L17 – Compactness – boundary and mean value inequalities

Note Title 4/14/2008

$\underline{u}^\nu : \underline{S} \to \underline{M}$ holomorphic quilts with fixed energy $\mathcal{E}(\underline{u}^\nu) = E$

<u>Last time</u>: $B_\delta \hookrightarrow S_k$ interior ball

small "energy" $\|du_k^\nu\|_{L^2(B_\delta)} \le \varepsilon_\delta \;\; \forall \nu \;\Rightarrow\; \exists\, \mathcal{C}^\infty$-convergent subsequence

<u>higher bounds with boundary conditions</u>

Assume $\mathcal{C}^0$-convergence (e.g. from $W^{1,p}$-bounds and $W^{1,p} \underset{\text{compact}}{\hookrightarrow} \mathcal{C}^0$ for $p>2$)

so we can use local coordinates on $\begin{matrix} M_{k_b} & \supset & \mathbb{C}^n \\ \cup & & \cup \\ L_{k_b} & \supset & \mathbb{R}^n \end{matrix}$ resp. $\begin{matrix} M_{k_\sigma} \times M_{k_\sigma'} & \supset & \mathbb{C}^n \\ \cup & & \cup \\ L_\sigma & \supset & \mathbb{R}^n \end{matrix}$

with $J|_L = i$

$$\partial_s u + J(u)\partial_t u = P(u), \quad u(s,t=0) \in \mathbb{R}^n \quad \forall s$$

$u = v + iw$; $v, w : B_\delta \cap \mathbb{H} \to \mathbb{R}^n$

$\Rightarrow \quad \Delta v + i \Delta w = \text{lower order}(v,w)$

$w|_{t=0} = 0$ $\rightsquigarrow$ Dirichlet problem for w

$\partial_t v|_{t=0} = \underbrace{-\partial_s w|_{t=0}}_{=0} + \underbrace{\mathrm{Im}\, P(v+iw)}_{\text{lower order}}$ $\rightsquigarrow$ Neumann problem for v

$\rightarrow$ same bootstrapping as in interior, starting from $W^{1,p>2}$-bounds

$\boxed{\text{mean value inequalities}}$

local energy : $\displaystyle \frac{1}{2}\int_{\substack{B_\delta \\ \text{or } B_\delta\cap\mathbb{H}}} |du - Y(u)|^2 = \int \underbrace{|\partial_s u - Y(u)(\partial_s)|^2}_{= |\partial_t u - Y(u)(\partial_t)|^2} = \int e(u)$

energy density $\quad e = e(u) : \; B_\delta \longrightarrow [0,\infty)$

$\qquad\qquad\qquad\qquad (s,t) \longmapsto |\partial_s u - Y(u)\partial_s|^2$

- $\displaystyle \Delta e = 2\underbrace{|\nabla(\partial_s u - Y(u)\partial_s)|^2}_{①} + 2\underbrace{\langle \partial_s u - Y(u)\partial_s, \Delta(\partial_s u - Y(u)\partial_s)\rangle}_{②}$

  $\geq -a e^2 - A_0 \quad ; \; a, A_0 > 0$ constants $\quad$ (estimate linear term $2e \leq 1 + e^2$)

[[MS, Lemma 4.3.1] $\quad$ using $\partial_s u + J\partial_t u$ = lower order

② $= \langle \partial_s u, \nabla_s\big((\nabla_{\partial_t u}J)\partial_s u - (\nabla_{\partial_s u}J)\partial_t u\big) - R(\partial_s u, \partial_t u)\partial_t u\rangle + l.o.$

$\geq -C(\nabla J, \nabla^2 J)\cdot|du|^4 - C(\nabla J)\cdot\underbrace{|du|^2|\nabla du|}_{\leq \underbrace{\frac{1}{2}\varepsilon^2|\nabla du|^2}_{\text{absorb in ①}} + \frac{1}{2}\varepsilon^{-2}|du|^4} + \; l.o. \qquad \boxed{2ab \leq a^2 + b^2}$

$\geq -a e^2 - A_1 e - A_0$

using $\quad |\partial_s u|, |\partial_t u| \leq \sqrt{e} + \|Y\|_\infty$]

