# L16 – Compactness – local estimates

Note Title 4/14/2008

Fix $\underline{S}$ quilted surface $\rightsquigarrow (S_k, j_k)$ surfaces (with strip-like ends)

$$\varphi_\sigma : I_\sigma \xrightarrow{\sim} I'_\sigma \quad \text{seams}, \qquad I_\sigma \subset \partial S_{k_\sigma},\ I'_\sigma \subset \partial S_{k'_\sigma}$$

$\underline{M}$ symplectic targets

$$\underline{LBS} = \begin{cases} (L_\sigma)_{\sigma \in \mathcal{S}} & \text{Lagrangian correspondences for seams } I_\sigma \simeq I'_\sigma \\ \cup \\ (L_b)_{b \in \mathcal{B}} & \text{Lagrangian submanifolds for boundaries } I_b \subset \partial S_{k_b} \end{cases}$$

$\underline{K}$ Hamiltonian perturbation $\rightsquigarrow Y_k \in \Omega^1(S_k, \Gamma(TM_k))$

$\underline{J}$ almost complex structure $\rightsquigarrow J_k \in \mathcal{C}^\infty(S_k, \mathcal{J}(M_k, \omega_k))$

Consider holomorphic quilts $\left(\underline{u}^\nu = (u_k^\nu) : \underline{S} \to \underline{M}\right)_{\nu \in \mathbb{N}}$ of fixed energy

$\left(\text{e.g. } \underline{u}^\nu \in \mathcal{M}^\ell(\underline{S}, \ldots, (\underline{x}_e)_{e \in \mathcal{E}^-}, (\underline{y}_e)_{e \in \mathcal{E}^+}) \ - \text{ index } \ell, \text{ end data } (\underline{x}_e), (\underline{y}_e) \text{ fixed}\right)$

i.e. $\forall \nu \in \mathbb{N} \quad \overline{\partial}_{\underline{J}, \underline{K}} \underline{u}^\nu = 0 \qquad \mathcal{E}(\underline{u}^\nu) = \frac{1}{2} \| d\underline{u}^\nu - \underline{Y}(\underline{u}) \|^2_{L^2} = E = E(\ell, (\underline{x}_e), (\underline{y}_e))$

i.e. $\forall \nu \in \mathbb{N}\ \forall k$

$$\begin{cases} J_k(u_k^\nu)\left(du_k^\nu - Y_k(u_k^\nu)\right) = \left(du_k^\nu - Y_k(u_k^\nu)\right) \circ j_k \\ u_{k_b}(I_b) \subset L_b \quad \forall b \in \mathcal{B}, \quad \left(u_{k_\sigma} \times (u_{k'_\sigma} \circ \varphi_\sigma)\right)(I_\sigma) \subset L_\sigma \\ \sum_k \frac{1}{2} \int_{S_k} |du_k^\nu - Y_k(u_k^\nu)|^2 = E \end{cases}$$

## equations in local coordinates on $\underline{S}$

**interior of $S_k$** : $B_\delta = \{(s,t) \in \mathbb{R}^2 \mid s^2 + t^2 < \delta^2\}$ $\quad j_k \partial_s = \partial_t$

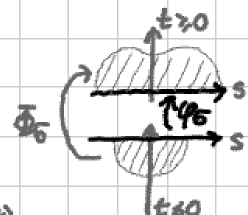
$$\partial_s u_k^\nu + J_k(u_k^\nu)\partial_t u_k^\nu = \underbrace{Y_k(u_k^\nu)(\partial_s) + J_k(u_k^\nu) Y_k(u_k^\nu)(\partial_t)}_{=: P_k(u_k^\nu)}$$

**near boundary** $I_b \subset S_{k=k_b}$ : $B_\delta \cap \mathbb{H} = \{s^2 + t^2 < \delta^2, \; t \geq 0\}$

$$\begin{cases} \partial_s u_k^\nu + J_k(u_k^\nu)\partial_t u_k^\nu = P_k(u_k^\nu) & \forall s,t \\ u_k^\nu(s,0) \in L_b & \forall s \end{cases}$$

**near seam** $I_\sigma \subset S_{k=k_\sigma}$ , $I_\sigma' \subset S_{k'=k_\sigma'}$ :

(pick $(s,t)$-coordinates $t \leq 0$ on $S_k$, $t \geq 0$ on $S_{k'}$)




$\varphi_\sigma : [-\delta, \delta] \to \mathbb{R}$
$\varphi_\sigma(0) = 0$
$\varphi_\sigma' > 0$ due to compatibility with ends

$$w^\nu(s,t) := \big(u_k^\nu(s,-t), \; u_{k'}^\nu(\overbrace{\varphi_\sigma(s), \varphi_\sigma'(s)\cdot t}^{\Phi_\sigma(s,t)})\big) : B_\delta \cap \mathbb{H} \longrightarrow M_k \times M_{k'}$$

