L15 - Floer Field Theory

Note Title 4/7/2008

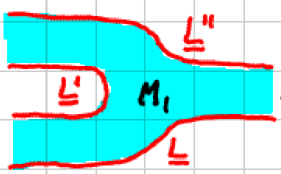
We defined a functor $Symp^{\#}_{monotone} \longrightarrow \mathcal{C}at$ by

(i) For each object, M symplectic, monotone, define the extended Donaldson-Fukaya category $Don^{\#}(M)$:

- Objects : generalized Lagrangians $pt \to \cdots \xrightarrow{\underline{L}} \cdots \to M$
- Morphisms : HF classes $Mor(\underline{L},\underline{L}') = HF(\underline{L},\underline{L}'^{T})$ $\quad pt \xrightarrow{\underline{L}} M \xleftarrow{\underline{L}'} pt, \quad M \xrightarrow{\underline{L}'^{T}} pt$
- composition :

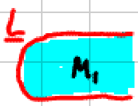
$HF(\underline{L}',\underline{L}'') \rightarrow$

$HF(\underline{L},\underline{L}') \rightarrow$



$\rightarrow HF(\underline{L},\underline{L}'')$

- identity :

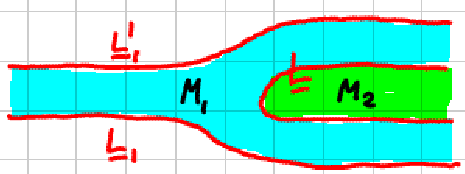


$\rightarrow 1_{\underline{L}} \in HF(\underline{L},\underline{L})$

(ii) For each generalized Lagrangian correspondence $M_1 \xrightarrow{\underline{L}} M_2$ define a functor $\Phi_{\underline{L}} : Don^{\#}(M_1) \longrightarrow Don^{\#}(M_2)$:

- on objects $\underset{pt \longrightarrow M_1}{\underline{L}_1} \longmapsto \underset{pt \longrightarrow M_1 \longrightarrow M_2}{\underline{L}_1 \# \underline{L}}$
- on morphisms $HF(\underline{L}_1,\underline{L}_1') \longrightarrow HF(\Phi_{\underline{L}}(\underline{L}_1),\Phi_{\underline{L}}(\underline{L}_1')) = HF(\underline{L}_1,\underline{L},\underline{L}^{T},\underline{L}_1'^{T})$

is given by the quilt invariant $-$



$+$

<u>Remark</u>: To define the functor $Symp^{\#} \to Cat$ it suffices to fix

(i) M symplectic, monotone $\longmapsto$ $Don^{\#}(M)$ category

(ii) $L_{12} \subset M_1^- \times M_2$ Lagrangian correspondence "simple" $\longmapsto$ $\Phi_{L_{12}} : Don^{\#}(M_1) \to Don^{\#}(M_2)$ functor

and check

(a) Any morphism of $Symp^{\#}_{monotone}$ can be decomposed into simple morphisms

$[\underline{L}] = [L_{01}] \circ [L_{12}] \circ \ldots \circ [L_{(k-1)k}]$; $L_{(j-1)j} \subset N_{j-1}^- \times N_j$ "simple".

(b) Any other decomposition $[\underline{L}] = [L'_{01}] \circ \ldots \circ [L'_{(k'-1)k'}]$

is obtained by a sequence of good moves.

Functoriality then determines for all $[\underline{L}]$

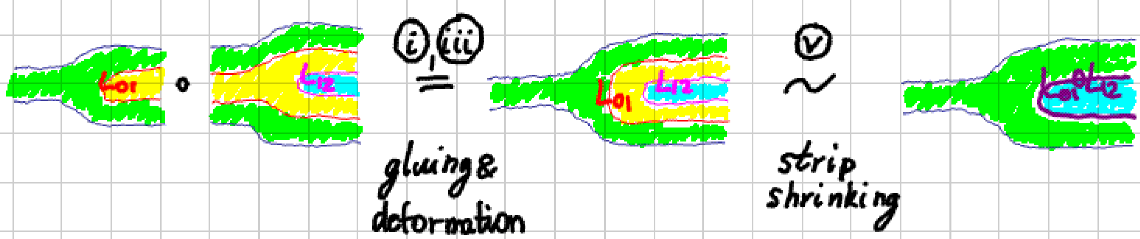
$$\Phi_{[\underline{L}]} = \Phi_{[L_{01}]\circ\ldots\circ[L_{(k-1)k}]} = \Phi_{L_{01}} \circ \ldots \circ \Phi_{L_{(k-1)k}}$$

