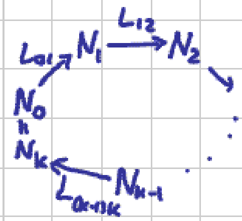
L7 – generalized Floer homology – quilts

Note Title 2/25/2008

$\underline{L} = (L_{01}, L_{12}, \dots, L_{(k-1)k})$ cyclic correspondence

$$\left(\begin{array}{l} L_{(j-1)j} \subset N_{j-1}^- \times N_j \quad \text{Lagrangian correspondences} \\ N_0, N_1, \dots, N_{k-1}, N_k = N_0 \quad \text{symplectic manifolds} \end{array} \right)$$



<u>Floer complex</u> $CF(\underline{L})$ is generated by

$\mathrm{Crit}\, A \cong \cap\, \underline{L} := \{ \underline{p} = (p_1, \dots, p_k = p_0) \in N_1 \times \dots \times N_k \mid (p_{j-1}, p_j) \in L_{(j-1)j} \ \forall j = 1..k \}$

<u>Floer differential</u> $\partial : CF(\underline{L}) \to CF(\underline{L})$ is defined by "counting"

(mod $\mathbb{R}$ isolated) <u>Floer trajectories</u> :

$$\boxed{\begin{array}{lcl} \underline{u} = (u_j)_{j=1..k} & \circledast & \left\{ \begin{array}{ll} \partial_s u_j + J_j(u_j)\, \partial_t u_j = 0 & \text{on } \mathbb{R} \times [0,1] \quad \forall j \\ (u_{j-1}(s,1), u_j(s,0)) \in L_{(j-1)j} & \forall s \in \mathbb{R} \quad \forall j = 1..k \end{array} \right. \\ u_j : \mathbb{R} \times [0,1] \to N_j & & \end{array}}$$

<u>Prop$^{\underline{n}}$</u>: Any Floer trajectory $\underline{u}$ with finite energy

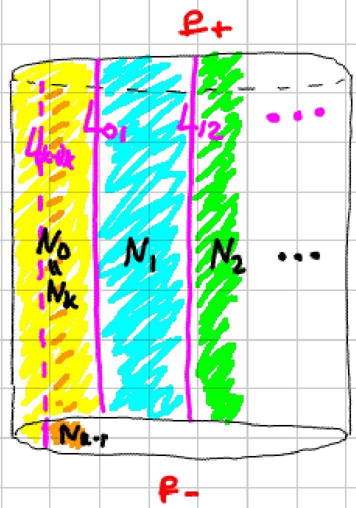
$$\mathcal{E}(\underline{u}) := \sum_{j=1}^{k} \int_{\mathbb{R} \times [0,1]} u_j^* \omega_j < \infty \qquad \text{converges (exponentially)}$$

to some $\underline{p}^{\pm} \in \cap\, \underline{L}$, i.e. $u_j(s,t) \xrightarrow[s \to \pm\infty]{} p_j^{\pm}$ uniformly in t.

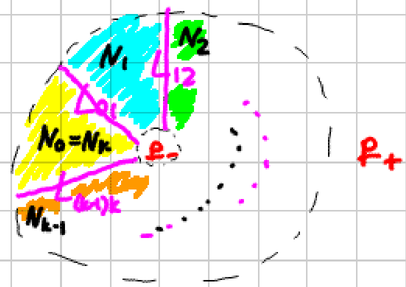
<u>"Proof"</u>: $\mathcal{E}(\underline{u}) = \sum \int \omega_j(\partial_s u_j, \overset{J\partial_s u_j}{\partial_t u_j}) = \sum \int |\partial_s u_j|^2 = \sum \int |\partial_t u_j|^2$

general picture of $\underline{u}$ as holomorphic quilt for $HF(\underline{\mathcal{L}})$

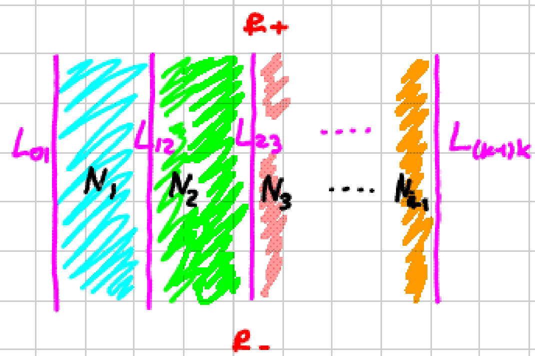
$\underline{\mathcal{L}}: N_0 \xrightarrow{L_{01}} N_1 \xrightarrow{L_{12}} N_2 \to \cdots \to N_{k-1} \xrightarrow{L_{(k-1)k}} N_k = N_0$



or



Ex. (iii) $N_0 = N_k = pt$



→ $CF(\underline{\mathcal{L}})$ is generated by "constant quilts" $\underline{u} \equiv \underline{p} \in \cap\underline{\mathcal{L}}$

