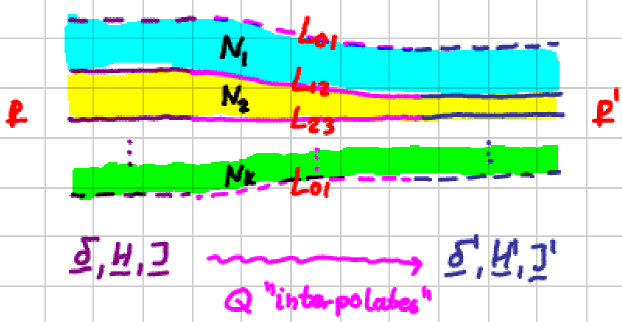
L11 - quilted Floer homology & geometric compositio

Note Title 2/25/2008

<u>Thm</u>: $HF(\underline{\mathcal{L}})$ is independent of $\underline{H}, \underline{\delta}, \underline{J}$.

"<u>Proof</u>": [Salamon, Lectures on F.H., §3.4.] [Schwarz, Morse Homology]

(i)
(ii) Counting holomorphic quilts $\tilde{\mathcal{M}}^0(\underline{\mathcal{L}}, Q, \underline{p}, \underline{p}')$ for regular "quilt data" Q



interpolating $(\underline{\delta}, \underline{H}, \underline{J})$ to $(\underline{\delta}', \underline{H}', \underline{J}')$

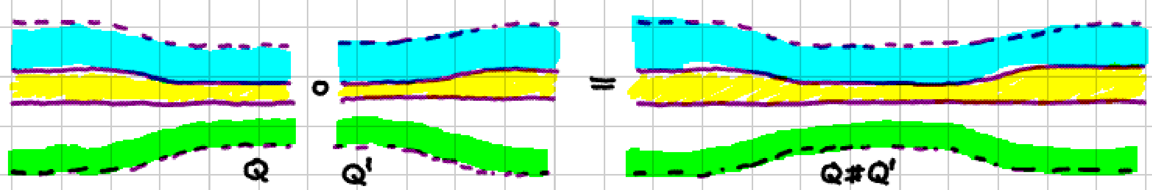
defines a map

$$H\Phi_Q : HF(\underline{\mathcal{L}}, \underline{H}, \underline{\delta}, \underline{J}) \to HF(\underline{\mathcal{L}}, \underline{H}', \underline{\delta}', \underline{J}')$$

Similarly construct $H\Phi_{Q'} : HF(\underline{\mathcal{L}}, \underline{H}', \underline{\delta}', \underline{J}') \to HF(\underline{\mathcal{L}}, \underline{H}, \underline{\delta}, \underline{J})$, then

(iii) $H\Phi_{Q'} \circ H\Phi_Q = H\Phi_{Q\#Q'} \circlearrowright HF(\underline{\mathcal{L}}, \underline{H}, \underline{\delta}, \underline{J})$

"because composition is given by gluing"



$$\left(\begin{array}{ccc} \tilde{\mathcal{M}}^0(Q, \underline{p}, \underline{p}') \times \tilde{\mathcal{M}}^0(Q', \underline{p}', \underline{q}) & \xrightarrow{\sim} & \tilde{\mathcal{M}}^0(Q \#_R Q', \underline{p}, \underline{q}) \\ (\underline{v}, \underline{w}) \xrightarrow{\text{pregluing}} & \underline{v} \#_R \underline{w} & \nearrow \text{implicit function theorem} \\ & & \text{for fixed large } R \end{array} \right)$$

(iv) $H\Phi_{Q\#Q'} = H\Phi_{Q_0}$ "because" $Q\#Q'$ is homotopic to trivial quilt $Q_0 = (\underline{\delta}, \underline{H}, \underline{J})$
below

(v) $H\Phi_{Q_0} = Id_{CF(\underline{\mathcal{L}},\underline{H})}$ since $\tilde{\mathcal{M}}^0(\underline{\mathcal{L}}, Q_0, \underline{p}, \underline{q})$ has an $\mathbb{R}$-action

$\Rightarrow$ solutions are index ≥ 1 or constant strip $\Rightarrow \# \tilde{\mathcal{M}}^0(\underline{p},\underline{q}) = \delta_{\underline{p},\underline{q}}$ $\Rightarrow \Phi = Id$

