L14 - The symplectic 2-category

Note Title 4/2/2008

holomorphic quilt invariants - summary

$\Phi_{\underline{S}} : \bigotimes_{\underline{e}^- \in \underline{\mathcal{E}}^-} HF(\underline{\mathcal{L}}_{\underline{e}^-}) \longrightarrow \bigotimes_{\underline{e}^+ \in \underline{\mathcal{E}}^+} HF(\underline{\mathcal{L}}_{\underline{e}^+})$ is defined by

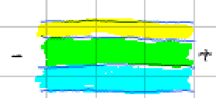
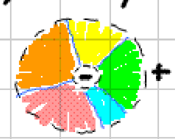
- a quilted surface – given as one surface (with boundary & +/- ends)
  - seams indicated by embedded non-intersecting 1-manifolds
- labeling patches / seams / boundary components by symplectic manifolds / Lagrangian correspondences / Lagrangian submanifolds

calculation rules

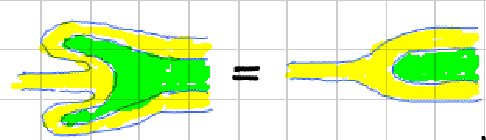
(i) composition is gluing $\qquad \Phi_{\underline{S}_0} \circ \Phi_{\underline{S}_1} = \Phi_{\underline{S}_0 \# \underline{S}_1}$

in $HF(\underline{\mathcal{L}}_{\underline{e}_0^+} = \underline{\mathcal{L}}_{\underline{e}_1^-})$ $\qquad$ at $\underline{e}_0^+ \sim \underline{e}_1^-$

(ii) for $\underline{S}$ = quilt of strips (with $\mathbb{R}$-symmetry) $\qquad \Phi_{\underline{S}} = Id$

 = Id $\qquad$  = Id

(iii) $\Phi_{\underline{S}}$ is invariant under deformation of quilt



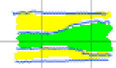
(deformation $Q_\tau = (\underline{S}_\tau, \underline{H}_\tau, \underline{J}_\tau)_{\tau \in [0,1]}$ with fixed ends induces chain homotopy equivalence $T_{\{Q_\tau\}}$, i.e. $\Phi_{\underline{S}_1} - \Phi_{\underline{S}_0} = (\oplus \partial_{\underline{e}^+}) \circ T + T \circ (\oplus \partial_{\underline{e}^-})$)

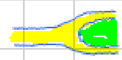
(iv) continuation maps intertwine between quilt invariants $\Phi_{\underline{S}}, \Phi_{\underline{S}'}$

for homotopic $\underline{S}, \underline{S}'$ with different end data $(\underline{\delta}_{\underline{e}}, \underline{H}_{\underline{e}}, \underline{J}_{\underline{e}})_{\underline{e}\in\mathcal{E}(\underline{S})}, (\underline{\delta}'_{\underline{e}'}, \ldots)_{\underline{e}'\in\mathcal{E}(\underline{S}')}$.

<u>Example</u>:

$$\begin{tikzcd}
\bigotimes_{\underline{e}^-} HF(\underline{\mathcal{L}}_{\underline{e}^-}, .. \underline{\delta}_{\underline{e}^-} \ldots) \arrow[r, "\Phi_{\underline{S}}"] \arrow[d, "\int \bigotimes_{\underline{e}^-} \Phi_{\underline{\delta}_{\underline{e}^-} \rightsquigarrow \underline{\delta}'_{\underline{e}^-}}"'] \arrow[dr, bend left, "a"] \arrow[dr, "b"'] & \bigotimes_{\underline{e}^+} HF(\underline{\mathcal{L}}_{\underline{e}^+}, .. \underline{\delta}_{\underline{e}^+} ..) \arrow[d, "\int \bigotimes_{\underline{e}^+} \Phi_{\underline{\delta}_{\underline{e}^+} \rightsquigarrow \underline{\delta}'_{\underline{e}^+}}"] \\
\bigotimes_{\underline{e}^-} HF(\underline{\mathcal{L}}_{\underline{e}^-}, .. \underline{\delta}'_{\underline{e}^-} \ldots) \arrow[r, "\Phi_{\underline{S}'}"] & \bigotimes_{\underline{e}^+} HF(\underline{\mathcal{L}}_{\underline{e}^+}, .. \underline{\delta}'_{\underline{e}^+} ..)
\end{tikzcd}$$





