii) $\mathcal{M}^1(p^-,p^+)$ can be compactified to a $\overset{\text{(oriented),}}{\text{1-manifold}}$ $\bar{\mathcal{M}}^1(p^-,p^+)$ with

boundary $\partial\bar{\mathcal{M}}^1(p^-,p^+) = \bigcup_{q\in\cap\underline{L}} \mathcal{M}^0(p^-,q)\times\mathcal{M}^0(q,p^+)$ "broken trajectories"

"''Proof''": as in Morse theory





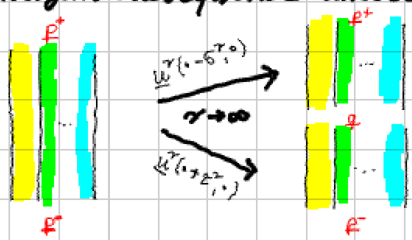
$\bar{\mathcal{M}}(p^-,p^+)$

''Proof of compactness'' Any sequence $\underline{u}^\nu \in \mathcal{M}^k(p^-,p^+)$ of bounded energy

$E(\underline{u}^\nu) = \tau(k+1) + c_{p^-,p^+}$ has a convergent subsequence unless

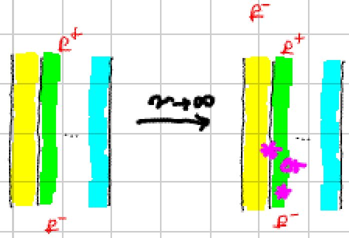
(a) energy escapes to $\pm\infty$

"breaking of trajectory"



(b) energy concentrates at a point *

"bubbling off"



In the image, a sphere $S^2\to N_j$ or disc $(D^2,\partial D^2)\to(N_{j-1}^-\times N_j, L_{(j-1)j})$ forms.

On the domain, $u^\nu\to u'$ converges on the complement of the point(s) *, and

the singularity can be removed to obtain a new solution $u'\in\mathcal{M}(p^-,p^+)$

with less energy $\underset{\text{monotonicity}}{\Longrightarrow}$ less index

(i) new index $\leq 1 - N_{\underline{L}} < 0 \Rightarrow \nexists u' \Rightarrow$ no bubbling

(ii) new index $\leq 2 - N_{L_{(j-1)j}} < 0 \Rightarrow \nexists u' \Rightarrow$ no bubbling "■"

Define $\partial : CF \to CF$ by $\quad \partial \langle \underline{p}^- \rangle := \sum_{\underline{p}^+ \in \cap_{\underline{H}} \underline{L}} \underbrace{\left(\sum_{\underline{u} \in \mathcal{M}^0(\underline{p}^-, \underline{p}^+)} \pm 1 \right)}_{\in \mathbb{Z}_2 \text{ or } \mathbb{Z} \text{ with orientations}} \langle \underline{p}^+ \rangle$

then $\quad \partial^2 \langle \underline{p}^- \rangle = \sum_{\underline{p}^+} \underbrace{\sum_{\underline{q}} \# \mathcal{M}^0(\underline{p}^-, \underline{q}) \cdot \# \mathcal{M}^0(\underline{q}, \underline{p}^+)}_{= \# \partial \overline{\mathcal{M}}^1(\underline{p}^-, \underline{p}^+) = 0} \langle \underline{p}^+ \rangle = 0 .$

<u>Thm</u>: Floer (co)homology groups $HF(\underline{L}) := \ker\partial / \operatorname{im}\partial$ are independent of the choice of $\underline{H}, \underline{\delta}, \underline{J}$; up to isomorphism.