L13 - quilt invariants

Note Title 3/31/2008

Thm: $\underline{S}$ quilted surface. Suppose $(\underline{M}, \underline{LBS})$ is monotone

(i.e. energy-index relation for all quilt maps $\underline{u}: \underline{S} \to \underline{M}$ satisfying $\underline{LBS}$ conditions)

Then there is a well defined "relative invariant"

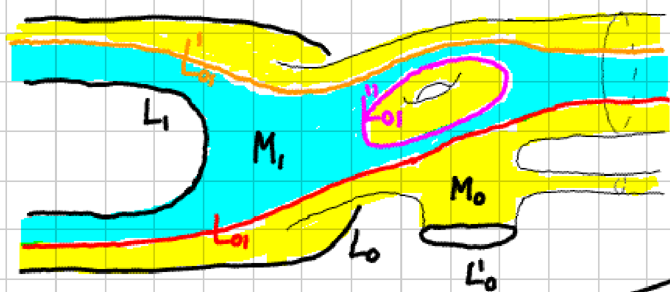
$$\Phi_{\underline{S}} : \bigotimes_{\underline{e} \in \mathcal{E}^-(\underline{S})} HF(\underline{\mathcal{L}}_{\underline{e}}) \longrightarrow \bigotimes_{\underline{e}^+ \in \mathcal{E}^+(\underline{S})} HF(\underline{\mathcal{L}}_{\underline{e}^+})$$

$$\bigotimes_{\underline{e}^-} \underset{\bigcap_{\underline{H}_{\underline{e}^-}} \underline{\mathcal{L}}_{\underline{e}^-}}{\underline{x}_{\underline{e}^-}} \longmapsto \sum_{\underline{y}_{\underline{e}^+} \in \bigcap_{\underline{H}_{\underline{e}^+}} \underline{\mathcal{L}}_{\underline{e}^+}} \# \left\{ \begin{array}{l} \text{perturbed } J\text{-holomorphic} \\ \text{quilt maps } \underline{u}: \underline{S} \to \underline{M} \\ \text{with limits } \underline{x}_{\underline{e}^-}, \underline{y}_{\underline{e}^+} \end{array} \right\} \otimes \underline{y}_{\underline{e}^+}$$

$HF(L_1, L_{01}', L_0)$

$\otimes$

$HF(L_0, L_{01}, L_1)$



$HF(L_{01}, L_{01}'^{T})$

$\otimes$

$HF(M_0) := HF(\Delta_{M_0})$

$\Phi_{\underline{S}}$ (also depends on L_0', L_{01}'')

Construction: (i) For each end $\underline{e} \in \mathcal{E}(\underline{S})$

- pick regular $\underline{H}_{\underline{e}} = (H_{k_i, e_i})$
- widths $\underline{\delta}_e$ are given by strip-like ends of $\underline{S}$
- pick regular $\underline{J}_{\underline{e}} = (J_{k_i, e_i})$

$\Bigg\}$ defines $HF(\underline{\mathcal{L}}_{\underline{e}})$

This determines for each end $\varepsilon_{k,e}: \mathbb{R}^{\pm}\times\left\{\begin{matrix}[0,\delta_{k,e}]\\ S^1\end{matrix}\right\} \hookrightarrow S_k$ a unique

- Hamiltonian $H_{k,e}\in\mathcal{C}^\infty\left(\left\{\begin{matrix}[0,\delta_{k,e}]\\ S^1\end{matrix}\right\}\times M_k\right)$
- ω_{M_k}-compatible almost complex structure $J_{k,e}\in\mathcal{C}^\infty\left(\left\{\begin{matrix}[0,\delta_{k,e}]\\ S^1\end{matrix}\right\}, \mathcal{E}nd(TM_k)\right)$

(ii) Pick "interpolating Hamiltonians" $\underline{K} = \left(K_k\in\Omega^1(S_k, \mathcal{C}^\infty(M_k))\right)_{k=1..n}$

- $\varepsilon_{k,e}^* K_k = H_{k,e}\ dt$ on $\underset{\ni s}{\mathbb{R}^{\pm}}\times\underset{\ni t}{\left\{\begin{matrix}[0,\delta_{k,e}]\\ S^1\end{matrix}\right\}}$
- $K_k|_{\partial S_k}\equiv 0$

and define the vector field valued 1-forms $\underline{Y} = \left(Y_k\in\Omega^1(S_k), \Gamma(TM_k)\right)$

by $\omega_{M_k}(Y_k, \cdot) = d^{M_k}K_k \in \Omega^1(S_k\times M_k)$. $\left(\text{Then } \varepsilon_{k,e}^* Y_k = X_{H_{k,e}}dt.\right)$

