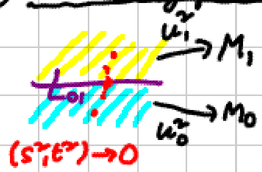
L20 – bubbling at seams

Note Title 5/5/2008

Ⅲ bubbling point on a seam



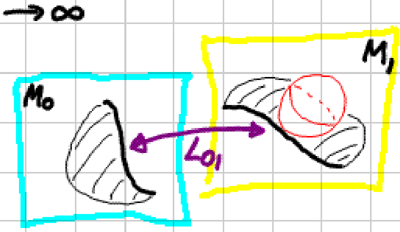
$$|\partial_s u_0^\nu(s_1^\nu, -|t^\nu|)|^2 + |\partial_t u_1^\nu(s_1^\nu, |t^\nu|)|^2 = (R^\nu)^2 \to \infty$$

fold →

$$|\partial_s w^\nu(s_1^\nu, |t^\nu|)| = R^\nu \to \infty$$

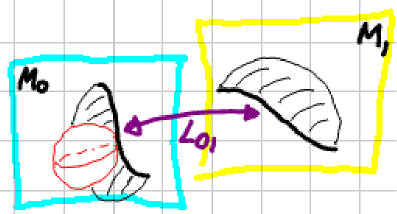
- subsequence with $t^\nu R^\nu \to \infty$ (and $|\partial_s u_1^\nu| \to \infty$)

  → bubble is a J_1-hol. sphere $v : S^2 \to M_1$



- subsequence with $t^\nu R^\nu \to -\infty$ (and $|\partial_s u_0^\nu| \to \infty$)

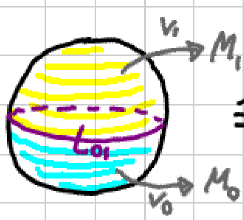
  → bubble is a J_0-hol. sphere $v : S^2 \to M_0$



- subsequence with $t^\nu R^\nu \to T$

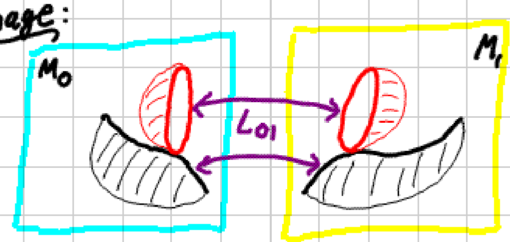
  → bubble is a $(-J_0, J_1)$-hol. disc $\underset{(v_0, v_1)}{w} : \mathbb{D} \to M_0^- \times M_1$, $\bar\partial_{(-J_0, J_1)} w = 0$, $w|_{\partial D} \in L_{01}$

or, equivalently, quilted sphere

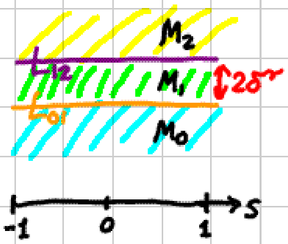


$\hat{=}$

image:



bubbling in shrinking strip



$u_2^\nu : [-1,1]\times[0,1) \to M_2$ $\quad \bar\partial_{J_2} u_2^\nu = 0$ $\quad (u_1^\nu(s,\delta^\nu), u_2^\nu(s,0)) \in L_{12}$

$u_1^\nu : [-1,1]\times[-\delta^\nu,\delta^\nu] \to M_1$ $\quad \bar\partial_{J_1} u_1^\nu = 0$ $\quad (u_0^\nu(s,1), u_1^\nu(s,-\delta^\nu)) \in L_{01}$

$u_0^\nu : [-1,1]\times(-1,0] \to M_0$ $\quad \bar\partial_{J_0} u_0^\nu = 0$

$$\sup_\nu \sum_{i=0}^{2} \int |\partial_s u_i^\nu|^2 < \infty$$

For simplicity consider the case of a bubbling sequence $(s^\nu,t^\nu)=(0,0)$ for u_1^ν

$$|du_1^\nu(0,0)| = R^\nu \to \infty \quad , \|du_i^\nu\|_{L^\infty} \le 2R^\nu \quad \text{for } i=0,1,2$$

