L18 - Compactness - up to energy concentration

Note Title 4/14/2008

LAST TIME: $u : \underset{\mathbb{R}^2}{\overset{\cup}{B_\delta}} \to M$

energy density $e = e(u) : B_\delta \to [0,\infty)$

$(s,t) \mapsto |\partial_s u - Y(u)\partial_s|^2$

Lemma (i): $\partial_s u + J(u)\partial_t u = Y(u)(\partial_s) + J(u)Y(u)(\partial_t)$

$\Rightarrow \quad \Delta e \geq -a e^2 - A_0 \qquad ; \; a, A_0 > 0$ constants

NEXT: $u : B_\delta \cap \mathbb{H} \to M$

Lemma (i) still holds

Lemma (ii): $u|_{t=0} \in L$ Lagrangian and $\partial_s u + J(u)\partial_t u = Y(u)(\partial_s) + J(u)Y(u)(\partial_t)$

$\Rightarrow \; -\partial_t e|_{t=0} \geq -b e^{3/2} - B_0 \qquad ; \; b, B_0 > 0$ constants

Proof: Note that $Y(u)(\partial_s)|_{t=0} = 0$ since $*K_k|_{\partial S_k} = 0$

$$\partial_t e|_{t=0} = \partial_t \, \omega\big(\partial_s u - Y(u)\partial_s, J(u)(\partial_s u - Y(u)\partial_s)\big)$$

pick any connection ∇

$$= \omega(\underbrace{\nabla_t \partial_s u}_{= \nabla_s \partial_t u = -\nabla_s J \partial_s u}, J(u)\partial_s u) + \omega(\partial_s u, J(u)\nabla_t \partial_s u) + \text{lower order}$$

e.g. $\nabla_{\partial_t u}\omega(\partial_s u, J\partial_s u)$

$$= \underbrace{2\omega(\underset{TL}{\partial_s u}, \nabla_s \underset{TL}{\partial_s u})}_{=0 \text{ if } L \text{ linear}} - \langle(\nabla_{\partial_s u} J)\partial_s u, \partial_s u\rangle + (\nabla_{\partial_t u}\omega)(\partial_s u, \partial_s u) + \ldots$$

(pick ∇ s.t. L geodesic, but then $\nabla J \neq 0$ or $\nabla \omega \neq 0$)

$$\leq C|du|^3 + \text{lower order} \quad \leq b e^{3/2} + B_0 \qquad \blacksquare$$

<u>Corollary</u>: $W^{1,\infty}$-bounds on balls of small energy

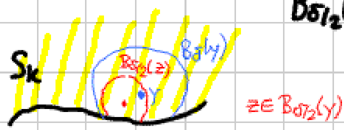
$\exists\, \mu = \mu(a,b) = \mu(\underline{S}, \underline{M}, \underline{L}, \underline{J}, \underline{Y}) > 0$ s.t.t.f.h.

$(\underline{u}^\nu : \underline{S} \to \underline{M})_{\nu\in\mathbb{N}}$ holomorphic quilts

- $y \in S_k$, $B_\delta(y) \subset S_k$ intersects no seams (but possibly a boundary component)

$$\int_{B_\delta(y)\subset S_k} |du_k^\nu - Y_k(u_k^\nu)|^2 \le \mu \quad \forall \nu \quad \Rightarrow \quad \sup_{B_{\delta/2}(y)} |du_k^\nu| \le C \quad \forall \nu$$

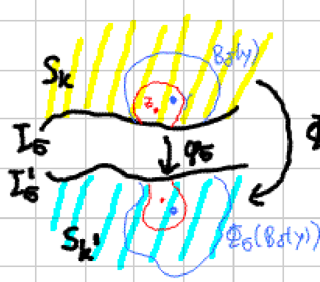
$$\| \qquad \int_{B_\delta(y)\cap H} e(u_k^\nu)$$



- $y \in S_k$, $B_\delta(y) \subset S_k$ intersects one seam $I_\sigma \subset \partial S_k$

$$\int_{B_\delta(y)\subset S_k} |du_k^\nu - Y_k(u_k^\nu)|^2 + \int_{\Phi_\sigma(B_\delta(y))} |du_{k'}^\nu - Y_{k'}(u_{k'}^\nu)|^2 \le \mu \quad \forall \nu \Rightarrow \sup_{B_{\delta/2}(y)} |du_k^\nu| + \sup_{\Phi_\sigma(B_{\delta/2}(y))} |du_{k'}^\nu| \le C \quad \forall \nu$$

$$\| \qquad \int_{B_\delta(y)\cap H} e(w_\sigma^\nu)$$



$$\Phi_\sigma(s,t) = (\varphi_\sigma(s), \varphi_\sigma'(s)\cdot t)$$

<u>Proof</u>: Apply mean value inequalities to $e(u_k^\nu) = |du_k^\nu - Y_k(u_k^\nu)|^2$

resp. $e(w_\sigma^\nu)(s,t) = |du_k^\nu - Y_k(u_k^\nu)|^2(s,-t) + |du_{k'}^\nu - Y_{k'}(u_{k'}^\nu)|^2(\Phi_\sigma(s,t))$

on balls of radius $\delta/2$ centered at $z \in B_{\delta/2}(y)$.