$T(S_k\times M_k) = TS_k\times TM_k$

(iii) Pick regular "interpolating almost complex structures"

$\underline{J} = \Big(J_k\in\mathcal{C}^\infty(S_k, \underbrace{\mathcal{J}(M_k,\omega_k)}_{\text{space of } \omega_k\text{-compatible almost complex structures}})\Big)$

- $\varepsilon_{k,e}^* J_k = J_{k,e}$
- The Fredholm section $\bar\partial_{\underline{H},\underline{J}}$ is transverse to 0.

$$\begin{aligned}\bar\partial : \{\text{quilt maps}\} &\longrightarrow \bigoplus_{k=1}^n \Omega^{0,1}(S_k, TM_k)\\ \underline{u} &\longmapsto \left(J_k(u_k)(du_k - Y_k(u_k)) - (du_k - Y_k(u_k))\circ j_k\right)\end{aligned}$$

$\boxed{i\partial_s = \partial_t}$

$$\boxed{\text{on ends: } = J_{k,e}(u_k)\begin{pmatrix}\partial_s u_k\ ds\\ +(\partial_t u_k - X_{H_{k,e}}(u_k))\,dt\end{pmatrix} - \begin{pmatrix}-\partial_s u_k\ dt\\ +(\partial_t u_k - X_{H_{k,e}}(u_k))\,ds\end{pmatrix}}$$

Consider the moduli spaces of <u>holomorphic quilts</u>

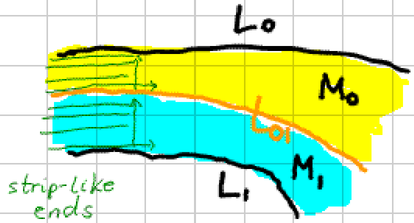
$\underline{u} : \underline{S} \to \underline{M}$ satisfying $\begin{cases} \overline{\partial}_{\underline{H},\underline{J}}\, \underline{u} = 0 \\ \underline{LBS}\text{ - conditions} \end{cases}$

- finite energy $E(\underline{u}) = \sum_{k=1}^{n} \int_{S_k} u_k^* \omega_k - d(H_k \circ u_k) = \frac{1}{2} \sum_{k=1}^{n} \int_{S_k} |du_k - Y_k|^2$

  $\Updownarrow$

  near each end $\underset{(k_i, z_{k_i,e_i})_{i=1..N}}{\underline{e}}$ $\underline{u}$ converges to $\cap_{\underline{H}_{\underline{e}}} \underline{\mathcal{L}}_{\underline{e}}$

[$\underline{u}$ near (z_{k_i,e_i}) is "half a Floer trajectory" $(u_{k_i} : \mathbb{R}^{\pm} \times [0, \delta_{k_i,e_i}] \to M_{k_i})$



so $u_{k_i}(s, \cdot) \xrightarrow[s \to \pm\infty]{} \gamma_{k_i} : [0, \delta_{k_i,e_i}] \to M_{k_i}$

$(\gamma_{k_i})_{i=1..N} \in \cap_{\underline{H}_{\underline{e}}} \underline{\mathcal{L}}_{\underline{e}}$]

- Gromov compactness & monotonicity:

  0-dim. moduli spaces are compact $\longrightarrow$ Count defines $\Phi_{\underline{S}}$ on chains

  1-dim. moduli spaces are compact up to "energy escaping off an end"

- gluing $\Rightarrow \left(\bigoplus_{\underline{e}^+ \in \underline{\mathcal{E}}^-} \partial_{\pm \underline{e}^+} \right) \circ \Phi_{\underline{S}} + \Phi_{\underline{S}} \circ \left(\bigoplus_{\underline{e}^- \in \underline{\mathcal{E}}^-} \partial_{\pm \underline{e}^-} \right) = 0$

  $\Rightarrow \Phi_{\underline{S}}$ descends to homology

$\blacksquare$

<u>Thm</u>: $\Phi_{\underline{S}}$ is independent of perturbations and only depends on $\underline{S}$ "up to homotopy", i.e. $\Phi_{\underline{S}}$ is determined by

- the surfaces $S_k = \overline{S}_k \setminus \{z_{k,e}\}$ up to diffeomorphism
- incoming (-) / outgoing (+) labels on ends
- combinatorial seams $\mathcal{S}$ and orientation of seam diffeomorphisms φ_σ.