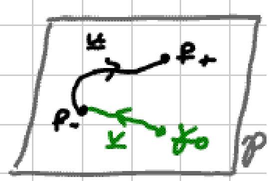
(each patch $u_j \equiv p_j \in N_j$ is a constant map)

→ $\partial \circlearrowright CF(\underline{\mathcal{L}})$ counts nonconstant holomorphic quilts (mod $\mathbb{R}$)

connecting (different) intersection points $\underline{p}_+, \underline{p}_- \in \cap\underline{\mathcal{L}}$

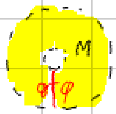
Note: A Floer trajectory with $\lim_{s\to\pm\infty} \underline{u}(s,\cdot) = \underline{p}_\pm$ has energy

$\mathcal{E}(\underline{u}) = \int \underline{u}^*\omega = \left(-\int \underline{v}^*\underline{\omega} + \int (\underline{u}\#\underline{v})^*\underline{\omega}\right) = A(\underline{p}_-) - A(\underline{p}_+)$.



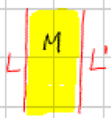
Alternative / "classical" definition

Ex. (0) For $\varphi \in \mathrm{Symp}\,\varphi$ $\quad HF(\underline{L} = (\mathrm{gr}\,\varphi)) = HF(\varphi)$



is the symplectic Floer homology by [Floer]

Ex. (i) For $L, L' \subset M$ $\quad HF(\underline{L} = (L, L')) = HF(L, L')$

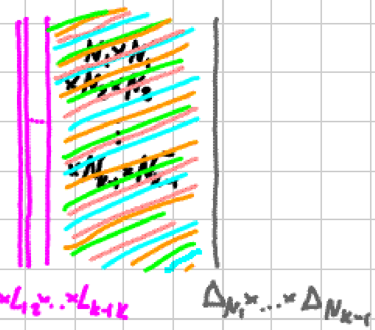


is the Lagrangian Floer homology by [Floer, Oh]

Ex. (iii) For $\underline{L} = (L_{01}, \ldots, L_{(k-1)k})$ with $N_0 = N_k = pt$

- $HF(\underline{L}) = HF\left(L_{01} \times L_{12} \times \ldots \times L_{(k-1)k},\ \Delta_{N_1} \times \Delta_{N_2} \times \ldots \times \Delta_{N_{k-1}}\right)$

by folding in middle of each strip :



$u_j : \mathbb{R} \times [0,1] \to (N_j, J_j) \quad \rightsquigarrow \quad v_j(s,t) := u_j(\tfrac{s}{2}, \tfrac{t}{2}) : \mathbb{R} \times [0,1] \to (N_j, J_j)$

$\qquad v_j'(s,t) := u_j(\tfrac{s}{2}, 1 - \tfrac{t}{2}) : \mathbb{R} \times [0,1] \to (N_j, -J_j)$

$\Rightarrow (v_1, v_1', \ldots, v_k, v_k')(s, 0) = (u_1(\tfrac{s}{2}, 0), u_1(\tfrac{s}{2}, 1), u_2(\tfrac{s}{2}, 0), \ldots) \in L_{01} \times L_{12} \times \ldots$

$\quad (v_1, v_1', \ldots, v_k, v_k')(s, 1) = (u_1(\tfrac{s}{2}, \tfrac{1}{2}), u_1(\tfrac{s}{2}, \tfrac{1}{2}), \ldots) \in \Delta \times \ldots$

- by folding at every seam :

$$HF(\underline{L}) = \begin{cases} HF\left(\underbrace{L_{01} \times L_{23} \times \ldots \times L_{(k-2)(k-1)}},\ \underbrace{L_{12} \times L_{34} \times \ldots \times L_{(k-1)k}}\right) & ; k \text{ even} \\ HF\left(\underbrace{L_{01} \times L_{23} \times \ldots \times L_{(k-1)k}},\ \underbrace{L_{12} \times L_{34} \times \ldots \times L_{(k-2)(k-1)}}\right) & ; k \text{ odd} \end{cases}$$

all Lagrangian submanifolds of $N_1 \times N_2 \times \ldots \times N_{k-1}$

$\qquad \underline{v}(s,0) = (u_1(s,0), u_2(s,1), u_3(s,0), \ldots) \in L_{01} \times L_{23} \times \ldots$