$(\underline{u}^\nu : \underline{S} \to \underline{M})_{\nu\in\mathbb{N}}$ holomorphic quilts of energy E

pick a j_k-compatible metric on each S_k and consider

$e(u_k^\nu)$ as function on S_k , $e(w_\sigma^\nu)$ as function on $S_{k\sigma}$

Claim: There exists a finite "bubbling set" $(z_i)_{i=1..N}$, $N < E/\mu$

and a subsequence $(\underline{u}^{\nu_i})_{i\in\mathbb{N}}$ such that

- for $z \in S_k \setminus \bigcup_\sigma I_\sigma \setminus \{z_1..z_N\}$ $\exists \delta_z > 0 : \int_{B_{\delta_z}(z)} e(u_k^{\nu_i}) \le \mu \quad \forall i$
- for $z \in I_\sigma \overset{\partial S_k}{\cup} \setminus \{z_1 .. z_N\}$ $\exists \delta_z > 0 : \int_{B_{\delta_z}(z)} e(w_\sigma^{\nu_i}) \le \mu \quad \forall i$

Proof: Either the claim holds with $N=0$ and the original sequence or

there is $z_1 \in S_k \setminus \cup I_\sigma$ resp. $z_1 \in I_\sigma$ and a subsequence $(\underline{u}^{\nu_i})_{i\in\mathbb{N}}$ s.t.

$\int_{B_{2^{-i}}(z_1)} e(u_k^{\nu_i})$ resp. $e(w_\sigma^{\nu_i}) > \mu \quad \forall i$ (and hence $E > \mu$)

Iteration:

Either the claim holds with $\{z_1,..,z_N\}$ and this subsequence or

there is another z_{N+1} and a further subsequence s.t.

$\int_{B_{2^{-i}}(z_j)} e(u_{k_j}^{\nu_i})$ resp. $e(w_{\sigma_j}^{\nu_i}) > \mu \quad \forall i \quad \forall j = 1..N+1$

(and hence $E > (N+1)\mu$ since $B_{2^{-i}}(z_{1..N+1})$ disjoint for $i \gg 1$)

Iteration stops since $E < \infty$. ∎

<u>Corollary</u>: There exists a subsequence that converges in

$$\mathcal{C}^\infty\Big(\bigsqcup_k S_k \setminus \bigcup_{i=1}^N z_i\Big) \qquad \big(\text{i.e. in } \mathcal{C}^\ell(K) \text{ for all } \ell\in\mathbb{N},\, K \text{ compact}\big)$$

<u>Proof</u>: For $j\in\mathbb{N}$ we can cover

$$\bigsqcup_k S_k \setminus \bigcup_{i=1}^N B_{2^{-j}}(z_i) \setminus \bigcup_{k,e\,\text{end}} \mathcal{E}_{k,e}(\{|s|>j\}) \qquad \mathcal{E}_{k,e} \subset S_k,\quad \{|s|>j\}\subset \mathbb{R}^\pm\times(S^1 \text{ or } [0,\delta_e])$$

by finitely many balls $B_{\delta_i/4}(z_i)$ $\big(\text{resp. } B_{\delta_i/4}(z_i) \cup \Phi_\sigma(B_{\delta_i/4}(z_i)) \text{ for } z_i\in I_\sigma\big)$

with small energy $(\leq\mu)$ on B_{δ_i}

$\Rightarrow$ $W^{1,\infty}$-bounds on $B_{\delta_i/2}$

$\Rightarrow$ $W^{k,p}$-bounds $\forall k,p$ on $B_{\delta_i/4}$

$\Rightarrow$ $\mathcal{C}^\infty$-convergent subsequence on $\bigcup_i B_{\delta_i/4}$ (fixed j)

Finally, take a diagonal subsequence over $j\in\mathbb{N}$ ■

<u>Note</u>: The above subsequence concentrates energy at every z_i

in the bubbling set: $\forall\delta>0 \quad \exists N_\delta : \quad \forall\nu\geq N_\delta$

$$\int_{B_\delta(z_i)} e(u^\nu_{k_i}) \text{ resp. } e(w^\nu_{\sigma_i}) \;>\mu$$

Hence the limit $\hat{\underline{u}}\in\mathcal{C}^\infty\big(\bigsqcup_k S_k \setminus \bigcup_{i=1}^N z_i\big)$ has energy

$$E(\hat{\underline{u}}) := \int_{\underline{S}\setminus\text{bubbling set}} \tfrac{1}{2}|d\hat{u}_k - Y_k d\hat{u}_k|^2 \;\leq E - \underbrace{N}_{\#\text{bubbling points}}\cdot\mu$$

<u>Thm (Removable Singularities)</u> $\hat{\underline{u}} \in \mathcal{C}^\infty(\bigsqcup_k S_k \setminus \bigcup_{i=1}^N z_i)$

"singular" holomorphic quilt with finite energy $E(\hat{\underline{u}}) < \infty$

$\Rightarrow \lim_{z \to z_i} u_{k_i}(z_i)$ resp. $w_{\sigma_i}(z_i)$ exists $\forall i = 1..N$ and defines a smooth extension to a holomorphic quilt $\tilde{\underline{u}} : \underline{S} \to \underline{M}$ $\in \mathcal{C}^\infty(\sqcup S_k)$ of energy $E(\tilde{\underline{u}}) = E(\hat{\underline{u}})$.

<u>Cor: "compactness" for monotone, minimal index moduli spaces</u>

Assume $\mathcal{M}^k := \mathcal{M}^k(\underline{S}, \underline{K}, \underline{J}, (\underline{x}_e)_{e \in \mathcal{E}^-}, (\underline{y}_e)_{e \in \mathcal{E}^+})$ satisfies

- monotonicity : $\mathcal{E}|_{\mathcal{M}^k} = const = E_k$; $\tilde{\underline{u}} \in \mathcal{M}^\ell$, $E(\tilde{\underline{u}}) < E_k \Rightarrow \ell < k$
- minimal index : $\mathcal{M}^\ell = \emptyset \quad \forall \ell < k_{min}$

Then $\mathcal{M}^{k_{min}}$ is compact w.r.t. $W^{m,p}(\sqcup S_k)$-norm (any m,p)

"up to breaking of trajectories on the cyl./strip-like ends."