- $\displaystyle \partial_t e\big|_{t=0} = \partial_t\, \omega\big(\partial_s u - Y(u)\partial_s, J(u)(\partial_s u - Y(u)\partial_s)\big)$

  $\displaystyle = \omega(\underbrace{\nabla_t\partial_s u}_{= \nabla_s\partial_t u = -\nabla_s J\partial_s u}, J(u)\partial_s u) + \omega(\partial_s u, J(u)\nabla_t\partial_s u)$ + lower order

  e.g. $\nabla_{\partial_t u}\omega(\partial_s u, J\partial_t u)$

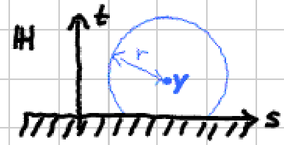
  $\displaystyle = \underbrace{2\omega(\underset{TL}{\partial_s u}, \nabla_s \underset{TL}{\partial_s u})}_{=0 \text{ if } L \text{ linear}} - \langle(\nabla_{\partial_s u}J)\partial_s u, \partial_s u\rangle + (\nabla_{\partial_t u}\omega)(\partial_s u, \partial_t u) + \ldots$

  (e.g. in coordinates, but then $\nabla J \neq 0$ or $\nabla\omega \neq 0$)

  $\leq C|du|^3$ + lower order $\quad \leq b e^{3/2} + B_0$

Thm: $\exists\, C \;,\; \forall a,b\geq 0 \;\exists\, \mu(a,b)>0 \;:\; \forall y\in\mathbb{H}^2,\ r>0$

If $e\in\mathcal{C}^2(B_r(y)\cap\mathbb{H}, [0,\infty))$ satisfies

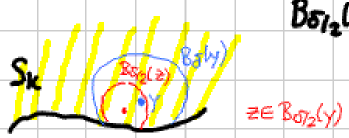


$$\begin{cases} \Delta e \geq -a e^2 - A_1 e - A_0 \\ -\frac{\partial}{\partial t} e\big|_{t=0} \geq -b e^{3/2} - B_1 e - B_0 \end{cases} \quad \text{and} \quad \int_{B_r(y)\cap\mathbb{H}} e \leq \mu(a,b),$$

then $\displaystyle e(y) \leq C\Big(A_0 r^2 + B_0 r + (A_1 + B_1^2 + r^{-2}) \int_{B_r(y)\cap\mathbb{H}} e \Big)$

Corollary: $W^{1,\infty}$-bounds on balls of small energy

- $\displaystyle \underbrace{\int_{B_\delta(y)\subset S_k} |du_k^\nu - Y_k(u_k^\nu)|^2}_{=\int_{B_\delta(y)\cap\mathbb{H}} e(u_k^\nu)} \leq \mu \quad \forall \nu \;\Rightarrow\; \sup_{B_{\delta/2}(y)} |du_k^\nu| \leq C \quad \forall\nu$



- $\displaystyle \underbrace{\int_{B_\delta(y)\subset S_k} |du_k^\nu - Y_k(u_k^\nu)|^2 + \int_{\Phi_\sigma(B_\delta(y))} |du_{k'}^\nu - Y_{k'}(u_{k'}^\nu)|^2}_{=\int_{B_\delta(y)\cap\mathbb{H}} e(w_\sigma^\nu)} \leq \mu \;\Rightarrow\; \sup_{B_{\delta/2}(y)} |du_k^\nu| + \sup_{\Phi_\sigma(B_{\delta/2}(y))} |du_{k'}^\nu| \leq C \quad \forall\nu$

Proof: Apply mean value inequalities to $e(u_k^\nu) = |du_k^\nu - Y_k(u_k^\nu)|^2$

resp. $e(w_\sigma^\nu)(s,t) = |du_k^\nu - Y_k(u_k^\nu)|^2(s,-t) + |du_{k'}^\nu - Y_{k'}(u_{k'}^\nu)|^2(\Phi_\sigma(s,t))$

on balls of radius $\delta/2$ centered at $z\in B_{\delta/2}(y)$.