which is independent of the decomposition since

<u>Thm</u>: good move in $(L_{(j-1)j})_{j=1..k}$ $\widehat{=}$ composition in $(\Phi_{L_{(j-1)j}})_{j=1..k}$

$L_{01} \circ L_{12}$ transverse, embedded $\Longrightarrow$ $\Phi_{L_{01}} \circ \Phi_{L_{12}} = \Phi_{L_{01} \circ L_{12}}$

<u>Proof</u>:



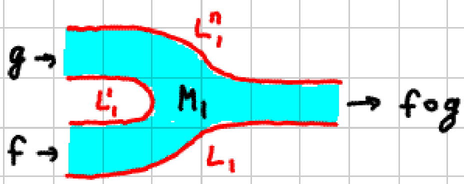
This reproduces the previous definition (ii)

$$\Phi_{\underline{L}} = \Phi_{L_{01}} \circ \Phi_{L_{12}} \circ \dots \circ \Phi_{L_{(k-1)k}}$$

<u>Homework</u>: Prove functoriality of $\Phi_{L_{12}} : \mathcal{D}on^{\#}(M_1) \to \mathcal{D}on^{\#}(M_2)$

$$\Phi_{L_{12}}(f \circ g) = \Phi_{L_{12}}(f) \circ \Phi_{L_{12}}(g) \qquad \forall f \in HF(L_1, L_1') \quad g \in HF(L_1', L_1'')$$

by pictures.



$\Phi_{L_{12}}$:

<u>Proposition</u>: $\mathcal{C}$ category with a subcollection of "simple morphisms"

To define a functor $\mathcal{C} \to Symp^{\#}$ it suffices to fix

(i) X object $\longmapsto M_X$ symplectic

(ii) $x_1 \xrightarrow{Y} x_2$ simple morphism $\longmapsto L_Y \subset M_{x_1}^- \times M_{x_2}$ Lagrangian corresp.,

and check

(a) Any morphism $\tilde{Y}$ of $\mathcal{C}$ can be decomposed into simple morphisms

$\tilde{Y} = Y_{01} \circ \ldots \circ Y_{(k-1)k}$, $Y_{(j-1)j}$ simple

(b) Any other decomposition $\tilde{Y} = Y'_{01} \circ \ldots \circ Y'_{(k'-1)k'}$, $Y'_{(j-1)j}$ simple

is obtained by a sequence of moves

$$\left\{ \begin{array}{l} \bullet\ Y_\alpha \circ Y_\beta = Y_\gamma \\ \bullet\ Y_\gamma = Y_\alpha \circ Y_\beta \\ \bullet\ Y_\alpha \circ Y_\beta = Y_\gamma \circ Y_\delta \end{array} \right\} \text{ that correspond to transverse embedded geometric composition } \left\{ \begin{array}{l} \bullet\ L_{Y_\alpha} \circ L_{Y_\beta} = L_{Y_\gamma} \\ \bullet\ L_{Y_\gamma} = L_{Y_\alpha} \circ L_{Y_\beta} \\ \bullet\ L_{Y_\alpha} \circ L_{Y_\beta} = L_{Y_\gamma} \circ L_{Y_\delta} \end{array} \right\}$$

<u>Corollary</u>: If all M_X and L_Y in (i),(ii) are monotone

then we obtain a functor $\mathcal{C} \longrightarrow Symp^{\#}_{monotone}$

and hence a categorification functor $\mathcal{C} \longrightarrow \mathcal{C}at$

factoring through $Symp^{\#}_{monotone}$.

