Note Title 2/7/2008

L3 - Lagrangian correspondences arising from moment maps

Ex.: $M=\mathbb{C}^{n+1}$, $S^1 = \mathbb{R}/\mathbb{Z} \circlearrowright M$ by rotation $\psi_{[t]}(z_0,\dots z_n) = (e^{2\pi i t} z_0, \dots, e^{2\pi i t} z_n)$

$\forall \xi \in \mathbb{R} = \mathfrak{g} \quad X_\xi(z_0 \dots z_n) = 2\pi i \xi \cdot (z_0, \dots, z_n)$

$\omega(X_\xi(\underline{z}), \underline{v}) = (i X_\xi(\underline{z})) \underset{\mathbb{R}^{2n}}{\cdot} \underline{v} = -2\pi \xi \; \underline{z} \cdot \underline{v}$

$= \xi \cdot d\mu(\underline{z}) \underline{v}$

with $\mu(\underline{z}) = -\pi |\underline{z}|^2 \quad \rightsquigarrow \mu^{-1}(0)/S^1 = \{0\}/S^1 = \{0\}$

or $\mu(\underline{z}) = \pi - \pi |\underline{z}|^2 \quad \rightsquigarrow \mu^{-1}(0)/S^1 = \{|\underline{z}|=1\}/S^1 = S^{2n+1} \subset \mathbb{C}^{n+1}/\text{rotation} = \mathbb{CP}^n$

(or any other constant)

Note: $\mu^{-1}(0) \subset M$ is G-invariant $\quad (\psi(g) : \mu^{-1}(0) \to \mu^{-1}(0) \quad \forall g)$

* Suppose G acts freely on $\mu^{-1}(0)$ $\quad [\psi_g(p) = p \in \mu^{-1}(0) \Rightarrow g = \mathbb{1}]$
and G is compact, connected

$\Rightarrow$ infinitesimal action is injective : $\quad X_\xi(p) = 0 \Rightarrow \xi = 0$

$\left[\frac{d}{dt} \psi_{\exp(t\xi)}(p) = d\psi_{\exp(t\xi)}(X_\xi(p)) = 0 \;\; \forall t \Rightarrow \exp(t\xi) = \mathbb{1} \Rightarrow \xi = 0 \right]$

$\Rightarrow$ 0 regular value of $\mu : M \to \mathfrak{g}$

$\Big[$ $d\mu(p) : T_pM \to \mathfrak{g}$ surj. $\forall p \in \mu^{-1}(0)$ since

$\xi \in (\text{im } d\mu(p))^\perp \Leftrightarrow \langle d\mu(p) v, \xi \rangle_{\mathfrak{g}} = 0 \quad \forall v \in T_pM$

$X_\xi(p) = 0 \Leftrightarrow \omega(X_\xi(p), v) = 0$ —— " —— $\Big]$

$\Rightarrow \mu^{-1}(0) \subset M$ is a coisotropic submanifold of $\dim = 2n - \dim G$

$$\left[\begin{array}{l} \text{Let } \mathcal{O}(p) := \{\psi_g(p) \mid g \in G\} \cong G \text{ be the orbit through } p \in \mu^{-1}(0) \\ T_p\mathcal{O} = \{X_\xi(p) \mid \xi \in \mathfrak{g}\} \subset T_p\,\mu^{-1}(0) \\ T_p\mu^{-1}(0) = \ker d\mu(p) = \{v \in T_pM \mid \underbrace{\langle d\mu(p)v, \xi\rangle}_{\omega(X_\xi(p), v)} = 0 \;\; \forall \xi \in \mathfrak{g}\} \\ \qquad = (T_p\mathcal{O})^\omega \\ \Rightarrow (T_p\mu^{-1}(0))^\omega = T_p\mathcal{O} \subset T_p\mu^{-1}(0) \qquad \text{coisotropic} \end{array}\right]$$

The isotropic leaves are the orbits $\mathcal{O}(p)$, so $(T_p\mu^{-1}(0))^\omega$ is a regular foliation

$\Rightarrow \mu^{-1}(0)/\sim \; = \mu^{-1}(0)/G$ is a symplectic manifold of $\dim = \dim M - 2\dim G$

$\mu^{-1}(0)/G \cong M/\!\!/G$ "symplectic quotient"

(Notation: We can shift μ by any central constant $\tau \in \mathfrak{g}$, $g^{-1}\tau g = \tau \;\; \forall g$
or, equivalently, take the quotient $M/\!\!/_G[\tau] := \mu^{-1}(\tau)/G$)

