L2 - Examples of Lagrangian correspondences and composition

Note Title 2/7/2008

<u>Def$^{\underline{n}}$</u>: $L_{01}\circ L_{12}$ is

- <u>transverse</u> if $(L_{01}\times L_{12}) \pitchfork (M_0\times\Delta_{M_1}\times M_2)$ i.e. $\forall (x_0,x_1,x_1,x_2)\in(\dots)\cap(\dots)$

  i.e. $\left(T_{(x_0,x_1)}L_{01}\times T_{(x_1,x_2)}L_{12}\right) + \left(T_{x_0}M_0\times T_{(x_1,x_1)}\Delta_{M_1}\times T_{x_2}M_2\right) = T_{(x_0,x_1,x_1,x_2)} M_0\times M_1\times M_1\times M_2$

- <u>embedded</u> if $\forall (x_0,x_2)\in L_{01}\circ L_{12}\ \exists! \, x_1\in M_1 : \begin{matrix}(x_0,x_1)\in L_{01}\\ (x_1,x_2)\in L_{12}\end{matrix}$

<u>Lemma</u>: (i) transverse $\Rightarrow$ $L_{01}\times_{M_1} L_{12}\subset M_0\times M_1\times M_1\times M_2$ is a submanifold

and $\pi_{02}: L_{01}\times_{M_1}L_{12}\longrightarrow M_0\times M_2$ is an immersion

(ii) transverse & embedded $\Rightarrow$ $L_{01}\circ L_{12}\subset M_0^-\times M_2$ is a Lagr. correspondence

(and π_{02} an embedding)

<u>Proof</u>: (i) implicit function theorem for $L_{01}\times L_{12}\longrightarrow M_1\times M_1 \xrightarrow{\text{local coordinates}} \mathbb{R}^{2n_1}$

$(x_0,x_1,x_1',x_2)\mapsto (x_1,x_1') \mapsto x_1-x_1'$

- $TL_{01}\times TL_{12}\longrightarrow TM_1$ surjective

  $(v_0,v_1,v_1',v_2)\longmapsto v_1-v_1'$

$\Leftrightarrow$ $(TL_{01}\times TL_{12})\times \underbrace{TM_0\ \ T\Delta_{M_1}\ \ TM_2} \longrightarrow TM_0\times TM_1\times TM_1\times TM_2$ surjective

$(v_0,v_1,v_1',v_2;\, w_0, w_1,w_1, w_2)\longmapsto (w_0, w_1+v_1, w_1+v_1', w_2)$

