L4 - generalized Lagrangian correspondences

Note Title 2/13/2008

$\boxed{\text{PREVIEW}}$ $\ell, \ell' \subset M/\!\!/G$ Lagrangian

Suppose $\ell = L \circ \widetilde{\mu^{-1}(0)}$ is the transverse & embedded composition with $L \subset M$ Lagrangian.

Then (under various compactness and monotonicity assumptions)

$$HF(\ell, \ell') \cong HF(L, \pi^{-1}(\ell'))$$

CF: $M/\!\!/G \supset \ell \cap \ell' \overset{1-1}{\longleftrightarrow} L \cap \pi^{-1}(\ell')$

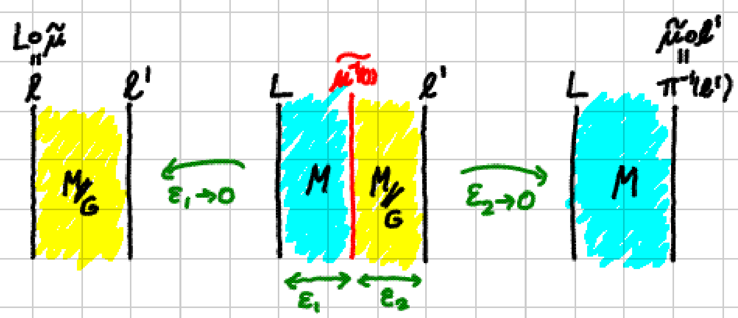
$$= \{ \underbrace{L \cap \mathcal{O}(p)}_{\text{none or unique point}} \mid \mathcal{O}(p) \in \ell' \}$$

CF ↺ ∂ : holomorphic curves in $M/\!\!/G$ $\longleftrightarrow$ holomorphic curves in M

"Proof" $(\ell, \ell') = (L \circ \widetilde{\mu^{-1}(0)}, \ell')$

$\updownarrow$ equivalence of generalized Lagrangian correspondences

$$(L, \widetilde{\mu^{-1}(0)} \circ \ell') = (L, \pi^{-1}(\ell'))$$



Composition and intersections:

1) $L_0 \subset M_0$, $L_1 \subset M_1$, $L_{01} \subset M_0^- \times M_1$ Lagrangian

$$\Rightarrow (L_0 \circ L_{01}) \cap L_1 \cong L_0 \cap (L_{01} \circ L_1)$$

$(L_0 \circ L_{01}) \cap L_1 = \{x_1 \in L_1 \mid \exists x_0 \in L_0 : (x_0, x_1) \in L_{01}\}$ ⇝ unique

$L_0 \cap (L_{01} \circ L_1) = \{x_0 \in L_0 \mid \exists x_1 \in L_1 : (x_0, x_1) \in L_{01}\}$ ⇝ unique

$x_1 \longmapsto x_0$

$x_1 \longleftarrow\!\shortmid x_0$

if $L_0 \circ L_{01}$ and $L_{01} \circ L_1$ are embedded.

2) $L \subset M = pt \times M$, $L' \subset M = M \times pt$ Lagrangian

- $L \circ L'$ is transverse if $L \times L' \pitchfork \Delta_M$ i.e. $L \pitchfork L'$

  $\Rightarrow L \times_{\Delta} L' = \{(x,x) \mid x \in L \cap L'\} \cong L \cap L'$ finite set

- $L \circ L'$ is embedded if $L \cap L' = \emptyset$ or point

<u>Def$^{\underline{n}}$</u>: M_0, M_1 symplectic manifolds

- A <u>generalized Lagrangian correspondence</u> from M_0 to M_1, $M_0 \xrightarrow{\underline{L}} M_1$ is a finite sequence $\underline{L} = (L_{01}, L_{12}, \ldots, L_{k-1\,k})$ of Lagrangian correspondences $L_{(j-1)j} \subset N_{j-1}^- \times N_j$ between an underlying sequence $M_0 = N_0, N_1, \ldots, N_{k-1}, N_k = M_1$ of symplectic manifolds.

$$M_0 = N_0 \xrightarrow{L_{01}} N_1 \xrightarrow{L_{12}} N_2 \to \ldots \to N_{k-1} \xrightarrow{L_{(k-1)k}} N_k = M_1$$

