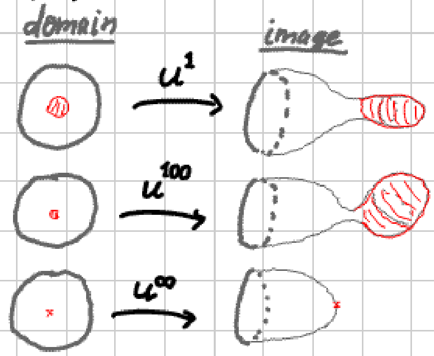
L19 – Compactness – bubbling

Note Title 4/24/2008

Compactness for monotone, minimal index moduli spaces

- $\mathcal{C}^\infty_{loc}$ – convergence on complement of bubbling points
- energy loss at bubbling points
- removal of bubbling singularities



skipping:
- index identities
- exponential decay on quilted ends

corrected

model case: $L_0 \pitchfork L_1 \subset M$ compact Lagrangian, J ω-comp. a.c.s.

$\exists\, C, \delta, \hbar > 0 : \forall T > 2,\ u : [-T,T]\times[0,1] \longrightarrow M$

$$\begin{cases} \partial_s u + J\partial_t u = 0 \\ u|_{t=0} \in L_0,\ u|_{t=1} \in L_1 \end{cases} \qquad \int_{[-T,T]\times[0,1]} |\partial_s u|^2 < \hbar$$

$$\Rightarrow \forall S \le T-1 \quad \sup_{[-S,S]\times[0,1]} d(u(s,t), L_0 \cap L_1) + \|\nabla u\|_{L^\infty([-S,S]\times[0,1])}$$

$$\le C e^{-\delta(T-S)} \Bigg(\int_{\boxed{\left(\substack{[-T,-T+1] \\ \cup [T-1,T]}\right)\times[0,1]}} |\partial_s u|^2 \Bigg)^{1/2}$$

Proof: $f(s) := \int_0^1 |\partial_s u(s,t)|^2\, dt$ satisfies $f'' \ge \delta^2 f$

<u>Cor.:</u> $\mathcal{M}^0$ compact in $W^{k,p}(\sqcup S_k)$-topology

$\mathcal{M}^1$ compact up to "one Floer trajectory breaking off at one end"

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geometric description of bubbling

(I) <u>interior</u> $u^\nu : B_1 = \{s^2+t^2<1\} \longrightarrow M$, $\partial_s u^\nu + J \partial_t u^\nu = 0$

$(s^\nu, t^\nu) \to 0$, $|du^\nu(s^\nu, t^\nu)| = R^\nu \underset{\nu\to\infty}{\longrightarrow} \infty$, $\sup_\nu \int_{B_1} |\partial_s u^\nu|^2 < \infty$

<u>Hofer trick</u>: can assume $\|du^\nu\|_{L^\infty(B_{\varepsilon^\nu}(s^\nu,t^\nu))} \leq 2R^\nu$; $\varepsilon^\nu R^\nu \to \infty$, $\varepsilon^\nu \to 0$

"<u>Little Lemma</u>" [Hofer–Zehnder Ch.6 Lemma 5]

X complete metric space , $f : X \to [0,\infty)$ continuous

$\forall x_0 \in X, \varepsilon_0 > 0 \ \exists x \in B_{2\varepsilon_0}(x_0)$, $\varepsilon \in (0, \varepsilon_0]$: $\sup_{y \in B_\varepsilon(x)} f(y) \leq 2 f(x)$

$\varepsilon f(x) \geq \varepsilon_0 f(x_0)$

(Apply this to $f = |du^\nu|$, $x_0 = (s_\nu, t_\nu)$, $\varepsilon_0 = R_\nu^{-1/2}$ to find $x = (s'_\nu, t'_\nu)$, $0 < \varepsilon_\nu < R_\nu^{-1/2}$, $R'_\nu = f(x) = |du^\nu(s'_\nu, t'_\nu)|$ with $\left\{ \begin{array}{l} (s'_\nu, t'_\nu) \to 0 \\ \varepsilon_\nu \to 0 \\ \varepsilon_\nu R'_\nu \geq R_\nu^{-1/2} R_\nu \to \infty \end{array} \right\}$)

<u>Rescaling</u>: $v^\nu : B_{\varepsilon^\nu R^\nu} \longrightarrow M$, $(\sigma, \tau) \longmapsto u^\nu(s^\nu + \frac{\sigma}{R^\nu}, t^\nu + \frac{\tau}{R^\nu})$

satisfies $\partial_\sigma v^\nu + J(v^\nu) \partial_\tau v^\nu = 0$, $\sup_\nu \int_{B_{\varepsilon^\nu R^\nu}} |\partial_s v^\nu|^2 = \sup_\nu \int_{B_{\varepsilon^\nu}} |\partial_s u^\nu|^2 < \infty$

$\|\partial_s v^\nu\|_{L^\infty} \leq 2$, $|\partial_s v^\nu(0)| = 1$

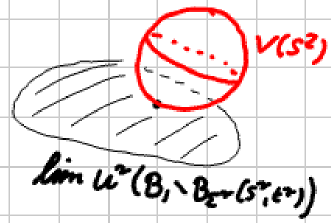
<u>Compactness</u>: $\exists$ subsequence $v^{\nu_i} \xrightarrow{\mathcal{C}^\infty_{loc}} v^\infty \in \mathcal{C}^\infty(\mathbb{R}^2 \cong S^2 \setminus pt, M)$

$\partial_s v^\infty + J(v^\infty)\partial_t v^\infty = 0$, $\int |\partial_s v^\infty|^2 < \infty$, $|\partial_s v^\infty(0)| = 1$

<u>Removal of singularity</u> $\Rightarrow$ "the bubble" is a J-hol. sphere

$v : S^2 \to M$, $\bar\partial_J v = 0$, nonconstant

$\Rightarrow$ $\frac{1}{2}\int_{S^2} |dv|^2 = \int_{S^2} v^*\omega \geq \hbar > 0$



$\hbar > 0$ positive generator of $\langle [\omega], \pi_2(M) \rangle \subset \mathbb{R}$

or $\hbar > 0$ by Gromov compactness

(II) boundary $u^\nu : B_1 \cap \{t \geq 0\} \longrightarrow M$, $u^\nu|_{t=0} \in L$

$\rightsquigarrow$ $v^\nu : B_{\varepsilon^\nu R^\nu} \cap \{\tau \geq -t^\nu R^\nu\} \longrightarrow M$, $v^\nu|_{\tau = -t^\nu R^\nu} \in L$

- subsequence with $t^\nu R^\nu \to \infty$ $\rightharpoonup$ limit $v^\infty : \mathbb{R}^2 \to M$

  $\to$ bubble is a J-hol. sphere $v : S^2 \to M$

- subsequence with $t^\nu R^\nu \to T < \infty$ $\rightharpoonup$ limit $v^\infty : \mathbb{R} \times [-T, \infty) \to M$, $v^\infty|_{\tau = -T} \in L$

  $\to$ bubble is a J-hol. disc $v : \underset{\{|z| \leq 1\} \subset \mathbb{C}}{\mathbb{D}} \to M$, $\begin{cases} \bar\partial_J v = 0 \\ v|_{\partial \mathbb{D}} \in L \end{cases}$

  $\Rightarrow$ $\frac{1}{2}\int |dv|^2 = \underset{\langle [\omega], \pi_2(M,L) \rangle}{\int v^*\omega} \geq \hbar > 0$