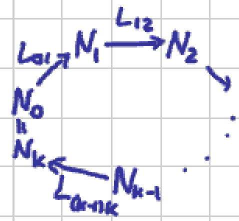
L5 – generalized Floer homology

Note Title 2/25/2008

We will define Floer homology $HF(\underline{\mathcal{L}})$ for a <u>cyclic correspondence</u> $\underline{\mathcal{L}} = (L_{01}, L_{12}, \dots, L_{(k-1)k})$ of Lagrangian correspondences $L_{(j-1)j} \subset N_{j-1}^- \times N_j$ with underlying symplectic manifolds $N_0, N_1, \dots, N_{k-1}, N_k = N_0$.

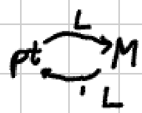


Ex.: (0) $L \subset M^- \times M$ Lagrangian (e.g. graph of symplectomorphism φ)

$\rightsquigarrow \underline{\mathcal{L}} = (L)$ with underlying $N_0 = N_1 = M$

(i) $L, L' \subset M$ Lagrangian submanifolds

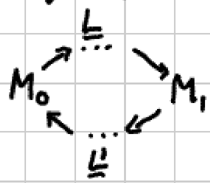
$\rightsquigarrow \mathcal{L} = (L, L')$



(underlying pt, M, pt (or M, pt, M))

(ii) $\underline{L}, \underline{L}'$ generalized Lagrangian correspondences from M_0 to M_1

$\rightsquigarrow \underline{\mathcal{L}} = (\underline{L}, \underline{L}'^t)$

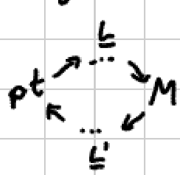


(underlying $M_0 = N_0, N_1 \dots N_k = M_1 = N'_{k'} \dots N'_1, N'_0 = M_0$)

(iii) $\underline{L}, \underline{L}'$ <u>generalized Lagrangian submanifolds of M</u>

(i.e. generalized Lagrangian correspondences from pt to M)

$\rightsquigarrow \underline{\mathcal{L}} = (\underline{L}, \underline{L}'^t)$



"Def$^{\underline{n}}$": $HF(\underline{\mathcal{L}}) := \frac{\ker\partial}{\operatorname{im}\partial}$ is the "Morse homology" (a la Witten, Floer)

on the (generalized) path space $\mathcal{P}$

of the (generalized) symplectic action functional $\mathcal{A} : \mathcal{P} \to \mathbb{R}/_{\sim}$

Floer complex CF generated by critical points of $\mathcal{A} : \mathcal{P} \to \mathbb{R}/_{\sim}$

Floer differential $\partial \circlearrowright CF$ defined by "counting" ~~gradient flow lines~~ of $\mathcal{A}$
Floer trajectories

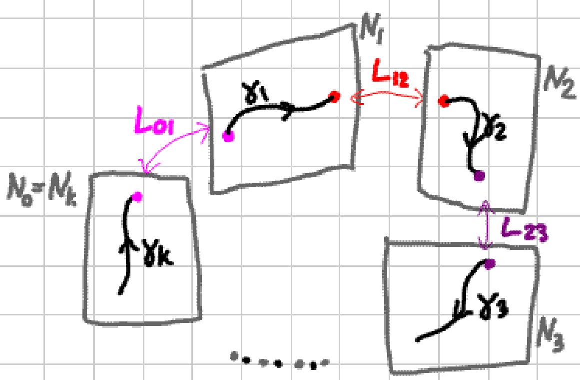
<u>path space</u>

$$\mathcal{P} = \left\{ \underline{\gamma} := (\gamma_1, \gamma_2, \dots, \gamma_{k-1}, \underset{\gamma_0}{\overset{\shortparallel}{\gamma_k}}) \;\middle|\; \gamma_j : [0,1] \to N_j \,,\, (\gamma_{j-1}(1), \gamma_j(0)) \in L_{(j-1)j} \quad \forall j = 1..k \right\}$$

<u>Ex (i)</u>: $\underline{\gamma} = \gamma_1 : [0,1] \to M \quad , \quad \gamma_1(0) \in L \quad , \quad \gamma_1(1) \in L'$

$\gamma_0 = \gamma_2 \equiv pt \qquad\qquad \gamma_1(0) = (\gamma_0(1), \gamma_1(0)) \qquad \gamma_1(1) = (\gamma_1(1), \gamma_2(0))$

<u>general</u>:



Symplectic action

$\mathcal{A} : \mathcal{P} \to \mathbb{R}/_{***}$ fix $\gamma^0 \in \mathcal{P}$

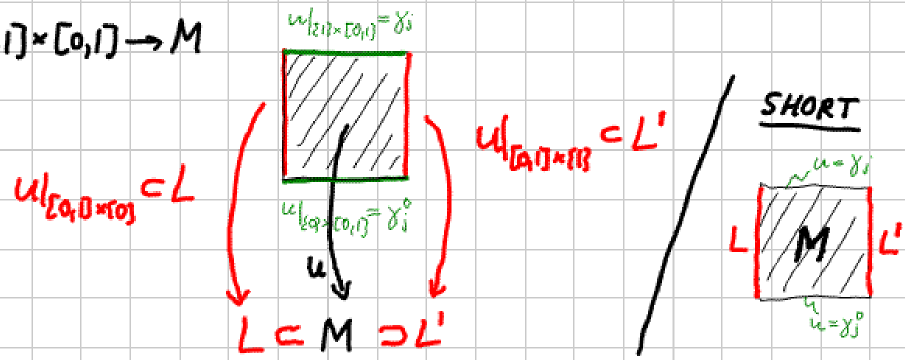
$$\gamma \longmapsto -\sum_{j=1}^{k} \int_{[0,1]^2} u_j^* \omega_{N_j}$$

