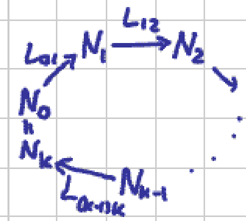
L6 – generalized Floer homology – trajectories

Note Title 2/25/2008

$\underline{\mathcal{L}} = (L_{01}, L_{12}, \ldots, L_{(k-1)k})$ cyclic correspondence



$$\left(\begin{array}{l} L_{(j-1)j} \subset N_{j-1}^- \times N_j \quad \text{Lagrangian correspondences} \\ N_0, N_1, \ldots, N_{k-1}, N_k = N_0 \quad \text{symplectic manifolds} \end{array} \right)$$

<u>critical points</u> of $\mathcal{A}$: $\mathcal{P} = \left\{ (\gamma_j : [0,1] \to N_j)_{j=1..k} \;\middle|\; \begin{array}{l} L_{j-1\,j} \\ \text{conditions} \end{array} \right\} \to \mathbb{R}/$[illegible]

$$\operatorname{Crit} \mathcal{A} = \left\{ \gamma \in \mathcal{P} \;\middle|\; d\mathcal{A}(\gamma)\xi = -\sum_{j=1}^{k} \int_0^1 \omega_j(\xi_j(t), \partial_t \gamma_j(t))\, dt = 0 \quad \forall \xi \in T_\gamma \mathcal{P} \right\}$$

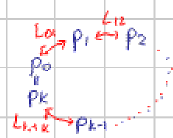
$$\cong \left\{ \underline{p} = (p_1, \ldots, \overset{=p_0}{p_k}) \in N_1 \times \ldots \times N_k \;\middle|\; (p_{j-1}, p_j) \in L_{(j-1)j} \quad \forall j = 1..k \right\}$$

$$=: \cap\, \underline{\mathcal{L}} \quad \text{generalized intersection}$$

<u>Ex.(i)</u>: $\operatorname{Crit} \mathcal{A} = \{ (pt, p, pt) \in pt \times M \times pt \mid \underset{\text{i.e. } p \in L}{(pt, p) \in L}, \underset{\text{i.e. } p \in L'}{(p, pt) \in L'} \} \cong L \cap L'$

<u>Ex.(ii)</u>: $\underline{\mathcal{L}} = (\operatorname{graph} \varphi)$, $\varphi \in \operatorname{Symp}(M) \rightsquigarrow \operatorname{Crit} \mathcal{A} = \{ (p_0 = p_1) \in \operatorname{gr} \varphi \} \cong \operatorname{Fix} \varphi$

<u>Ex (iii) / general</u>: "quilt" with each component $u_j \equiv p_j \in N_j$ constant



"<u>Def$^{\underline{n}}$</u>:" Floer homology $HF(\underline{\mathcal{L}}) := \ker / \operatorname{im} \partial$

$$CF(\underline{\mathcal{L}}) := \bigoplus_{\underline{p} \in \operatorname{Crit} \mathcal{A}} \mathbb{Z} \langle \underline{p} \rangle \qquad (\text{assuming } \cap \underline{\mathcal{L}} \text{ is finite})$$

$\partial : CF \to CF$ defined by $\underbrace{\text{"counting" Floer trajectories}}_{\text{to be defined}}$

<u>gradient:</u> To define $\nabla \mathcal{A}$, fix a metric on $\mathcal{P}$

For $j=1..k$ pick J_j ω_j-compatible almost complex structure

$J : M \to End(TM)$ smooth (but not neccessarily $\nabla J = 0$)

$J^2 = -Id$, $g_J(X,Y) := \omega(X, JY)$ is a metric on M
(i.e. symmetric, positive definite)

<u>Thm</u>: The space of such J is nonempty, contractible

L^2-metric on $\mathcal{P}$: $\underline{\xi}, \underline{\eta} \in T_{\underline{\gamma}}\mathcal{P}$ (i.e. $\xi_j, \eta_j \in \Gamma(\gamma_j^* TN_j)$ with $TL_{\alpha_{j-1}\alpha_j}$ conditions)

$$\langle \underline{\xi}, \underline{\eta} \rangle := \sum_{j=1}^{k} \int_0^1 g_{J_j}(\xi_j(t), \eta_j(t))\, dt$$

$\forall \underline{\xi} \in T_{\underline{\gamma}}\mathcal{P}$

$$\langle \underline{\xi}, \nabla \mathcal{A}(\underline{\gamma}) \rangle = d\mathcal{A}(\underline{\gamma})\underline{\xi} = \sum_{j=1}^{k} -\int_0^1 \omega_j(\xi_j(t), \partial_t\gamma_j(t))\, dt = \sum_{j=1}^{k} \int_0^1 g_j(\xi_j(t), J(\gamma_j(t))\partial_t\gamma_j(t))\, dt$$

$$\Rightarrow \nabla \mathcal{A}(\underline{\gamma}) = \underline{J}(\underline{\gamma})\partial_t\underline{\gamma} = \left(J_j(\gamma_j)\partial_t\gamma_j\right)_{j=1..k}$$

<u>Note</u>: $\nabla \mathcal{A}$ cannot really be viewed as vector field on $\mathcal{P}$.

