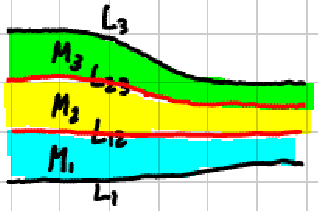
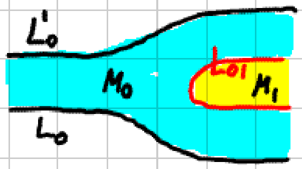
L12 - Pseudoholomorphic quilts

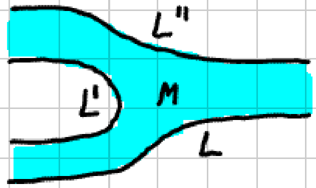
Note Title 3/31/2008



$HF(\underline{L}, .. \underline{\delta} ..) \to HF(\underline{L}, .. \underline{\delta}' ..)$



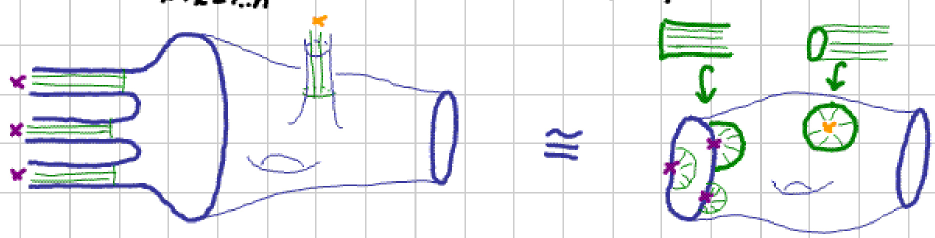
$HF(L_0, L_0') \to HF(L_0, L_{01}, L_{01}^T, L_0')$



$HF(L, L') \otimes HF(L', L'') \to HF(L, L'')$

Def$^{\underline{n}}$: A quilted surface $\underline{S} = \big((\bar{S}_k), (z_{k,e}), (j_k), (\varepsilon_{k,e}), (\delta_{k,e}), \mathcal{S}, (\varphi_\sigma)\big)$ consists of

① patches: $(S_k)_{k=1..n}$ surfaces with strip-like ends



a) $\bar{S}_k$ compact Riemann surface with boundary

$\mathcal{E}_k^b = \mathcal{E}_k^{b+} \cup \mathcal{E}_k^{b-}$, $\mathcal{E}_k^i = \mathcal{E}_k^{i+} \cup \mathcal{E}_k^{i-}$ finite sets

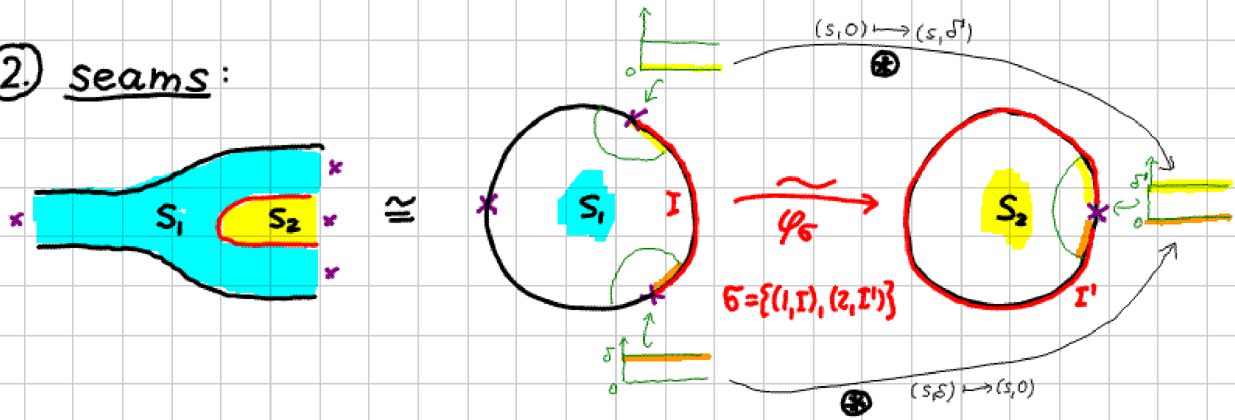
$z_{k,e} \in \partial \bar{S}_k \ \forall e \in \mathcal{E}_k^b$, $z_{k,e} \in \bar{S}_k \setminus \partial \bar{S}_k \ \forall e \in \mathcal{E}_k^i$ distinct marked points

b) j_k complex structure on $S_k = \bar{S}_k \setminus \{ z_{k,e} \mid e \in \mathcal{E}_k^b \cup \mathcal{E}_k^i \}$

c) strip-like ends $\varepsilon_{k,e}$: $\mathbb{R}^{\pm} \times [0, \delta_{k,e}] \hookrightarrow S_k$ $e \in \mathcal{E}_k^{b\pm}$, width $\delta_{k,e} > 0$
$\phantom{\varepsilon_{k,e}:}$ $\mathbb{R}^{\pm} \times S^1 \hookrightarrow S_k$ $e \in \mathcal{E}_k^{i\pm}$