Example: Floer field theory in 2+1 dimension (almost 2+1 TQFT)

from moduli spaces of "central curvature & fixed determinant" bundles

- $\mathcal{C} = \mathcal{C}ob_{2+1}$ objects: Σ Riemann surface (closed, oriented 2-mfds) !connected!

  morphisms: Y 3dim cobordism $\partial Y = \Sigma_0^- \sqcup \Sigma_1$

(i) [Narasimhan-Seshadri] smooth, monotone symplectic manifolds:

$$M_\Sigma := \left\{ \begin{array}{l} A \ U(r)\text{-connection on } \Sigma \\ (F_A)_{su(r)} = 0 \ , \det(A) = \delta \end{array} \right\} \Big/ \text{gauge} \cong \left\{ \begin{array}{l} \rho : \pi_1(\Sigma \setminus pt) \to SU(r) \\ \rho(\circlearrowleft pt) = -\mathbb{1} \end{array} \right\} \Big/ SU(r)$$

(degree d part)

$\boxed{\text{fix } r \in \mathbb{N},\ u(r) = su(r) \oplus u(1) \ ;\ \delta\ U(1) \text{ bundle on } \Sigma, \text{ degree } d \text{ coprime to } r}$

$$\cong \left\{ \begin{array}{l} \alpha_1 \ldots \alpha_g, \beta_1 \ldots \beta_g \in SU(r)^{2g} \\ \prod_{j=1}^{g} \alpha_j \beta_j \alpha_j^{-1} \beta_j^{-1} = -\mathbb{1} \end{array} \right\} \Big/ SU(r)$$

(ii) $L_Y := \left\{ (\tilde{A}|_{\Sigma_0}, \tilde{A}|_{\Sigma_1}) \;\middle|\; \tilde{A}\ U(r)\text{-connection on } \Sigma,\ (F_{\tilde{A}})_{su(r)} = 0,\ \det(\tilde{A}) = \tilde{\delta} \right\} \Big/ \text{gauge}$

$= \left\{ (\rho_0, \rho_1) \in M_{\Sigma_0} \times M_{\Sigma_1} \;\middle|\; \exists \text{ extension } \tilde{\rho} : \pi_1(Y \setminus \text{line}) \to SU(r) \right\}$

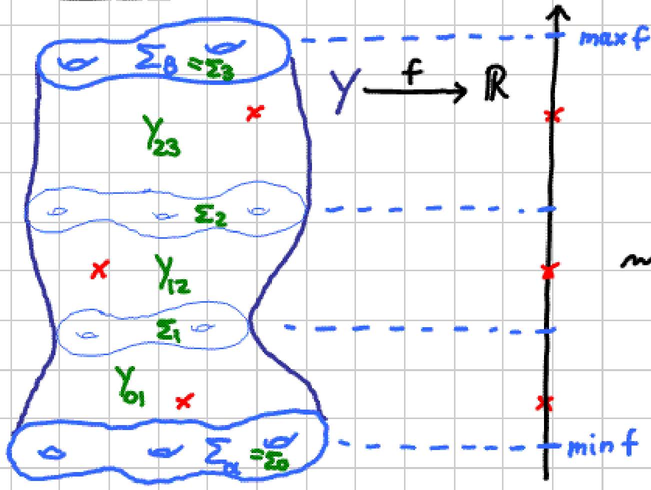
is in general not smooth

(ii') $L_Y \subset M_{\Sigma_0}^- \times M_{\Sigma_1}$ smooth, monotone Lagrangian correspondence for

- simple morphisms Y = cylinder $\Sigma \times [0,1]$ or handle attachment

  (i.e. $\exists f : Y \to \mathbb{R}$ Morse, 0 or 1 crit.pt., maximal on Σ_1, minimal on Σ_0)