$u_j : [0,1]\times[0,1] \to N_j$

$u_j(0,\cdot) = \gamma_j^0, \quad u_j(1,\cdot) = \gamma_j$

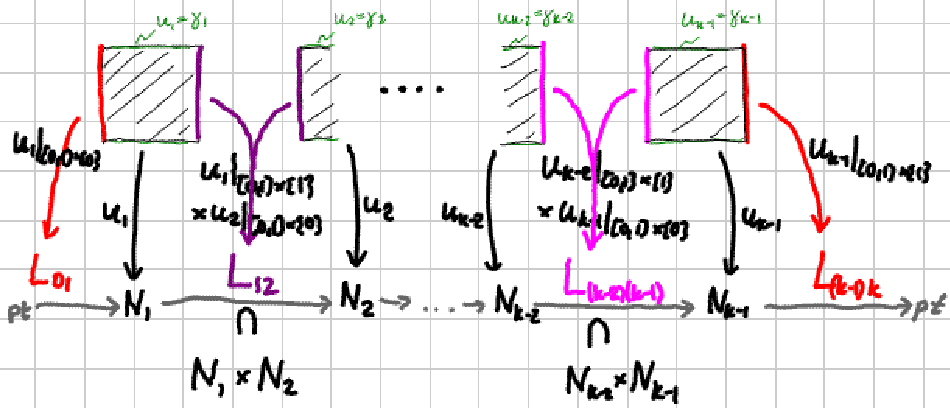
$(u_j(s,\cdot))_{j=1..k} \in \mathcal{P} \quad \forall s \in [0,1]$

<u>Ex. (i):</u> $u_1 : [0,1]\times[0,1] \to M$

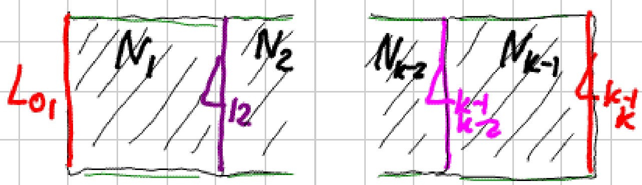


$***$ $\mathcal{A}$ well defined up to $\left\{ \int w^*\omega \;\middle|\; \begin{array}{l} w : S^1\times[0,1] \to M \\ S^1\times\{0\} \to L \\ S^1\times\{1\} \to L' \end{array} \right\}$

Ex (iii): $\underline{\mathcal{L}} = (L_{01},\dots,L_{k-1\,k})$ $\quad N_0 = N_k = pt$



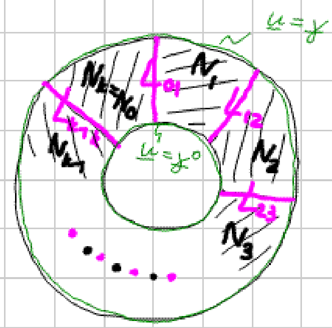
SHORT:



general case: picture $\underline{u}=(u_1,\dots,u_k)$ as "quilt"

*** $A =$ "$-\int \underline{u}^*\underline{\omega}$" is well defined up to

$$\left\{\sum_{j=1}^{k} w_j^*\omega_{N_j} \;\middle|\; w_j : S^1\times[0,1]\to N_j,\ (w_j(s,\cdot))_{j=1..k}\in \mathcal{P}\ \forall s\in S^1\right\}$$



<u>critical points</u>:

$$dA(\underline{\gamma}) : T_{\underline{\gamma}}\mathcal{P} \longrightarrow \mathbb{R}$$

$$(\underline{\xi}) \longmapsto -\int_0^1 \sum_{j=1}^{k} \omega_j(\xi_j(t), \partial_t\gamma_j(t))\, dt \qquad \begin{pmatrix} \xi_j \in \Gamma(\gamma_j^* TN_j) \\ (\xi_{j-1}(1), \xi_j(0)) \in T_{(\gamma_{j-1}(1),\gamma_j(0))} L_{(j-1)j} \end{pmatrix}$$

$$\left(\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \Big(A(\exp_{\underline{\gamma}}(\underline{\xi})) - A(\underline{\gamma})\Big) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} -\int_{[0,\varepsilon]\times[0,1]} \underbrace{\overbrace{\exp_{\underline{\gamma}}(s\,\underline{\xi})}^{\underline{u}}{}^{*}\,\underline{\omega}}_{\sum_{j=1}^{k} \omega_j(\partial_s u_j,\, \partial_t u_j)\, ds\, dt} \right)$$

$$\partial_s u_j = \xi_j, \quad \partial_t u_j = \partial_t\gamma_j \quad \text{at } s=\varepsilon=0$$

$$\underline{\gamma} \in \mathrm{Crit}\, A \iff dA(\underline{\gamma})\,\underline{\xi} = 0 \ \ \forall \underline{\xi} \iff \partial_t \gamma_j = 0 \ \ \forall j$$

$$\Rightarrow \mathrm{Crit}\, A = \{ \underline{p} = (p_1, \dots, p_k \overset{:= p_0}{}) \in N_1 \times \dots \times N_k \mid (p_{j-1}, p_j) \in L_{(j-1)j} \ \ \forall j = 1..k \}$$

$$= \cap\, \underline{\mathcal{L}} \quad \text{generalized intersection}$$

<u>Ex.(i)</u>: $\mathrm{Crit}\, A = \{ (pt, p, pt) \in pt \times M \times pt \mid \underset{\text{i.e. } p \in L}{(pt, p) \in L},\ \underset{\text{i.e. } p \in L'}{(p, pt) \in L'} \} = L \cap L'$