$\iota \times \pi : \mu^{-1}(0) \longrightarrow M^- \times M/\!\!/_G$ embeds to a Lagrangian correspondence

$$\widetilde{\mu^{-1}(0)} := \{(p, \mathcal{O}(q)) \in M \times M/\!\!/_G \mid \mu(p) = 0 = \mu(q),\; \mathcal{O}(p) = \mathcal{O}(q)\}$$

previous Example: $\mu^{-1}(\pi) \cong S^{2n+1} \hookrightarrow \mathbb{C}^{n+1} \times \mathbb{CP}^n$

$\underline{z} \longmapsto (\underline{z}, [z_0 : \ldots : z_n])$

Composition with $\mu^{-1}(0) \subset M^- \times M/\!\!/G$

$\pi : \mu^{-1}(0) \to \mu^{-1}(0)/G = M/\!\!/G$

- $\underset{\underset{pt^- \times M}{\shortparallel}}{L \subset M}$ Lagrangian $\rightsquigarrow L \circ \widetilde{\mu^{-1}(0)} = \pi\big(\underbrace{L \cap \mu^{-1}(0)}_{\subset M}\big) \subset \underset{\underset{pt^- \times M/\!\!/G}{\shortparallel}}{M/\!\!/G}$

  - transverse if $L \pitchfork \mu^{-1}(0) \subset M$

  - embedded if $L \cap \mathcal{O}(p) = $ point or $\emptyset$ for all orbits $\mathcal{O}(p)$

  <u>Ex.:</u> $\mathbb{R}^n \subset \mathbb{C}^n \rightsquigarrow \mathbb{R}^n \circ \widetilde{\mu^{-1}(\pi)} = \pi\big(\underbrace{\mathbb{R}^n \cap S^{2n+1}}_{S^{n-1} \subset \mathbb{R}^n}\big) = \mathbb{RP}^n \subset \mathbb{CP}^n$

  composition is transverse but not embedded ($S^{n-1} \to \mathbb{RP}^n$ is a double cover)

- $\widetilde{\mu^{-1}(0)} \circ \widetilde{\mu^{-1}(0)}^t = \{(x,y) \in M \times M \mid \mu(x) = \mu(y) = 0,\ \pi(x) = \pi(y) \in M/\!\!/G\}$

  is always "transverse" (since $\pi : \mu^{-1}(0) \to M/\!\!/G$ surjective)

  and "embedded" (since $\pi(x) \in M/\!\!/G$ uniquely determined by x).

- $\widetilde{\mu^{-1}(0)}^t \circ \widetilde{\mu^{-1}(0)} = \{(p,q) \in M/\!\!/G \times M/\!\!/G \mid \exists x \in M : p = \pi(x) = q\} = \Delta_{M/\!\!/G}$

  is smooth but neither transverse nor embedded :

$$\widetilde{\mu^{-1}(0)}^t \times_{\Delta_M} \widetilde{\mu^{-1}(0)} = \{(\pi(x), x, x, \pi(x)) \mid x \in M\} \cong M \hookleftarrow G \text{ fiber}$$

$$\downarrow \pi_{02} \qquad \pi_{02}^{-1}(\pi(x), \pi(x)) = \{(\pi(x), \psi_g(x), \psi_g(x), \pi(x)) \mid g \in G\}$$

$$\widetilde{\mu^{-1}(0)}^t \circ \widetilde{\mu^{-1}(0)} = \Delta_{M/\!\!/G} \qquad \ni (\pi(x), \pi(x))$$

- $\underset{\underset{pt \times M/\!\!/G}{\shortparallel}}{\ell \subset M/\!\!/G} \rightsquigarrow \ell \circ \widetilde{\mu^{-1}(0)}^t = \pi^{-1}(\ell) \subset \underset{\underset{pt \times M}{\shortparallel}}{M}$

  always transverse, embedded

<u>Note</u>: Any Lagrangian $\ell \subset M/\!\!/G$ is the composition with $\widetilde{\mu^{-1}(0)}$ of a Lagrangian $L \subset M$.

E.g. $L = \ell \circ \widetilde{\mu^{-1}(0)}^t \rightsquigarrow L \circ \widetilde{\mu^{-1}(0)} = \ell \circ \widetilde{\mu^{-1}(0)}^t \circ \widetilde{\mu^{-1}(0)} = \ell \circ \Delta_{M/\!\!/G} = \ell$

<u>Question</u>: Which ℓ are transverse (& embedded) compositions?

(Is there $L \subset M$ s.t. $L \pitchfork \mu^{-1}(0)$ and $\pi(L \cap \mu^{-1}(0)) = \ell$?)