<u>Proof</u>: • continuation maps intertwine between quilt invariants $\Phi_{\underline{S}}, \Phi_{\underline{S}'}$ with different end data $(\underline{\delta}_{\underline{e}}, \underline{H}_{\underline{e}}, \underline{J}_{\underline{e}})_{\underline{e}\in\mathcal{E}(\underline{S})}$, $(\underline{\delta}'_{\underline{e}}, \underline{H}'_{\underline{e}}, \underline{J}'_{\underline{e}})_{\underline{e}\in\mathcal{E}(\underline{S}')}$

$$\begin{array}{ccc}
\bigotimes_{\underline{e}^-} HF(\ldots \underline{\delta}_{\underline{e}^-} \ldots) & \xrightarrow{\Phi_{\underline{S}}} & \bigotimes_{\underline{e}^+} HF(\ldots \underline{\delta}_{\underline{e}^+} \ldots) \\
\wr\downarrow \bigotimes_{\underline{e}^-} \Phi_{\underline{\delta}_{\underline{e}^-} \rightsquigarrow \underline{\delta}'_{\underline{e}^-}} & & \wr\downarrow \bigotimes_{\underline{e}^+} \Phi_{\underline{\delta}_{\underline{e}^+} \rightsquigarrow \underline{\delta}'_{\underline{e}^+}} \\
\bigotimes_{\underline{e}^-} HF(\ldots \underline{\delta}'_{\underline{e}^-} \ldots) & \xrightarrow{\Phi_{\underline{S}'}} & \bigotimes_{\underline{e}^+} HF(\ldots \underline{\delta}'_{\underline{e}^+} \ldots)
\end{array}$$

- composition is gluing: $\Phi_{\underline{S}} \circ \Phi^{+}_{\underline{\delta}\to\underline{\delta}'} \underset{(Q_\tau)}{\sim} \Phi^{-}_{\underline{\delta}\to\underline{\delta}'} \circ \Phi_{\underline{S}'}$
- homotopies $(Q_\tau)_{\tau\in[0,1]} = (\underline{S}_\tau, \underline{H}_\tau, \underline{J}_\tau)_{\tau\in[0,1]}$ between quilted surfaces $\underline{S}_0, \underline{S}_1$ and perturbation data $(\underline{H}_0, \underline{J}_0), (\underline{H}_1, \underline{J}_1)$ with "fixed ends" (widths $(\delta_{k,e})$ and Floer data $(\underline{H}_{\underline{e}}), (\underline{J}_{\underline{e}})$ fixed) provide chain homotopy equivalences $\Phi_{\underline{S}_0, \underline{H}_0, \underline{J}_0} \sim \Phi_{\underline{S}_1, \underline{H}_1, \underline{J}_1}$ ∎

UPSHOT: We can define maps $\Phi_{\underline{S}} : \bigotimes_{\underline{e}^-} HF(\underline{L}_{\underline{e}^-}) \longrightarrow \bigotimes_{\underline{e}^+} HF(\underline{L}_{\underline{e}^+})$

by drawing a quilted surface as one surface (with boundary & ends)

with seams indicated by embedded non-intersecting 1-manifolds ($\cong \mathbb{R}$ or S^1)

and labeling the patches / seams / boundary components by

symplectic manifolds / Lagrangian correspondences / Lagrangian submanifolds.

We have calculation rules

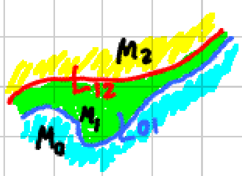
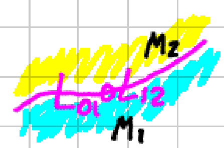
- composition is gluing $\Phi_{\underline{S}_0} \circ \Phi_{\underline{S}_1} = \Phi_{\underline{S}_0 \# \underline{S}_1}$

  in $HF(\underline{L}_{\underline{e}_0^+} = \underline{L}_{\underline{e}_1^-})$ at $\underline{e}_0^+ \sim \underline{e}_1^-$

- for $\underline{S}$ = quilt of strips (with $\mathbb{R}$-symmetry after deformation)

  $\Phi_{\underline{S}} = Id$

- $\Phi_{\underline{S}}$ is invariant under

  - deformation of quilt
  - replacing a strip  by a seam 

    with transverse & embedded composition