<u>Proof</u>:

$a = \Phi_{\underline{S}} \circ \Phi^{+}_{\delta \to \delta'} \overset{(i)}{=} \Phi_{Q_0} \underset{\text{homotopy } (Q_\tau)}{\overset{(iii)}{=}} \Phi_{Q_1} \overset{(i)}{=} \Phi^{-}_{\delta \to \delta'} \circ \Phi_{\underline{S}'} = b$

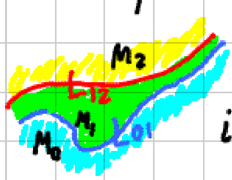
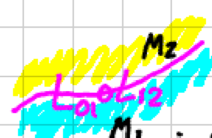
glue

glue

(v) isomorphisms $HF(\underline{\mathcal{L}}_{\underline{e}}) \cong HF(\underline{\mathcal{L}}'_{\underline{e}'})$ between equivalent generalized

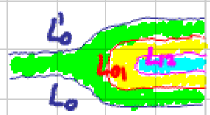
Lagrangian correspondences $\left(\text{related by a simple move } \begin{matrix} \underline{\mathcal{L}}_{\underline{e}} = (.. \; L_{01}, L_{12} \; ..) \\ \underline{\mathcal{L}}'_{\underline{e}'} = (.. \; L_{01} \circ L_{12} \; ..) \end{matrix}\right)$

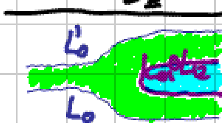
intertwine between quilt invariants $\Phi_{\underline{S}}, \Phi_{\underline{S}'}$

where a strip  in $\underline{S}$ is replaced by a seam  in $\underline{S}'$.

<u>Example</u>:

$$\begin{tikzcd}
HF(L_0, L_0') \arrow[r, "\Phi_{\underline{S}}"] \arrow[d, "\int"] & HF(L_0, L_{01}, L_{12}, L_{12}^T, L_{01}^T, L_0') \arrow[d, "\int \; (2 \text{ simple moves})"] \\
HF(L_0, L_0') \arrow[r, "\Phi_{\underline{S}'}"] & HF(L_0, L_{01}\circ L_{12}, \underset{L_{12}^T \circ L_{01}^T}{\overset{\shortparallel}{(L_{01}\circ L_{12})^T}}, L_0')
\end{tikzcd}$$





The (monotone) symplectic 2-category

($\tau = 1$ by scaling of ω)

__objects__: (M,ω) closed symplectic manifold, monotone : $[\omega] = c_1$

__morphisms__: $Mor(M_0, M_1)$ = generalized Lagrangian correspondences $M_0 \xrightarrow{\underline{L}} M_1$
modulo equivalence generated by good moves
(i.e. transverse, embedded geometric composition) ; monotone

__composition__: $Mor(M_0,M_1) \times Mor(M_1,M_2) \longrightarrow Mor(M_0,M_2)$

concatenation $\left[M_0 \xrightarrow{\underline{L}_{01}} M_1\right], \left[M_1 \xrightarrow{\underline{L}_{12}} M_2\right] \longmapsto \left[M_0 \xrightarrow{\underline{L}_{01}\#\underline{L}_{12}} M_1 \longrightarrow M_2\right]$

⊛ associativity

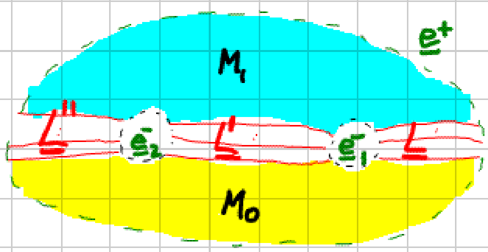
__identity__ : $1_M := [\Delta_M] \in Mor(M,M)$

__2-morphisms__: ${}^2Mor\left(M_0 \xrightarrow{\underline{L}} M_1,\ M_0 \xrightarrow{\underline{L}'} M_1\right) = HF(\underline{L}\#(\underline{L}')^T)$ $\quad M_0 \overset{\underline{L}}{\underset{(\underline{L}')^T}{\rightleftarrows}} M_1$