(iv) A "homotopy $(Q_\tau)_{\tau\in[0,1]}$ of quilts" with fixed ends $(\underline{H}_i, \underline{\delta}_i, \underline{J}_i)$, $i=1,2$

defines a chain homotopy equivalence $T: CF(\underline{\mathcal{L}}, \underline{H}_1) \to CF(\underline{\mathcal{L}}, \underline{H}_2)$

$\boxed{\Phi_{Q_1} - \Phi_{Q_2} = \partial_2 \circ T + T \circ \partial_1}$, thus $HF_{Q_1} = HF_{Q_2} : HF(\underline{\mathcal{L}}, \underline{H}_1, \ldots) \to HF(\underline{\mathcal{L}}, \underline{H}_2, \ldots)$.

We construct $T\langle \underline{p}_1 \rangle := \sum_{\underline{p}_2} \# \hat{\mathcal{M}}^{-1}(\underline{\mathcal{L}}, \{Q_\tau\}, \underline{p}_1, \underline{p}_2) \langle \underline{p}_2 \rangle$

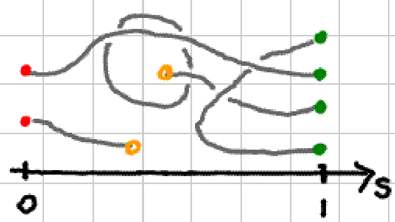
from the index $k=-1$ moduli spaces

$$\hat{\mathcal{M}}^k(\underline{\mathcal{L}}, \{Q_\tau\}, \underline{p}_1, \underline{p}_2) := \{ (\tau, \underline{u}) \mid \tau \in [0,1], \ \underline{u} \in \tilde{\mathcal{M}}^k(\underline{\mathcal{L}}, Q_\tau, \underline{p}_1, \underline{p}_2) \}$$

(index $D_{\underline{u}} = k$)

The $\boxed{\text{identity}}$ follows from $\hat{\mathcal{M}}^0$ having

- true boundary $\tilde{\mathcal{M}}^0(Q_0) \cup \tilde{\mathcal{M}}^0(Q_1)$
- ends ($\to$ compactified boundary)



$$\mathcal{M}^0(\underline{H}_1 \ldots \underline{p}_1, \underline{q}_1) \times \hat{\mathcal{M}}^{-1}(\{Q_\tau\}, \underline{q}_1, \underline{p}_2) \ \cup \ \hat{\mathcal{M}}^{-1}(\{Q_\tau\}, \underline{p}_1, \underline{q}_2) \times \mathcal{M}^0(\underline{H}_2 \ldots, \underline{q}_2, \underline{p}_2)$$

index $\quad 1 \ + \ -1 \quad$ or $\quad -1 \ + \ 1 \ = 0$

<u>Thm</u> [W-Woodward]: $\underline{\mathcal{L}} \sim \underline{\mathcal{L}}'$ equivalent cyclic correspondences

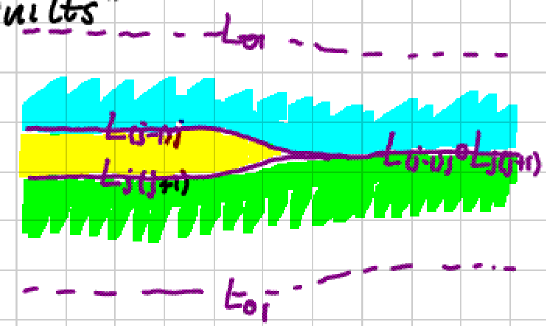
$\Rightarrow HF(\underline{\mathcal{L}}) \cong HF(\underline{\mathcal{L}}')$ isomorphism (induced by sequence of good moves from $\underline{\mathcal{L}}$ to $\underline{\mathcal{L}}'$).

<u>Dream Proof</u> for a <u>good</u> move $\underline{\mathcal{L}} = (\ldots\ L_{(j-1)j}\,,\, L_{j(j+1)}\,,\ldots)$

define $HF(\underline{\mathcal{L}}) \xrightarrow{\sim} HF(\underline{\mathcal{L}}')$ $\qquad \underline{\mathcal{L}}' = (\ldots\ L_{(j-1)j} \circ L_{j(j+1)}\,,\ldots)$

transverse & embedded

by counting "quilts"



...but cannot make sense of  ...

