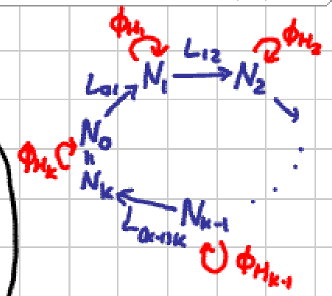
L8 - quilted Floer homology

Note Title 2/25/2008

$\underline{L} = (L_{01}, L_{12}, \dots, L_{(k-1)k})$ cyclic correspondence



$$\left(\begin{array}{l} L_{(j-1)j} \subset N_{j-1}^- \times N_j \ \text{Lagrangian correspondences} \\ N_0, N_1, \dots, N_{k-1}, N_k = N_0 \ \text{symplectic manifolds} \end{array} \right)$$

Construction of quilted Floer homology $HF(\underline{L})$

① Choose Hamiltonians $\underline{H} = (H_j) \in \bigoplus_{j=1}^{k} C^\infty(N_j, \mathbb{R})$ such that

$$(L_{01} \times L_{12} \times \dots \times L_{(k-1)k}) \pitchfork \tau(gr\,\phi_{H_1} \times \dots \times gr\,\phi_{H_k}) \subset N_0 \times N_1 \times N_1 \times \dots \times N_k$$

where $gr\,\phi_H \subset N \times N$ is the time 1-flow of the Hamiltonian vector field $X_H = J\nabla H$

and $\tau : N_1 \times N_1 \times \dots \times N_k \times N_k \to N_0 \times N_1 \times N_1 \times \dots \times N_k$; $(x_1, y_1, \dots, x_k, y_k) \mapsto (y_k, x_1, y_1, \dots x_k)$.

Prop^{n}: Such $\underline{H}$ exist and make the perturbed generalized intersection a finite set.

$$\cap_{\underline{H}} \underline{L} := \{ \underline{p} = (p_1, \dots, p_k^{p_0}) \in N_1 \times \dots \times N_k \mid (\phi_{H_{j-1}}(p_{j-1}), p_j) \in L_{(j-1)j} \ \forall j \}$$

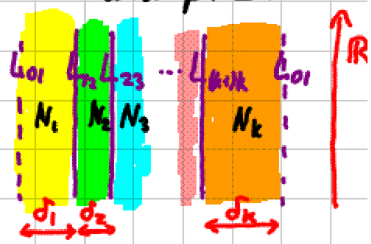
$$\cong \{ \underline{y} = (y_j : [0,1] \to N_j)_{j=1..k} \mid \dot{y}_j = X_{H_j}(y_j) \ , (y_{j-1}(1), y_j(0)) \in L_{(j-1)j} \ \forall j \} \subset \mathcal{P}$$

$$\cong (L_{01} \times L_{12} \times \dots \times L_{(k-1)k}) \pitchfork \tau(gr\,\phi_{H_1} \times \dots \times gr\,\phi_{H_k}) \cong \underbrace{\cap \left((\phi_{H_{j-1}}^{-1} \times Id_{N_j})(L_{(j-1)j}) \right)_{j=1..k}}_{\text{unperturbed intersection of perturbed Lagrangians}}$$

$\rightarrow$ $CF(\underline{L}, \underline{H}) := \bigoplus_{\underline{p} \in \cap_{\underline{H}} \underline{L}} \underset{\sim}{\mathbb{Z}_2} \langle \underline{p} \rangle$ is a finitely generated complex

or $\mathbb{Z}$ (with orientations for $\mathcal{M}^0, \mathcal{M}^1$)

② Choose strip widths $\underline{\delta} = (\delta_1,\dots,\delta_k) \in (0,\infty)^k$ and $p>2$.



For $\underline{\rho}^\pm \in \pitchfork_{\underline{H}} \underline{\mathcal{L}}$ define a Banach manifold

$$\mathcal{B}(\underline{\mathcal{L}},\underline{\rho}^-,\underline{\rho}^+) := \{\underline{u} \mid (i),(ii)\} \subset \bigoplus_{j=1}^{k} W^{1,p}_{loc}(\mathbb{R}\times[0,\delta_j], N_j)$$

(i) $u_j(s,t) \xrightarrow[s\to\pm\infty]{} \gamma_j^\pm(t/\delta_j) = \phi_{H_j}^{t/\delta_j}(\rho_j)$ uniformly $\forall t\in[0,\delta_j]$ $\forall j=1..k$

(ii) $(u_{j-1}(s,1), u_j(s,0)) \in L_{(j-1)j}$ $\forall s\in\mathbb{R}$ $\forall j=0..k$ (with $u_0 := u_k$)

and a Banach bundle $\mathcal{E}\to\mathcal{B}(\underline{\mathcal{L}},\underline{\rho}^-,\underline{\rho}^+)$ with fibers

$$\mathcal{E}_{\underline{u}} = \bigoplus_{j=1}^{k} L^p(\mathbb{R}\times[0,\delta_j], u_j^*TN_j).$$

③ Choose a "t-dependent split almost complex structure" $\underline{J}=(J_j)_{j=1..k}$

$J_j\in\mathcal{C}^\infty([0,\delta_j],\mathrm{End}(TN_j))$, where each $J_j(t)$ is an ω_j-compatible almost complex structure.