- Its <u>dual</u> is the reversed sequence $\underline{L}^t := (L^t_{k-1\,k}, \ldots, L^t_{01})$.
- The <u>algebraic composition</u> of gen. Lagr. corr. $M_0 \xrightarrow{\underline{L}} M_1$, $M_1 \xrightarrow{\underline{L}'} M_2$ is the concatenation $M_0 \xrightarrow{\underline{L} \# \underline{L}'} M_2$, given by

$$\underline{L} \# \underline{L}' := (L_{01}, \ldots, L_{k-1\,k}, L'_{01}, \ldots, L'_{k'-1\,k'})$$

with underlying $M_0 = N_0, N_1, \ldots N_{k-1}, N_k = M_1 = N'_0, N'_1, \ldots, N'_{k'-1}, N'_{k'} = M_2$.

$$\underbrace{M_0 = N_0 \xrightarrow{L_{01}} N_1 \to \ldots \to N_{k-1} \xrightarrow{L_{k-1\,k}} N_k = M_1}_{\underline{L}} = \underbrace{N'_0 \xrightarrow{L'_{01}} N'_1 \to \ldots \to N'_{k'-1} \xrightarrow{L'_{k'-1\,k'}} N'_{k'} = M_2}_{\underline{L}'}$$

Def^n: Two generalized Lagr. corresp. $M_0 \xrightarrow{\underline{L}} M_1$ and $M_0 \xrightarrow{\underline{L}'} M_1$ are equivalent if they are connected by a sequence of "good moves". A "good move" takes

$$M_0 \to \dots \to N_{\ell-1} \xrightarrow{L_{\ell-1\,\ell}} N_\ell \xrightarrow{L_{\ell\,\ell+1}} N_{\ell+1} \to \dots \to M_1$$

to

$$M_0 \to \dots \to N_{\ell-1} \xrightarrow{L_{\ell-1\,\ell} \circ L_{\ell\,\ell+1}} N_{\ell+1} \to \dots \to M_1$$

(or vice versa), where $L_{\ell-1\,\ell} \circ L_{\ell\,\ell+1}$ is transverse & embedded.

Example: $M_0 \xrightarrow{L_{01}} N_1 \xrightarrow{L_{12}} N_2 \xrightarrow{L_{23}} M_1$

$\wr\wr$

$M_0 \xrightarrow{L_{01} \circ L_{12}} N_2 \xrightarrow{L_{23}} M_1 \quad \underset{\substack{\text{if} \\ \pitchfork, \hookrightarrow}}{\approx} \quad M_0 \xrightarrow{L_{01} \circ L_{12} \circ L_{23}} M_1$

$\wr\wr$

$M_0 \xrightarrow{L_{01} \circ L_{12}} N_2 \xrightarrow{L'_{23}} N_3 \xrightarrow{L'_{34}} M_1$

$\wr\wr$

$M_0 \xrightarrow{L_{01} \circ L_{12} \circ L'_{23}} N_3 \xrightarrow{L'_{34}} M_1$

Homework: Let $L, L' \subset M_0^- \times M_1$ be (simple) Lagrangian correspondences. If $L \neq L'$ show that $L \not\approx L'$ as generalized Lagr. corresp.

We define the <u>symplectic category</u> *Symp* by

- objects : (M,ω) symplectic manifold $\left(\begin{array}{l}\text{finite dimensional} \\ \text{could specify to e.g. compact}\end{array}\right)$
- morphisms $Mor(M_0, M_1)$: $\left(\begin{array}{c}\text{generalized Lagrangian correspondences } M_0 \xrightarrow{\underline{L}} M_1 \\ \text{modulo equivalence}\end{array}\right)$
- composition – algebraic as above
- identity $Mor(M,M) \ni 1_M := \Delta_M \subset M^- \times M$ diagonal

TO CHECK : • composition is associative,

- $1_M \circ \underline{L} = (\Delta_M, L_{01}, \dots, L_{k-1\,k}) \sim (\underbrace{\Delta_M \circ L_{01}}_{= L_{01}}, \dots, L_{k-1\,k}) = \underline{L}$

  $\underline{L} \circ 1_M = \dots \sim \underline{L}$

<u>Next</u>: extend *Symp* to a 2-category, i.e. make

morphism space $Mor(M_0, M_1)$ a category

composition $Mor(M_0, M_1) \times Mor(M_1, M_2) \to Mor(M_0, M_2)$ a functor

PREVIEW : ${}^2Mor(\underset{\underset{Mor(M_0,M_1)}{\cap}}{\underline{L}}, \underset{\cap}{\underline{L}'}) := HF(\underline{L}, \underline{L}')$