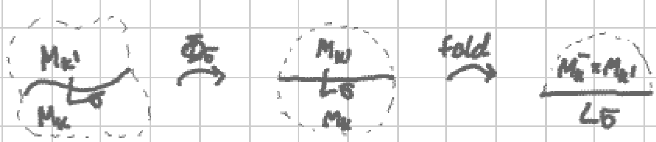
satisfies
$$\begin{cases} w^\nu(s,0) \in L_\sigma & \forall s \\ \partial_s w^\nu + \tilde{J}(w^\nu)\, \partial_t w^\nu = \tilde{P}(w^\nu) & \forall s, t \geq 0 \end{cases}$$

$$\partial_s w^\nu = \begin{pmatrix} \partial_s u_k^\nu \\ \varphi_\sigma' \cdot \partial_s u_{k'}^\nu \end{pmatrix} \qquad \partial_t w^\nu = \begin{pmatrix} -\partial_t u_k^\nu \\ \varphi_\sigma' \cdot \partial_t u_{k'}^\nu \end{pmatrix}$$

$$\tilde{J}(s,t,\underbrace{v,v'}_{M_k \times M_{k'}}) = -J_k(s,t,v) \oplus J_{k'}(\varphi_\sigma(s), \varphi_\sigma'(s)\cdot t, v') \quad \in \mathrm{End}(T_v M_k \times T_{v'} M_{k'})$$

$$\tilde{P}(s,t,\overbrace{v,v'}) = P_k(s,-t,v) + \varphi_\sigma' \cdot P_{k'}(\varphi_\sigma(s), \varphi_\sigma'(s)\cdot t, v') \quad \in T_v(M_k \times M_{k'})$$

<u>Rmk</u>: this uniformizes and folds the seam



energy: $E \geq \frac{1}{2} \int_{B_\delta} |d\tilde{u}_k - Y_k(\tilde{u}_k)|^2$ ; $\sup_{z \in S_k, u \in M_k} |Y_k(z,u)| = C_Y < \infty$

$\Rightarrow \|d\tilde{u}_k\|^2_{L^2(B_\delta)} \leq 2E + C_Y^2 \pi \delta^2$

i.e. $W^{1,2}$-bound on $\tilde{u}$

higher bounds ($W^{k,p} \subset W^{1,2}$) from Laplace equation

$$\partial_s u + J(u)\partial_t u = P(u)$$

$$\Rightarrow (\partial_s^2 + \partial_t^2) u = \underbrace{(\partial_s - J\partial_t)(\partial_s + J\partial_t) u}_{\nabla_{\partial_s u} P(u) - J(u)\nabla_{\partial_t u} P(u)} - \nabla_{\partial_t u} J(u) \cdot \partial_t u + J(u) \nabla_{\partial_t u} J(u) \partial_t u$$

Calderon-Zygmund : $\|\nabla^2 v\|_{L^p} \leq C_{p,n} \|\Delta v\|_{L^p}$ $\quad \forall v \in W^{2,p}(\mathbb{R}^n)$, $1 < p < \infty$

$$\Rightarrow \|u\|_{W^{2,p}} \leq C_p \|du\|_{L^p} + C_J \| |du|^2 \|_{L^p} + C \|u\|_{W^{1,p}}$$

[ $v = h \cdot u : \mathbb{R}^2 \to M \hookrightarrow \mathbb{R}^N$


$\|u\|_{W^{2,p}(B_{\delta/2})} \leq \|\nabla^2(h \cdot u)\|_{L^p} + \|u\|_{W^{1,p}} \leq C_{p,2} \|h \cdot \Delta u\|_{L^p} + C_h \|u\|_{W^{1,p}}$ ]

To get $W^{2,p}$-bounds on $\tilde{u}_k$ we need

- $W^{1,2p}$-bounds on $\tilde{u}_k$ ; $1<p<\infty$ $\qquad \| |du|^2 \|_{L^p} \leq \|du\|^2_{L^{2p}}$

or

- $\|d\tilde{u}_k\|_{L^2}$ small ; $1<p<2$ $\qquad \| |du|^2 \|_{L^p} \leq \|du\|_{L^2} \cdot \|du\|_{L^{\frac{2p}{2-p}}} \leq \underbrace{\|du\|_{L^2} C}_{\text{small}} \|u\|_{W^{2,p}}$

(Hölder $L^2 \cdot L^q \hookrightarrow L^p$ and Sobolev embedding $W^{1,p} \hookrightarrow L^q$ with $\frac{1}{p} = \frac{1}{2} + \frac{1}{q}$) $\rightarrow$ absorb into LHS

<u>bootstrapping</u>:

small energy $\|du_k^\gamma\|_{L^2} \le \varepsilon \quad \forall \gamma$ on $B_\delta \subset S_k$

$\rightarrow W^{2,p}$-bounds $(p<2)$ on $(u_k^\gamma)_{\gamma\in\mathbb{N}}$

$\rightarrow W^{1,\frac{2p}{2-p}}$-bounds ($=: q > 2$)

$\rightarrow W^{2,q}$-bounds

$W^{k,q}$-bounds, $q>2 \rightarrow W^{k+1,q}$-bounds   from Calderon-Zygmund $(v = \partial_s^{k-1} u)$

$\rightarrow W^{k,q}$-bounds $\forall k$

$\rightarrow W^{k-1,q}$-convergent subsequence $\forall k$

$\xrightarrow[\text{subsequence}]{\text{diagonal}}$ $\mathcal{C}^\infty$-convergent subsequence

($\mathcal{C}^\ell$-convergence on all compact subsets $\forall \ell$)

So compactness holds on interior balls of small energy.

However, we still need to understand

- compactness near the boundary
- dependence of energy quantum $\varepsilon$ on ball radius $\delta$