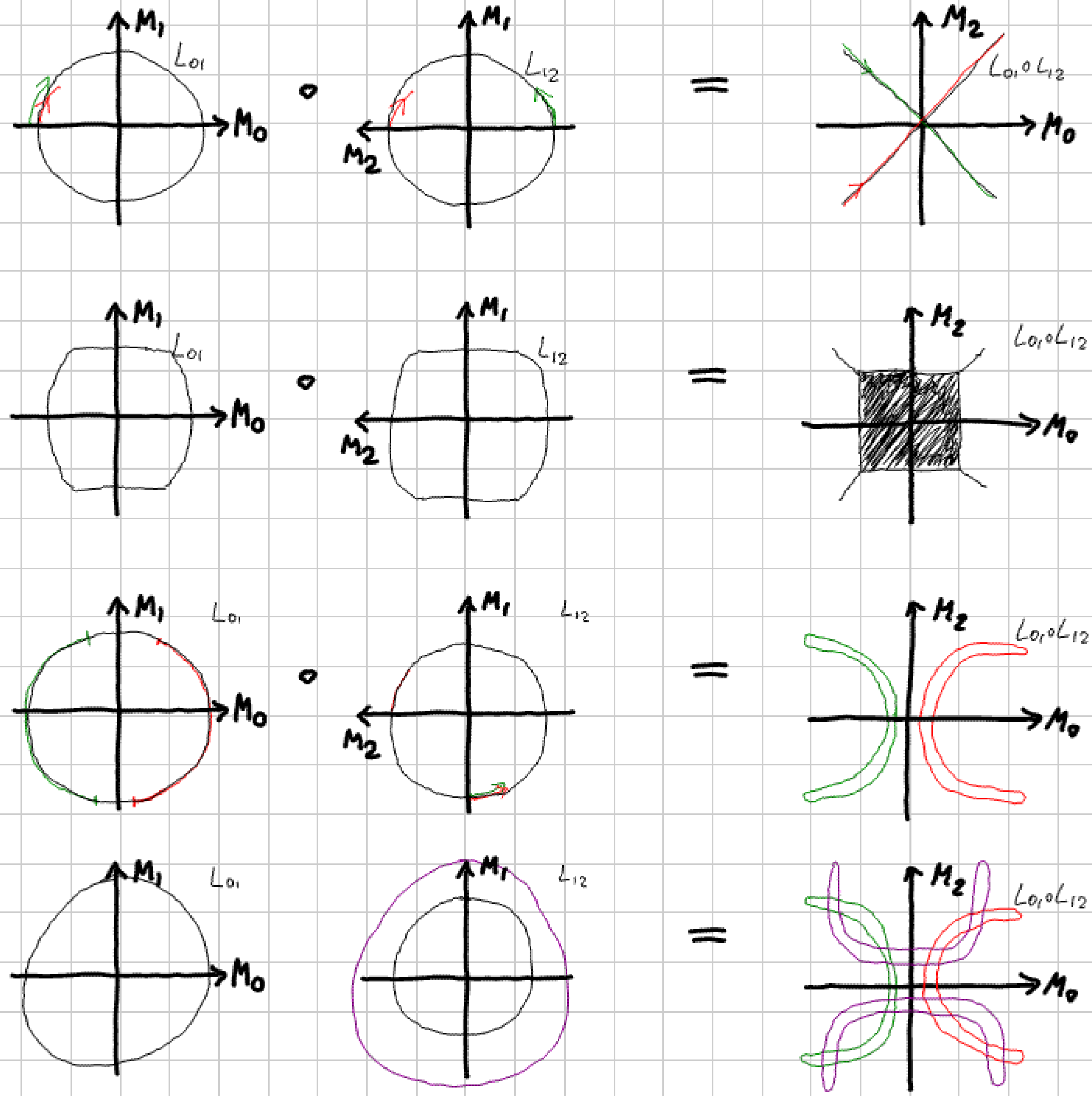
$\Updownarrow$ transversality

- immersion:

$\ker d\pi_{02} \underset{\text{as in linear Lemma}}{\cong} T(M_0\times M_1\times M_1\times M_2) \Big/ \left((TL_{01}\times TL_{12})+(TM_0\times T\Delta_1\times TM_2)\right) = \{0\}$

(ii) $\pi_{02}: L_{01}\times_{M_1}L_{12}\longrightarrow M_0^-\times M_2$ is injective immersion ■

Examples $M_i = \mathbb{R}$ (not symplectic)



Question: Can transversality be achieved by shifting L_{01}, L_{12} with Hamiltonian diffeomorphisms on M_0, M_2?

No?! $\rightsquigarrow$ no easy definition of $L_{01} \circ L_{12}$ as immersed $\dfrac{\text{Lagrangian}}{\text{Ham. diffeom.}}$

Lagrangian correspondences arising from fibered coisotropics

[McDuff-Salamon §5.3]

(M,ω) symplectic mfd. , $\dim M = 2n$

- $C \subset M$ coisotropic submanifold

$$\left[\forall x \in C \quad T_xC^{\omega} := \{ v \in T_xM \mid \omega(v, T_xC) \equiv 0 \} \subset T_xC \right]$$

$$\left(\begin{array}{l} \Rightarrow \dim C = n+k \;\; ; k \geq 0 \quad , \quad \dim T_xC^{\omega} = n-k \\ \Rightarrow \dim {TC}/{TC^{\omega}} = 2k \quad \text{and} \quad \left({TC}/{TC^{\omega}}, \omega \right) \text{ is symplectic} \end{array} \right)$$

__Lemma:__ The "null foliation" $(TC)^{\omega} \subset TC$ is integrable,

i.e. locally $(TC)^{\omega} = TN$; $N \subset C$ submanifold ("isotropic leaf").