For $\nabla \mathcal{A}(\underline{\gamma}) \in T_{\underline{\gamma}}\mathcal{P}$ the linearized conditions in Ex. (i) are

$J(\gamma(0))\partial_t\gamma(0) \in T_{\gamma(0)}L$, $J(\gamma(1))\partial_t\gamma(1) \in T_{\gamma(1)}L'$ (i.e. $\partial_t\gamma(0) \perp TL$, $\partial_t\gamma(1) \perp TL'$)

but a general $\gamma \in \mathcal{P}$ only satisfies $\gamma(0) \in L$, $\gamma(1) \in L'$.

We can still try to study the flow lines of $\nabla \mathcal{A}$ on the subset

$\{\underline{\gamma} \in \mathcal{P} \mid \nabla \mathcal{A}(\underline{\gamma}) \in T_{\underline{\gamma}}\mathcal{P}\}$.

<u>negative gradient flow lines</u>: $\eta : \mathbb{R} \to \mathcal{P}, \quad \frac{d}{ds}\eta = -\nabla\mathcal{A}(\gamma)$

i.e. $\eta_j : \mathbb{R} \to \mathcal{C}^\infty([0,1], N_j), \begin{cases} \frac{d}{ds}\eta_j(s) = -J_j(\eta_j(s))\frac{d}{dt}\eta_j(s) & \text{on } [0,1]\ \forall s \in \mathbb{R} \\ \left(\eta_{j-1}(s)|_{t=1}, \eta_j(s)|_{t=0}\right) \in L_{(j-1)j} & \forall j = 1..k, \forall s \in \mathbb{R} \end{cases}$

<u>Note</u>: If $\gamma = \eta(s_0)$ is a point on a neg. gradient flow line $\eta : (0, s_0] \to \mathcal{P}$ (in Ex. (i)) then $J(\gamma(0))\frac{d}{dt}\gamma(0) = \frac{d}{ds}\eta(s_0)|_{t=0} \in T_{\gamma(0)}L$

since $s \mapsto \eta(s_0)|_{t=0}$ is a path in L

However, the neg. gradient flow equation is still not well posed.

For existence would need $\mathcal{P}' := \{\gamma \in \mathcal{P} \mid \nabla\mathcal{A}(\gamma) \in T_\gamma\mathcal{P}\}$ at least complete.

If we take a completion $\mathcal{P}^\ell := \overline{\mathcal{P}'}^{W^{\ell,2}} \subset \bigoplus_{j=1}^{k} W^{\ell,2}([0,1], N_j)$ then for $\gamma \in \mathcal{P}^\ell$ the gradient again is not necessarily a tangent vector – for analytic reasons:

$\nabla\mathcal{A}(\gamma) = \underline{J}\partial_t\gamma \subset \bigoplus_{j=1}^{k} W^{\ell-1,2}([0,1], \gamma_j^* TN_j)$ whereas $T_\gamma\mathcal{P}^\ell \subset \bigoplus W^{\ell,2}(\ldots)$.

Conley-Zehnder solved this by using a different metric to define $\nabla\mathcal{A}$.

Floer got inspired by Gromov and noticed that the L^2-gradient flow lines are holomorphic curves.

<u>Floer trajectories</u> : view $\frac{d}{ds}\underline{p} + \nabla\mathcal{A}(\underline{p}) = 0$

as PDE $\partial_s \underline{u} + \underline{J}(\underline{u})\,\partial_t \underline{u} = 0$ for $\underline{u}(s,t) = \underline{p}(s)(t)$

$$\boxed{\begin{array}{ll} \underline{u} = (u_j)_{j=1..k} & \circledast \left\{ \begin{array}{ll} \partial_s u_j + J_j(u_j)\,\partial_t u_j = 0 & \text{on } \mathbb{R}\times[0,1] \quad \forall j \\ (u_{j-1}(s,1), u_j(s,0)) \in L_{(j-1)j} & \forall s \in \mathbb{R} \quad \forall j = 1..k \end{array} \right. \\ u_j : \mathbb{R}\times[0,1] \to N_j & \end{array}}$$

- trivial solutions : $\underline{u}(s,t) = \underline{p} \in \cap \underline{L}$
- $\mathbb{R}$-symmetry : if $\underline{u}$ is a solution then so is $(\sigma * \underline{u})(s,t) := \underline{u}(\sigma + s, t)$ for any $\sigma \in \mathbb{R}$.