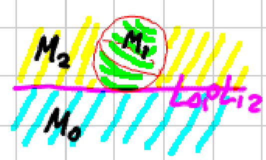
<u>Rescaling</u>: $v_1^\nu : [-R^\nu,R^\nu]\times[-\delta^\nu R^\nu, \delta^\nu R^\nu] \to M_1$, $(\sigma,\tau) \mapsto u_1^\nu(R^\nu\sigma, R^\nu\tau)$

$v_0^\nu : [-R^\nu,R^\nu]\times(-R^\nu,0] \to M_0$

$v_2^\nu : [-R^\nu,R^\nu]\times[0,R^\nu) \to M_2$

$$\|dv_i^\nu\|_{L^\infty} \le 2 \quad , \quad |dv_1^\nu(0)| = 1$$

<u>bubble in domain</u>



- $\delta^\nu R^\nu \to \infty$: limit $v_1^\infty : \mathbb{R}^2 \to M_1$

  bubble $v : S^2 \to M_1$

- $\delta^\nu R^\nu \to 0$: limit $v_0^\infty : \mathbb{R}\times(-\infty,0] \to M_0$

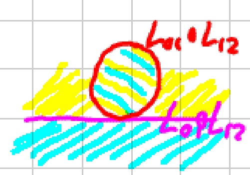
  $v_1^\infty : \mathbb{R} \to M_1$

  $v_2^\infty : \mathbb{R}\times[0,\infty) \to M_2$

$$(v_0^\infty(s,0), v_1^\infty(s)) \in L_{01} \; , \; (v_1^\infty(s), v_2^\infty(s,0)) \in L_{12} \iff (v_0^\infty(s,0), v_2^\infty(s,0)) \in L_{01}\circ L_{12}$$

$\xrightarrow{\text{fold}}$ $\quad w^\infty = (v_0^\infty(-\cdot,\cdot), v_2^\infty) : \mathbb{R}\times[0,\infty) \to M_0^- \times M_2$

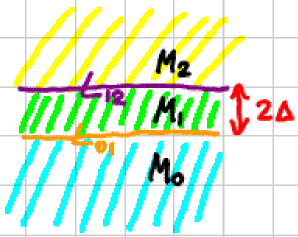
$$\begin{cases} \bar\partial_{(-J_0,J_2)} w = 0 \\ w|_{s=0} \in L_{01}\circ L_{12} \end{cases} \quad \rightsquigarrow \text{bubble } w : D^2 \to M_0^- \times M_2$$



- $\delta^2 R^2 \longrightarrow \Delta > 0$: limit $v_0^\infty : \mathbb{R}\times(-\infty,0] \to M_0$ $\quad (v_0^\infty(s,0), v_1^\infty(s,-\Delta)) \in L_{01}$

$v_1^\infty : \mathbb{R}\times[-\Delta,\Delta] \to M_1$

$v_2^\infty : \mathbb{R}\times[0,\infty) \to M_2$ $\quad (v_1^\infty(s,\Delta), v_2^\infty(s,0)) \in L_{12}$



finite energy: $\sum_{i=0}^{2} \int |dv_i^\infty|^2 < \infty$

nonconstant: $|dv_1^\infty(0,0)| = 1$ $\quad \left(\text{in general } |dv_i^\infty(0,\tau)| = 1 \text{ for some } i,\tau\right)$

$\boxed{\underline{\text{Conjecture}}: \lim_{s^2+t^2\to\infty} v_i^\infty(s,t) = p_i \in M_i \quad \text{for } i = 0,1,2}$

__Remark__: This is true in the trivial cases

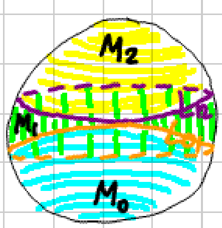
$M_1 = pt$ $\quad$ by removal of singularity for discs v_0^∞ and v_2^∞ since $L_{01} \subset M_0$, $L_{12} \subset M_2$ are simple Lagrangian boundary conditions