- disjoint images
- $\varepsilon_{k,e}(\pm\infty, \cdot) = z_{k,e}$, $\varepsilon_{k,e}(\cdot, \{0,1\}) \subset \partial \bar{S}_k$
- $\varepsilon_{k,e}^* j_k = i$ standard complex structure on $\mathbb{R} \times \mathbb{R} = \mathbb{C}$ $(S^1 = \mathbb{R}/\mathbb{Z})$

② seams:



a) $\mathcal{S}$ collection of pairwise disjoint 2-element subsets

$$\sigma=\{(k_\sigma,I_\sigma),(k'_\sigma,I'_\sigma)\}\subset\bigcup_{k=1}^{n}\{k\}\times\pi_0\left(\partial\bar S_k\setminus\{z_{ke}\,|\,e\in\mathcal{E}_k^b\}\right)$$

$\cup$

I_σ, I'_σ connected components $\cong\mathbb{R}$ or S^1

b) for each $\sigma\in\mathcal{S}$ a diffeomorphism $\varphi_\sigma: I_\sigma\xrightarrow{\sim}I'_\sigma$

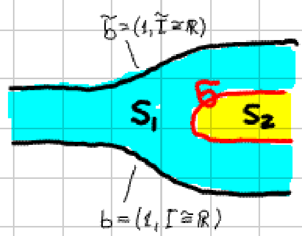
⊛ compatible with strip-like ends $\left(\text{i.e. } \varepsilon_{k'e'}^{-1}\circ\varphi_\sigma\circ\varepsilon_{ke}:(s,0)\mapsto(s,\delta_{k'e'}) \text{ or } (s,\delta_{ke})\mapsto(s,0)\right)$

③ various orderings (for orientations)

Def^n: The boundary of S is

$$\mathcal{B}:=\bigcup_{k=1}^{n}\{k\}\times\pi_0\left(\partial\bar S_k\setminus\{z_{ke}\,|\,e\in\mathcal{E}_k^b\}\right)\setminus\bigcup_{\sigma\in\mathcal{S}}\sigma\,.$$

It indexes the true boundary components $b=(k_b,I_b)$.
$I_b\cong\mathbb{R}$ or S^1

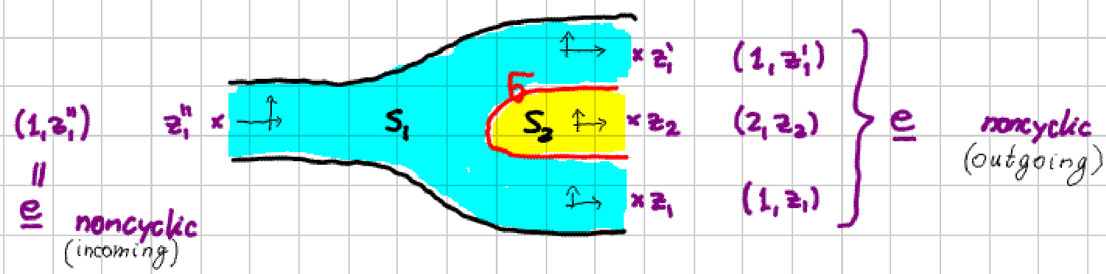


Def$^{\underline{n}}$: The ends of $\underline{S}$ are $\underline{e} \in \mathcal{E}(\underline{S}) = \mathcal{E}^+(\underline{S}) \sqcup \mathcal{E}^-(\underline{S})$

given by maximal sequences $\underline{e} = (k_i, z_{k_i,e_i})_{i=1..N}$ of

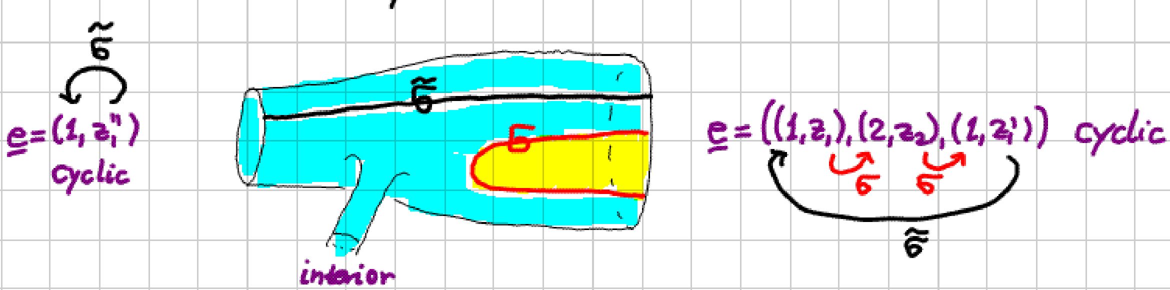
marked points $z_{k_i,e_i} \in \overline{S}_{k_i}$ identified by seams $\sigma_i = \{(k_i, I_i), (k_{i+1}, I'_{i+1})\}$, φ_{σ_i}

$\left(\text{i.e. } \lim_{z \to z_{k_i e_i}} \varphi_{\sigma_i}(z) = z_{k_{i+1}, e_{i+1}} \right)$.