<u>'Def^n'</u>:

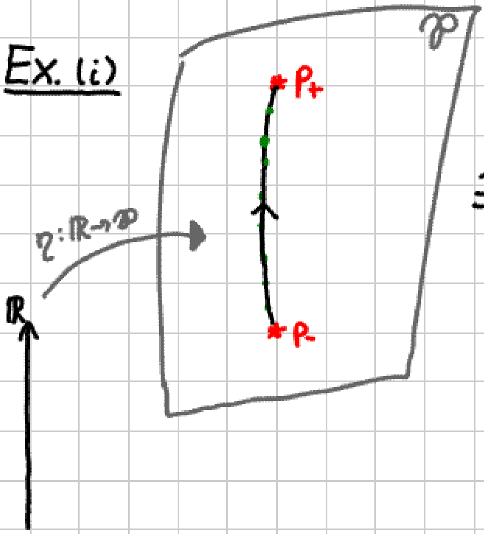
$$CF(\underline{L}) := \bigoplus_{\substack{\underline{p} \in \mathrm{Crit}\,\mathcal{A} \\ = \cap \underline{L}}} \mathbb{Z}\langle \underline{p} \rangle \qquad (\text{assuming } \cap \underline{L} \text{ is finite})$$

∂ linear and

$$\partial \langle \underline{p}_+ \rangle := \sum_{\underline{p}_- \in \mathrm{Crit}\,\mathcal{A}} \# \left\{ \underline{u} \in \bigoplus_{j=1}^{k} C^\infty(\mathbb{R}\times[0,1], N_j) \;\middle|\; \circledast, \lim_{s\to\pm\infty} \underline{u}(s,\cdot) = \underline{p}_\pm \right\} \Big/ \mathbb{R}\text{-translation}$$

(signed count of isolated trajectories from $\underline{p}_-$ to $\underline{p}_+$
(assuming transversality, compactness, etc.)
$= 0$ if moduli space $\{\ldots\}/_{\mathbb{R}}$ has dimension > 0
To Be Defined)

Ex. (i)



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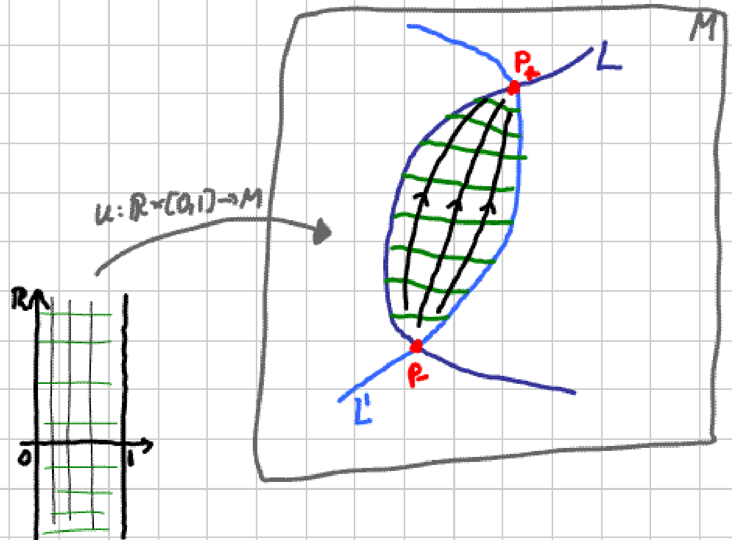
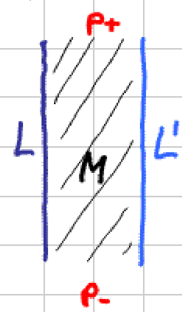
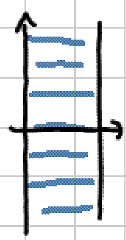


picture this as holomorphic strip



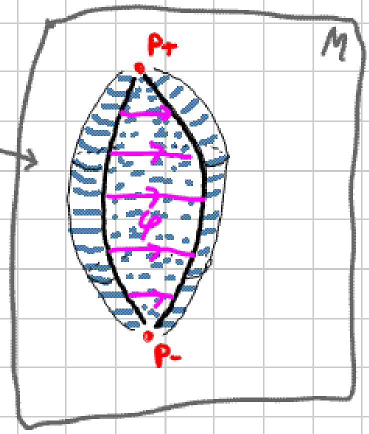
Ex. (0):



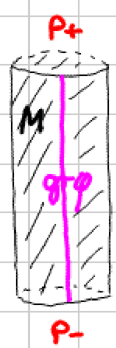
$u : \mathbb{R} \times [0,1] \to M$

$u(s,1) = \varphi(u(s,0))$

$\lim_{s \to \pm\infty} u(s,\cdot) \equiv p_\pm \in \mathrm{Fix}\,\varphi$



SHORT:



or