$M_0 = M_2 = pt$: by finite energy $v_1^\infty(s,t) \xrightarrow[s\to\pm\infty]{} p_1^\pm \in M_1$ uniformly in $t \in [-\Delta,\Delta]$

embeddedness of $L_{01} \circ L_{12} \subset pt \times pt$ means $L_{01} \cap L_{12} \subset M_1$ is a single point so, since $p_1^+, p_1^- \in L_{01} \cap L_{12}$, they are automatically equal.

If Conj. is true then $(p_0,p_1) \in L_{01}$, $(p_1,p_2) \in L_{12}$ $(\Rightarrow (p_0,p_2) \in L_{01} \circ L_{12})$

and $(v_0^\infty, v_1^\infty, v_2^\infty)$ can be compactified to a <u>"figure 8 bubble"</u>



$v_0 : D_0 \to M_0$ $\quad (v_0,v_1)|_{\partial D_0} \in L_{01}$

$v_1 : S^2 \setminus (\mathring{D}_0 \cup \mathring{D}_2) \to M_1$

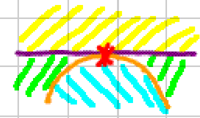
$v_2 : D_2 \to M_2$ $\quad (v_1,v_2)|_{\partial D_2} \in L_{12}$

and hence $(v_0,v_2)|_{D_0 \cap D_2} \in L_{01} \circ L_{12}$

$D_0, D_2 \subset S^2$ closed discs with $D_0 \cap D_2 = pt$

We call this "singularly quilted sphere" figure eight because, with smaller discs, the seams touch like a figure eight 8.

The conjecture is a removable singularity statement for a quilt with tangentially intersecting seams at the singularity *.



Complete list of possible bubbles:

$S^2 \to M_0$ $\quad$ $D^2 \to M_0^- \times M_1$ $\quad$ $S^2 \to M_1$ $\quad$ $D^2 \to M_1^- \times M_2$ $\quad$ $S^2 \to M_2$

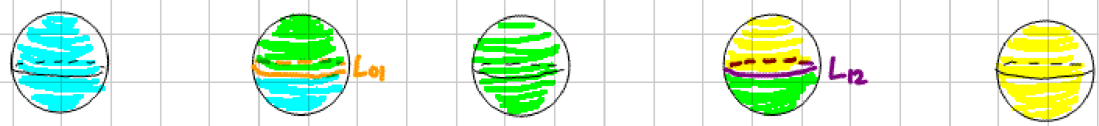


figure 8 $\quad$ $D^2 \to M_0^- \times M_2$

Note: Can view all others as degeneration of figure 8.

Energy Quantization in case of figure 8 bubbling:

$$\liminf \sum_{i=1}^{3} \int_{B_{\varepsilon^\nu R^\nu}} |du_i^\nu|^2 \geq \liminf \sum_{i=0}^{2} \int |dv_i^\infty|^2 \geq \hbar_\Delta > 0$$

follows from mean value inequality (for bubbling point $(0,0)$ in u_1^ν)

If $\int |dv_1^\infty|^2 < \hbar$ then $\quad 1 = |dv_1^\infty(0,0)|^2 \leq C + C\Delta^{-2} \int_{B_\Delta(0,0)} |dv_1^\infty|^2$

$$\Rightarrow \int |dv_1^\infty|^2 \geq \Delta^2 (C^{-1} - 1),$$

otherwise $\int |dv_1^\infty|^2 \geq \hbar$; so in any case $\int |dv_1^\infty|^2 \geq \hbar_\Delta := \min(\hbar, \Delta^2(C^{-1}-1))$.