__2-composition__ : $\Phi_{\underline{S}}$: ${}^2Mor(\overbrace{\underline{L}, \underline{L}'}^{\underline{\varepsilon}_{e_1^-}}) \otimes {}^2Mor(\overbrace{\underline{L}', \underline{L}''}^{\underline{\varepsilon}_{e_2^-}}) \longrightarrow {}^2Mor(\overbrace{\underline{L}, \underline{L}''}^{\underline{\varepsilon}_{e^+}})$

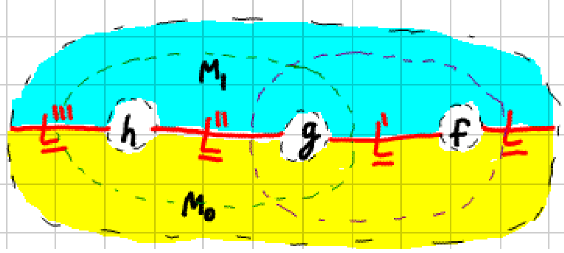
$(f,g) \longmapsto f \circ g$

$\underline{S}$:



⊛ associativity:

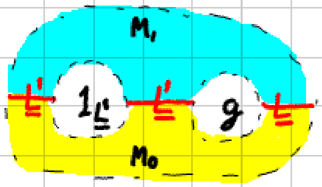
$(f\circ g)\circ h \ \underset{\text{gluing}}{=}$



$\underset{\text{gluing}}{=} \ f\circ(g\circ h)$

2-identity: $1_{\underline{L}} = \Phi_{\underline{S}} \in {}^2Mor\left(M_0 \xrightarrow{\underline{L}} M_1,\ M_0 \xrightarrow{\underline{L}} M_1\right)$

⊛ $\forall g \in HF(\underline{L},\underline{L}')\quad g \circ 1_{\underline{L}'} = g \quad$ and $\quad 1_{\underline{L}} \circ g = g$



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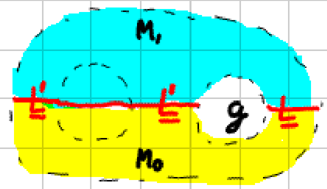

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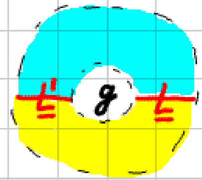
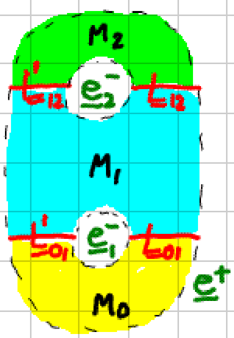


strips with $\mathbb{R}$-action

$\approx Id(g)$

composition functor: $Mor(M_0,M_1) \times Mor(M_1,M_2) \longrightarrow Mor(M_0,M_2)$

$\underline{S}$:



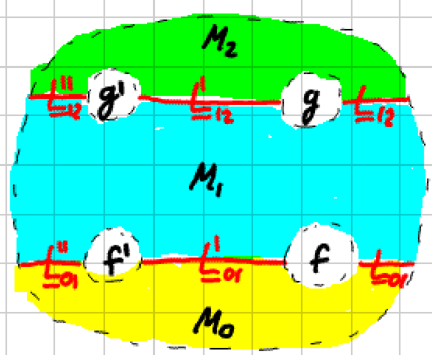
$\underline{L}_{01} \qquad \underline{L}_{12} \longmapsto \underline{L}_{01} \# \underline{L}_{12}$

$\underline{L}'_{01} \qquad \underline{L}'_{12} \longmapsto \underline{L}'_{01} \# \underline{L}'_{12}$

$$\Phi_{\underline{S}}: \underbrace{HF(\underline{L}_{01},\underline{L}'_{01})}_{\mathcal{I}_{e_1^-}} \otimes \underbrace{HF(\underline{L}_{12},\underline{L}'_{12})}_{\mathcal{I}_{e_2^-}} \longrightarrow \underbrace{HF(\underline{L}_{01}\#\underline{L}_{12},\ \underline{L}'_{01}\#\underline{L}'_{12})}_{\mathcal{I}_{e^+}}$$