$v_{2j+1} = u_{2j+1},\ v_{2j}(s,t) := u_{2j}(s, 1-t) : \mathbb{R} \times [0,1] \to (N_{2j}, -J_{2j}) \quad \rightsquigarrow \quad \underline{v}(s,1) = (u_1(s,1), u_2(s,0), \ldots) \in L_{12} \times \ldots$

in general

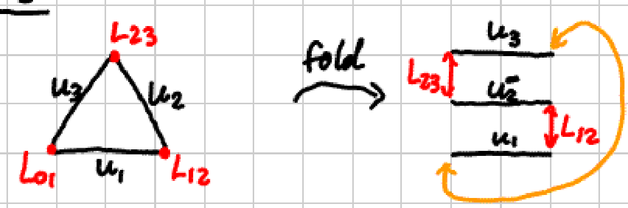
- $HF(\underline{\mathcal{L}}) = HF(L_{01} \times L_{12} \times \ldots \times L_{(k-1)k}\, ,\, \widetilde{\Delta})$ for

$$\widetilde{\Delta} = \tau(\Delta_{N_1} \times \Delta_{N_2} \times \ldots \times \Delta_{N_{k-1}} \times \Delta_{N_k}) \quad ; \quad \tau : N_1 \times N_1 \times \ldots \times N_k \times N_k \to N_0 \times N_1 \times N_1 \times \ldots \times N_k$$
$$(p_1, q_1, \ldots, p_k, q_k) \mapsto (q_k, p_1, q_1, \ldots, p_k)$$

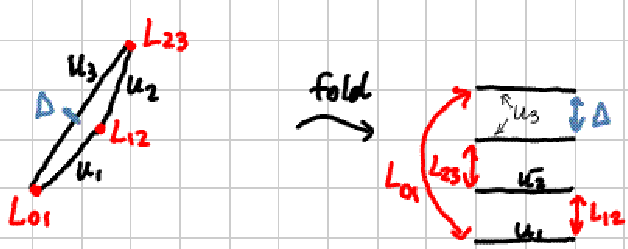
- $$HF(\underline{\mathcal{L}}) = \begin{cases} HF(L_{01} \times L_{23} \times \ldots \times L_{(k-2)(k-1)}\, ,\, \tau(L_{12} \times L_{34} \times \ldots \times L_{(k-1)k})) \; ; & k \text{ even} \\ HF(L_{01} \times L_{23} \times \ldots \times L_{(k-1)k}\, ,\, \tau(L_{12} \times L_{34} \times \ldots \times L_{(k-2)(k-1)} \times \Delta_{N_k})) \; ; & k \text{ odd} \end{cases}$$

⚠ For odd k we fold the N_k-strip in the middle, but not the others. So in order to obtain one strip (of width 1) in the product manifold, we need to start from a quilt $u_j : \mathbb{R} \times [0,1] \to N_j \; ; j = 1..k-1$, $u_k : \mathbb{R} \times [0,2] \to N_k$ containing strips of different widths.

Ex. $k=3$



→ not a strip in $N_1 \times N_2 \times N_3$ with simple boundary conditions

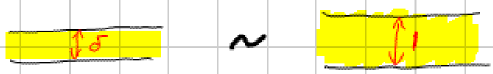


→ strip in $N_1 \times N_2 \times N_3 \times N_3$ with boundary conditions in $\tau(L_{01} \times L_{23})$, $L_{12} \times \Delta_{N_3}$

We will hence define the "quilted Floer homology" $HF(\underline{L})$ by allowing any widths $\underline{\delta} = (\delta_j)_{j=1..k} \in (0,\infty)^k$ and counting the Floer trajectories

$$\underline{u} = (u_j)_{j=1..k} \qquad \begin{cases} \partial_s u_j + J_j(u_j)\,\partial_t u_j = 0 & \text{on } \mathbb{R}\times[0,\delta_j] \ \forall j \\ (u_{j-1}(s,\delta_{j-1}), u_j(s,0)) \in L_{(j-1)j} & \forall s\in\mathbb{R} \ \ \forall j=1..k \end{cases}$$

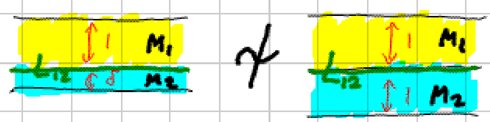
$$u_j : \mathbb{R}\times[0,\delta_j] \to N_j$$



__Note__: $(\mathbb{R}\times[0,\delta], j_{std}) \sim (\mathbb{R}\times[0,1], j_{std})$ is biholomorphic,
$(s,t) \longmapsto (\delta^{-1}s, \delta^{-1}t)$

hence the moduli spaces of holomorphic discs of different widths can be identified.

The moduli space of holomorphic quilted strips with different widths cannot be identified, since separate rescaling of strips destroys the seam condition:



$\Phi : (s_1,t_1,s_2,t_2) \longmapsto (s_1,t_1,\delta^{-1}s_2,\delta^{-1}t_2)$

If (u_1,u_2) satisfies $(u_1(s,1), u_2(s,0)) \in L_{12}$ $\forall s$ then $\Phi^*(u_1,u_2) =: (v_1,v_2)$ satisfies $(v_1(s,1), v_2(\delta s,0)) \in L_{12}$ $\forall s$, but the seam condition is $(v_1(s,1), v_2(s,0)) \in L_{12}$ $\forall s$. This rescaling preserves the seam condition only for correspondences $L_{12} = L_1 \times L_2$ of split form; $L_1 \subset M_1$, $L_2 \subset M_2$.