Note • By compatibility of seams with strip-like ends, all marked points in an end $\underline{e}$ are either incoming ($\underline{e} \in \mathcal{E}^-(\underline{S})$) or outgoing ($\underline{e} \in \mathcal{E}^+(\underline{S})$).

• Ends can be
- interior $\underline{e} = (k, z_{k,e} \in \overline{S}_k \setminus \partial \overline{S}_k)$ without seams
- cyclic : with seam $\sigma_N = \{(k_N, z_{k_N, e_N}), (k_1, z_{k_1, e_1})\}$
- noncyclic : without ——— " ———



<u>Symplectic targets</u> for $\underline{S}$ is a tuple $\underline{M} = (M_k)_{k=1..n}$
of symplectic manifolds M_k for each patch S_k.

<u>Lagrangian boundary and seam conditions</u> for $(\underline{S}, \underline{M})$
is a collection $\underline{LBS} = (L_b)_{b\in\mathcal{B}} \cup (L_\sigma)_{\sigma\in\mathcal{S}}$ of

- Lagrangian submanifolds $L_b \subset M_{k_b}$ for each boundary component $b \in \mathcal{B}$,
- Lagrangian correspondences $L_\sigma \subset M^-_{k_\sigma} \times M_{k'_\sigma}$ for each seam $\sigma \in \mathcal{S}$.

A <u>quilt map</u> from $\underline{S}$ to $(\underline{M}, \underline{LBS})$ is a tuple $\underline{u} = (u_k)_{k=1..n}$
of maps $u_k : S_k \to M_k$ satisfying

- Lagrangian boundary conditions $u_{k_b}(I_b) \subset L_b \quad \forall b = (k_b, I_b) \in \mathcal{B}$
- Lagrangian seam conditions $(u_{k_\sigma} \times (u_{k'_\sigma} \circ \varphi_\sigma))(I_\sigma) \subset L_\sigma$

$$\forall \quad \sigma = \{(k_\sigma, \underbrace{I_\sigma), (k'_\sigma, I'_\sigma}_{\varphi_\sigma})\} \in \mathcal{S}.$$

Fix almost complex structures $\underline{J} = (J_k \in \mathrm{End}(TM_k))$, then

<u>$\underline{u}$ is $\underline{J}$-holomorphic</u> iff $\overline{\partial}_{j_k, J_k} u_k = 0 \qquad \forall k = 1..n$.

$$\overline{\partial}_{j_k, J_k} u_k = \tfrac{1}{2}\left(du_k \circ j_k - J_k \circ du_k\right) \in \Omega^1(S_k, u_k^* TM_k)$$

<u>Note</u>: <u>LBS</u> associates a cyclic generalized Lagr. correspondence to each end $\underline{e} = (k_i, z_{k_i, e_i})_{i=1..N}$

- $\underline{e}$ interior : $\underline{\mathcal{L}}_{\underline{e}} = \emptyset$ (~ Hamiltonian Floer homology with $\cap_H \underline{L}_{\underline{e}} = \mathrm{Fix}\, \varphi_H$)
- $\underline{e}$ cyclic with seams $\sigma_1, \ldots \sigma_{N-1}, \sigma_N$: $\underline{\mathcal{L}}_{\underline{e}} = (L_{\sigma_N}, L_{\sigma_1}, \ldots, L_{\sigma_{k-1}})$
- $\underline{e}$ noncyclic with seams $\sigma_1, \ldots, \sigma_{N-1}$ and boundary components b_0, b_N (adjacent to z_{k_1, e_1} , z_{k_N, e_N})
  $\underline{\mathcal{L}}_{\underline{e}} = (L_{b_0}, L_{\sigma_1}, \ldots, L_{\sigma_{N-1}}, L_{b_N})$

<u>Thm</u>: For any "monotone" $(\underline{M}, \underline{LBS})$ $\underline{S}$ induces a relative invariant

$$\Phi_{\underline{S}} : \bigotimes_{\underline{e} \in \mathcal{E}^-(\underline{S})} HF(\underline{\mathcal{L}}_{\underline{e}}) \longrightarrow \bigotimes_{\underline{e}^+ \in \mathcal{E}^+(\underline{S})} HF(\underline{\mathcal{L}}_{\underline{e}^+})$$