<u>Conj.</u>: This holds with $\hbar > 0$ independent of $\Delta > 0$.

possible reason: $\sum_{i=0}^{2} \int |dv_i^\infty|^2 = \sum_{i=0}^{2} \int v_i^* \omega_i \in \langle [\omega_0, \omega_1, \omega_2], \underset{\text{homotopy class}}{\text{figure 8}} \rangle$

We do obtain compactness for strip shrinking under the assumption of monotonicity and minimal index from

<u>Lemma</u>: $\exists \hbar > 0$ s.t.t.f.h.

$(u_0^\nu, u_1^\nu, u_2^\nu)_{\nu \in \mathbb{N}}$ any sequence as above with $\delta^\nu \to 0$

If $\liminf_{\nu \to \infty} \sum_i \| du_i^\nu \|_{L^\infty(B_\varepsilon(0))} = \infty \quad \forall \varepsilon > 0$

then $\exists$ subsequence $(\nu_j)_{j \in \mathbb{N}}$ and $\varepsilon_j \to 0$ s.t. $\liminf_{j \to \infty} \sum_i \int_{B_{\varepsilon_j}(0)} |du_i^{\nu_j}|^2 \geq \hbar$.

(these domains need to be somewhat enlarged, depending on δ^ν, as a result of folding)

<u>Sketch of proof by contradiction</u>

- Find a (diagonal) sequence $(u_0^\nu, u_1^\nu, u_2^\nu)$ with $\delta^\nu \to 0$, $\sum \| du_i^\nu \|_{L^\infty} = R^\nu \to \infty$

  but $\sum \int |du_i^\nu|^2 \to 0$.

- Deduce $\delta^\nu R^\nu \to 0$ from width-dependent energy quantization.

- Show that the limit is $w_{02} = (u_0^\infty, u_2^\infty) : \mathbb{R} \times [0, \infty) \to M_0^-, M_2$

$$\begin{cases} \bar\partial_{(-J_0, J_2)} w_{02} = 0 \\ w_{02}|_{t=0} \in L_{01} \circ L_{12} \end{cases} \qquad \int |dw_{02}|^2 = 0 \quad \text{but} \quad \underbrace{|dw_{02}(0)| > 0}_{\text{need to prove } \mathcal{C}^1\text{-convergence}} \quad \lightning$$

QED

More open questions

Is
$$\left\{ \begin{array}{lll} v_0^\infty : \mathbb{R}\times(-\infty,0] \to M_0 & (v_0^\infty(s,0), v_1^\infty(s,-\Delta)) \in L_{01} & \\ v_1^\infty : \mathbb{R}\times[-\Delta,\Delta] \to M_1 & & \bar\partial_{J_i} v_i^\infty = 0 \\ v_2^\infty : \mathbb{R}\times[0,\infty) \to M_2 & (v_1^\infty(s,\Delta), v_2^\infty(s,0)) \in L_{12} & \end{array} \right\}$$

a Fredholm problem?

If so, what does the moduli space of figure 8 bubbles look like?

(dimension, transversality, ...)

Is there a gluing map?

$$\left\{ \begin{array}{l} \text{moduli space of} \\ \text{figure 8 bubbles} \end{array} \right\} \times \left\{ \begin{array}{l} \text{holomorphic quilts with marked} \\ \text{point on } L_{01}\circ L_{12} \text{ seam} \end{array} \right\} \to \left\{ \begin{array}{l} \text{holomorphic quilts} \\ \text{with } L_{01}\text{- and } L_{12}\text{-seam} \\ \text{at distance } \Delta \end{array} \right\}$$

What algebraic structure (à la FOOO obstructions / A_∞-algebra) results from that? E.g. the canonical map

$$I : \underset{\circlearrowleft \delta_\Delta}{CF(\dots L_{01}, L_{12} \dots)} \longrightarrow \underset{\circlearrowleft \delta_0}{CF(\dots L_{01}\circ L_{12} \dots)}$$

should intertwine the differentials δ_Δ and δ_0 up to a count of figure 8 bubbles.

... now build a general symplectic A_∞-2-category, allowing non-monotone symplectic and Lagrangian manifolds!