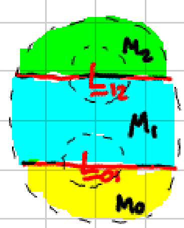
$$(f,g) \longmapsto f \# g$$

⊛ $(f \circ f') \# (g \circ g') =$



$= (f\#g) \circ (f'\#g')$

⊛ $1_{\underline{L}_{01}} \# 1_{\underline{L}_{12}} =$



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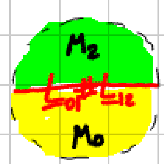


$= 1_{\underline{L}_{01}\#\underline{L}_{12}}$

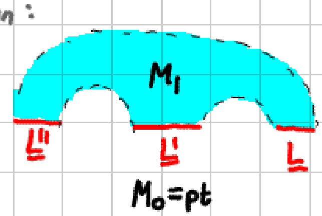
Corollary (Categorification): There exists a functor

$$\mathrm{Symp}^{\#}_{\text{monotone}} \longrightarrow \mathcal{C}at \qquad \begin{bmatrix} \text{Objects: categories} \\ \text{Morphisms: functors} \\ \text{– composition \& identity functor} \end{bmatrix}$$

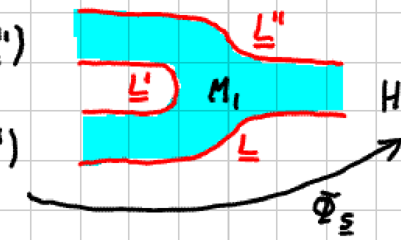
(i) M symplectic $\longmapsto$ $Mor(pt, M) := Don^{\#}(M)$ Donaldson-Fukaya category

Objects: generalized Lagrangians $pt \to \ldots \to M$
Morphisms: HF-classes

composition:

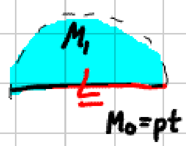


$\widehat{=}$ $HF(\underline{L}', \underline{L}'') \otimes HF(\underline{L}, \underline{L}')$

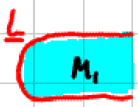


$HF(\underline{L}, \underline{L}'')$

identity:



$\widehat{=}$

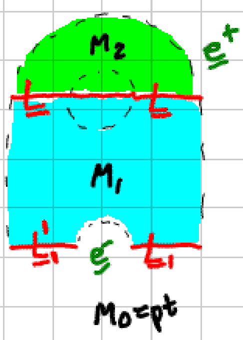


$\in HF(\underline{L}, \underline{L})$

(ii) $M_1 \xrightarrow{\underline{L}} M_2$ generalized Lagrangian correspondence $\longmapsto$ $\Phi_{\underline{L}} : Don^{\#}(M_1) \to Don^{\#}(M_2)$

$\underline{L}_1 \mapsto \underline{L}_1 \# \underline{L}$ $\quad pt \to .. \to M_1 \to .. M_2$

$HF(\underline{L}_1, \underline{L}_1') \ni f \longmapsto f \# 1_{\underline{L}} \in HF(\underline{L}_1 \# \underline{L}, \underline{L}_1' \# \underline{L})$



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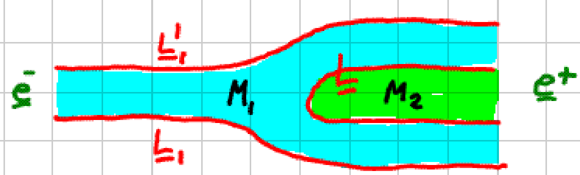

<u>Proof</u>: Given any 2-category $\mathcal{C}$ and distinguished object p_0 there is a functor of 1-categories $\mathcal{C} \to \mathcal{C}at$ given by

object $p \longmapsto \mathcal{M}or(p_0, p)$ is a category

morphism $p \xrightarrow{h} q \longmapsto$

$$\begin{tikzcd} \mathcal{M}or(p_0,p) \arrow[rr] \arrow[dr, "Id \times (h, 1_h)"'] & & \mathcal{M}or(p_0,q) \\ & \mathcal{M}or(p_0,p) \times \mathcal{M}or(p,q) \arrow[ur, "\text{composition functor of } \mathcal{C}"'] & \end{tikzcd}$$

is a functor

We picked $p_0 = pt$ as distinguished object in $\mathcal{S}ymp^{\#}$.