<u>INSTEAD</u>: Define $\Phi : CF(\underline{\mathcal{L}}, \underline{H}) \to CF(\underline{\mathcal{L}}', \underline{H}')$ from a canonical

isomorphism $\cap_{\underline{H}} \underline{\mathcal{L}} \xrightarrow{\sim} \cap_{\underline{H}'} \underline{\mathcal{L}}'$, $(p_1 \ldots p_{j-1}, p_j, p_{j+1} \ldots p_k) \longmapsto (p_1 \ldots p_{j-1}, p_{j+1} \ldots p_k)$

• choose $\underline{H}'$ regular for $\underline{\mathcal{L}}'$, then $\underline{H} := (\ldots H'_{j-1}, H_j = 0, H'_{j+1}, \ldots)$ is regular for $\underline{\mathcal{L}}$

$L_{01} \times \ldots L_{(j-1)j} \circ L_{j(j+1)} \ldots L_{(k-1)k} \pitchfork \tau(\ldots \Delta_{j-1} \times \Delta_{j+1} \ldots)$

$L_{01} \times \ldots \underbrace{L_{(j-1)j} \times L_{j(j+1)}} \ldots L_{(k-1)k} \pitchfork \tau(\ldots \Delta_{j-1} \times \Delta_j \times \Delta_{j+1} \ldots)$

$T(\ldots) \cong T(L_{(j-1)j} \circ L_{j(j+1)}) \oplus D_j$; $D_j \pitchfork \Delta_j \subset N_j^- \times N_j$ since $L_{(j-1)j} \circ L_{j(j+1)}$ is <u>transverse</u>

• $\cap_{\underline{H}} \underline{\mathcal{L}} \xrightarrow{\sim} \cap_{\underline{H}'} \underline{\mathcal{L}}'$ is bijective since $(\varphi(p_{j-1}), p_{j+1}) \in \underline{L_{(j-1)j} \circ L_{j(j+1)}}$ embedded

has a unique $p_j \in N_j$ with $(\varphi(p_{j-1}), p_j) \in L_{(j-1)j}$, $(p_j, p_{j+1}) \in L_{j(j+1)}$

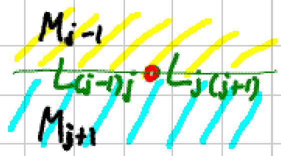
Clearly, $\Phi \circ \Phi^{-1} = Id = \Phi^{-1} \circ \Phi$, but we need to show that Φ descends to homology $H\Phi : HF(\underline{L}) \to HF(\underline{L}')$, i.e. $\partial = \partial'$ on $CF(\underline{L}) \cong CF(\underline{L}')$.

- Fix regular $\underline{H}', \underline{\delta}', \underline{J}'$ for $\underline{L}'$, then $\underline{H} := (H_1' \ldots H_j = 0 \ldots)$,

  $\underline{\delta} := (\delta_1' \ldots \delta_j = \varepsilon \ldots)$, $\underline{J} := (J_1' \ldots J_j(t) = \text{some fixed } J_j \in End(TN_j) \ldots)$

  is regular for $\underline{L}$ for all $\varepsilon > 0$ sufficiently small.
- For $\varepsilon > 0$ suff. small there is a (oriented) bijection

$$\mathcal{M}^0(\underline{L}, \underline{H}, \underline{\delta}, \underline{J}, \underline{p}^-, \underline{p}^+) \xrightarrow{\sim} \mathcal{M}^0(\underline{L}', \underline{H}', \underline{\delta}', \underline{J}', \Phi(\underline{p}^-), \Phi(\underline{p}^+))$$



"■"

Corollary: We can define groups of 2-morphisms in the symplectic category $Symp$: For $M_0, M_1 \in Ob$ (sympl. mfds) and $[\underline{L}], [\underline{L}'] \in Mor(M_0, M_1)/\sim$ (generalized correspondences $M_0 \to M_1$)

$${}^2Mor([\underline{L}], [\underline{L}']) := HF(\underline{L} \# (\underline{L}')^t) \quad \text{is well defined.}$$

$$\begin{tikzcd} & \underline{L} \arrow[dr] & \\ M_0 \arrow[ur] & & M_1 \arrow[dl] \\ & \underline{L}' \arrow[ul] & \end{tikzcd}$$