ⓐ decomposition of general morphism $\partial Y = \overline{\Sigma_\alpha} \amalg \Sigma_\beta$



$$\rightsquigarrow Y = \left\{ \begin{array}{l} Y_{23} \\ \cup_{\Sigma_2} \\ Y_{12} \\ \cup_{\Sigma_1} \\ Y_{01} \end{array} \right. \quad \overset{(ii)}{\rightsquigarrow} \quad \underline{L}_Y = \begin{array}{l} M_{\Sigma_3} \\ \uparrow L_{Y_{23}} \\ M_{\Sigma_2} \\ \uparrow L_{Y_{12}} \\ M_{\Sigma_1} \\ \uparrow L_{Y_{01}} \\ M_{\Sigma_0} \end{array}$$

cut between levels of critical points

ⓑ moves between decompositions : Cerf moves for Morse functions

- cancellation of critical points $\rightsquigarrow$

$$Y_{g \to g+1} \cup Y_{g+1 \to g} \cong [0,1] \times \Sigma_g$$

$$\boxed{\begin{array}{l} L_{Y_{g\to g+1}} = \left\{ \left(\begin{smallmatrix}\alpha_i\\ \beta_i\end{smallmatrix}\right)_{i=1..g}, \left(\begin{smallmatrix}\alpha'_i\\ \beta'_i\end{smallmatrix}\right)_{i=1..g+1} \;\middle|\; \alpha'_i = \alpha_i,\ \beta'_i = \beta_i,\ \beta'_{g+1} = 1\!\!1 \right\} \\ L_{Y_{g+1\to g}} = \left\{ \left(\begin{smallmatrix}\alpha'_i\\ \beta'_i\end{smallmatrix}\right)_{i=1..g+1}, \left(\begin{smallmatrix}\alpha_i\\ \beta_i\end{smallmatrix}\right)_{i=1..g} \;\middle|\; \alpha'_i = \alpha_i,\ \beta'_i = \beta_i,\ \alpha'_{g+1} = 1\!\!1 \right\} \\ L_{Y_{g\to g+1}} \circ L_{Y_{g+1\to g}} = \Delta_{M_{\Sigma_g}} = L_{[0,1]\times\Sigma_g} \end{array}}$$

- change of order
- handle slide

$$Y_\alpha \cup_\Sigma Y_\beta = Y \cong Y_\gamma \cup_\Sigma Y_\delta \rightsquigarrow \boxed{L_{Y_\alpha} \circ L_{Y_\beta} = L_Y = L_{Y_\gamma} \circ L_{Y_\delta}}$$

- cancelation of trivial (no crit. pt.) cobordism $\quad Y_\alpha \cup [0,1]\times\Sigma \cong Y_\alpha \rightsquigarrow \boxed{L_{Y_\alpha} \circ \Delta_{M_\Sigma} = L_{Y_\alpha}}$

Corollary: Fix Riem. surfaces $\Sigma_\alpha, \Sigma_\beta$ and (gen.) Lagrangians $L_\alpha \subset M_{\Sigma_\alpha}, L_\beta \subset M_{\Sigma_\beta}$ then we have a topological invariant

Y cobordism from Σ_α to Σ_β

$\downarrow$

$\underline{L}_Y$ gen. Lagr. corresp. $M_{\Sigma_\alpha} \to M_{\Sigma_\beta}$

$\downarrow$

$HF(\underset{pt \to M_{\Sigma_\alpha} \to M_{\Sigma_\beta} \to pt}{L_\alpha \# \underline{L}_Y \# L_\beta})$ quilted Floer homology

Ex.: $\Sigma_\alpha = \Sigma_\beta = T^2$

$pt \subset M_{T^2} = pt$

Y 3-manifold

$\downarrow$

$Y \# [0,1] \times T^2 =: \tilde{Y}$ cobordism

$\downarrow$

$HF(\underline{L}_{\tilde{Y}})$

Conj: $HF(\underline{L}_{Y \# [0,1] \times T^2})$ is closely related to [Kronheimer-Mrowka] invariants of Y from singular instantons
Collin-Steer