__Proof:__ $X, Y \in \Gamma(TC)$ vector fields

need to check: $X, Y \in TC^{\omega}$ in nbhd of $p \in C$ $\Rightarrow [X,Y](p) \in T_pC^{\omega}$

$\forall Z \in \Gamma(TC) \qquad \omega([X,Y], Z)$

$$= \omega([X,Y],Z) + \omega(\underset{TC}{[Y,Z]}, \underset{TC^{\omega}}{X}) + \omega(\underset{TC}{[Z,X]}, \underset{TC^{\omega}}{Y}) + \mathcal{L}_X \omega(\underset{TC}{Z}, \underset{TC^{\omega}}{Y}) + \mathcal{L}_Y \omega(\underset{TC^{\omega}}{X}, \underset{TC}{Z}) + \mathcal{L}_Z \omega(\underset{\substack{TC^{\omega} \\ TC}}{Y}, \underset{\substack{TC \\ TC^{\omega}}}{X})$$

$$= d\omega\,(X,Y,Z) = 0$$

$$\Rightarrow [X,Y] \in TC^{\omega} \quad \blacksquare$$

- Suppose $(TC)^{\omega}$ is <u>regular</u> : all leafs are compact submanifolds.

  Then $B := C/_{(p\sim q \text{ if } p,q \text{ on same leaf})}$ is a symplectic manifold

  with ω_B induced by ω on $TB = TC/_{TC^{\omega}}$.

  So we have a fibration $\pi : C \twoheadrightarrow B$ with $\pi^*\omega_B = \omega|_C$.

  We also have the embedding $\iota : C \hookrightarrow M$ and

  $(\iota \times \pi)(C) \subset M^- \times B$ is a Lagrangian correspondence.

<u>CHECK</u>: $\iota \times \pi$ embeds, $\dim C = n + k = \frac{1}{2}(\dim M + \dim B)$

$(\iota \times \pi)^*(-\omega \oplus \omega_B) = -\omega|_C + \pi^*\omega_B = 0$ //

Lagrangian correspondences arising from moment maps

[McD-Sal. 5.2, 5.3]

(M,ω) symplectic mfd

G Lie group ; $\mathfrak{g} := T_{\mathbb{1}}G$ Lie algebra , fix G-invariant inner product $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$

$\psi : G \to \mathrm{Symp}(M)$ Hamiltonian group action with moment map $\mu : M \to \mathfrak{g}$

- $\psi(\mathbb{1}) = \mathrm{Id}_M$, $\psi(gh) = \psi(g)\circ\psi(h)$
- $d_{\mathbb{1}}\psi : \mathfrak{g} \longrightarrow \Gamma(TM)$, $\xi \longmapsto X_\xi(p) = \frac{d}{dt}\Big|_{t=0} \psi(\exp(t\xi))p$ maps to Hamiltonian vector fields $\omega(X_\xi,\cdot)$ exact 1-form
- $\omega(X_\xi,\cdot) = d\big(\underbrace{\langle\mu,\xi\rangle_{\mathfrak{g}}}_{H_\xi : M \to \mathbb{R}\ ;\ p \mapsto \langle\mu(p),\xi\rangle_{\mathfrak{g}}}\big) \quad \forall \xi\in\mathfrak{g}$
- ($\xi \longmapsto H_\xi$ is a Lie algebra homomorphism) $\Updownarrow$ μ is equivariant : $\mu(\psi(g)x) = g\,\mu(x)\,g^{-1}$

Ex.: Any Hamiltonian S^1-action has a moment map.

$$\left[\begin{array}{l} \frac{d}{dt}\Big|_{t=0} \psi_{e^{i\lambda t}}(p) = \lambda X_H \quad (\lambda\in\mathbb{R}) \quad ; \quad \omega(X_H,\cdot) = d\lambda H \\ \rightsquigarrow \mu : M \to \mathfrak{g} \cong \mathbb{R} \quad \text{given by} \quad \mu(p) = H(p) \end{array}\right]$$

__Note__: $\mu^{-1}(0) \subset M$ is G-invariant ($\psi(g) : \mu^{-1}(0) \to \mu^{-1}(0) \quad \forall g$)

* Suppose G acts freely on $\mu^{-1}(0)$ $[\ \psi_g(p) = p \in \mu^{-1}(0) \Rightarrow g = \mathbb{1}\ ]$
and G is compact, connected

__Claim:__

$\mu^{-1}(0) \subset M$ is a coisotropic submanifold

orbits $\{\psi_g(p) \mid g \in G\}$ are the isotropic leaves

$M/\!\!/_G := \mu^{-1}(0)/_G$ is a symplectic manifold "symplectic quotient"

$\iota \times \pi : \mu^{-1}(0) \longrightarrow M^{-} \times M/\!\!/_G$ embeds to a Lagrangian correspondence