<u>Prop^n</u>: $\bar\partial := \bar\partial_{\underline{J},\underline{H},\underline{\delta}}: \mathcal{B}(\underline{\mathcal{L}},\underline{\rho}^-,\underline{\rho}^+) \to \mathcal{E}$, $\underline{u}\mapsto \left(\partial_s u_j + J_j(\partial_t u_j - \delta_j^{-1}X_{H_j})\right)_{j=1..k}$

is a Fredholm section $\left(\mathcal{C}^1\text{-map}, \underbrace{(T_{\underline{u}}\bar\partial)^{ver}}_{\text{using a connection } T_\xi\mathcal{E}\cong T_{\underline u}\mathcal{B}\times\mathcal{E}_{\underline u}\ \forall\xi\in\mathcal{E}_{\underline u}}: T_{\underline u}\mathcal{B}\to\mathcal{E}_{\underline u} \text{ Fredholm } \forall \underline u\right)$.

There exists $\underline J$ such that $\bar\partial$ is transverse to the 0-section $\forall \underline\rho^\pm\in\pitchfork_{\underline H}\underline{\mathcal L}$,

i.e. $D_{\underline u} := (T_{\underline u}\bar\partial)^{ver}$ surjective $\forall \underline u\in\bar\partial^{-1}(0)$.

$\underset{\text{implicit function thm}}{\Longrightarrow}$ $\mathcal{M}(\underline{\mathcal L},\underline\rho^-,\underline\rho^+,\underline H,\underline\delta,\underline J) := \bar\partial^{-1}(0)\subset\mathcal B(\underline{\mathcal L},\underline\rho^-,\underline\rho^+)$

is a smooth manifold of local dimension

$\dim_{\underline u}\mathcal M := \dim(\text{neighborhood of } \underline u \text{ in } \mathcal M) = \operatorname{index} D_{\underline u} = \dim\ker D_{\underline u}$

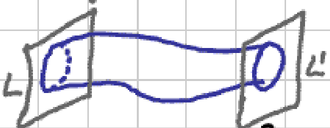
$\boxed{\text{Index \& Energy}}$

<u>energy</u> : $\mathcal{M}(\underline{L}, \underline{p}^-, \underline{p}^+, \underline{H}, \underline{\delta}, \underline{J}) \longrightarrow [0,\infty)$

• $\mathcal{E}(\underline{u}) := \sum_{j=1}^{k} \int |\partial_s u_j|^2 = \sum \int \omega_j(\partial_s u_j, \partial_t u_j - \delta_j^{-1} X_H)$ $\qquad \big(\omega(\cdot, X_H) = -dH\big)$

$$= \sum_{j=1}^{k} \int u_j^*\omega_j + d(u_j^* H_j \cdot \delta_j^{-1} dt) = \sum_{j=1}^{k} \int u_j^*\omega_j + \sum_{j=1}^{k} \big(H_j(p_j^+) - H_j(p_j^-) \big)$$

is determined by $\underline{p}^\pm$ up to



$$\Big\{ \int \underline{v}^*\underline{\omega} = \sum_{j=1}^{k} \int v_j^*\omega_j \;\Big|\; v_j : S^1 \times [0,1] \to N_j \,,\, (v_{j-1}(s,1), v_j(s,0)) \in L_{(j-1)j} \Big\}$$

• <u>index</u> : $\mathcal{M}(\underline{L}, \underline{p}^-, \underline{p}^+, \underline{H}, \underline{\delta}, \underline{J}) \longrightarrow \mathbb{N}_0$ is determined by $\underline{p}^\pm$ up to

$$\Big\{ I_{Maslov}(\underline{v}) \;\Big|\; v_j : S^1 \times [0,1] \to N_j \,,\, (v_{j-1}(s,1), v_j(s,0)) \in L_{(j-1)j} \Big\} = N_{\underline{L}} \cdot \mathbb{Z}$$

$$\Vert$$

$$\sum_{j=1}^{k} I_{Maslov}\Big(\underbrace{(v_{j-1}|_{t=1} \times v_j|_{t=0})^* TL_{(j-1)j}}_{S^1 \to \mathcal{L}ag(\underbrace{TN_{j-1} \times TN_j}_{\cong \mathbb{C}^N})} \Big)$$

using trivializations of $v_j^* TN_j \ \forall j$

$\Big(I_{Maslov} : \pi_1(\mathcal{L}ag(\mathbb{C}^N)) \xrightarrow{\sim} \mathbb{Z}$ is the Maslov index [McDuff-Salamon, Intr. to Symp. Top., §2.3] $\Big)$

<u>Monotonicity Assumption</u>

$\exists\, \tau > 0$ s.t. $\int \underline{v}^*\underline{\omega} = \tau \cdot I_{Maslov}(\underline{v})$ $\quad \forall\, \underline{v}$ as above

(For experts this means in particular each N_j and each $L_{(j-1)j}$ is τ-monotone.)