L10 - quilted Floer homology - invariance

Note Title 2/25/2008

Thm (Gromov compactness) $\underline{L}$ monotone cyclic correspondence

(i) $\mathcal{M}^0(\underline{p}^-,\underline{p}^+)$ is compact if $N_{\underline{L}} \geq 2$ ($\Rightarrow N_{L_{(j+1)j}} \geq 2$)

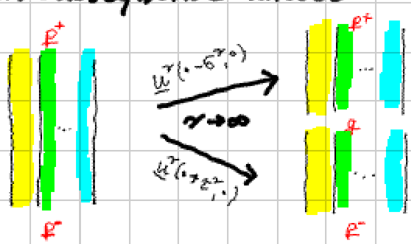
(ii) $\mathcal{M}^1(\underline{p}^-,\underline{p}^+)$ is compact "up to breaking of trajectories" if $N_{L_{(j-1)j}} \geq 3$

"'Proof'": Any sequence $\underline{u}^\nu \in \mathcal{M}^k(\underline{p}^-,\underline{p}^+)$ has bounded energy $E(\underline{u}^\nu) = \tau \underbrace{(k+1)}_{\text{index } D_{\underline{u}^\nu}} + c_{\underline{p}^-,\underline{p}^+}$

"and hence" (!analysis!) has a convergent subsequence unless

(a) energy escapes to $\pm\infty$

"breaking of trajectory"



$\underline{w} \in \mathcal{M}(\underline{q},\underline{p}^+)$

$\underline{v} \in \mathcal{M}(\underline{p}^-,\underline{q})$

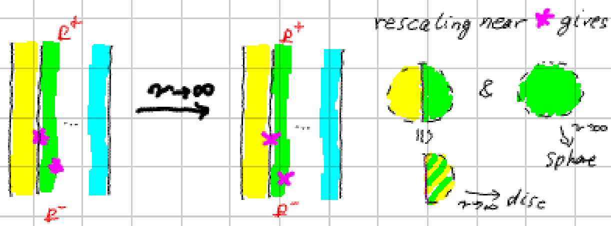
(i): $\underline{v}$ or $\underline{w}$ has index 0 ↯ constant has no energy

index $D_{\underline{u}^\nu}$ = index $D_{\underline{v}}$ + index $D_{\underline{w}}$

(ii): 2 = 1 + 1

(b) energy concentrates at point(s) $*$

"bubbling off"



In the image, a sphere $S^2 \to N_j$ or disc $(D^2,\partial D^2) \to (N_{j-1} \times N_j, L_{(j-1)j})$ forms.

On the domain, $\underline{u}^\nu \to \underline{u}'$ converges on the complement of the point(s) $\times$, and the singularity can be removed to obtain a new solution $\underline{u}' \in \mathcal{M}(\underline{p}^-,\underline{p}^+)$

with less energy $\xRightarrow{\text{monotonicity}}$ less index

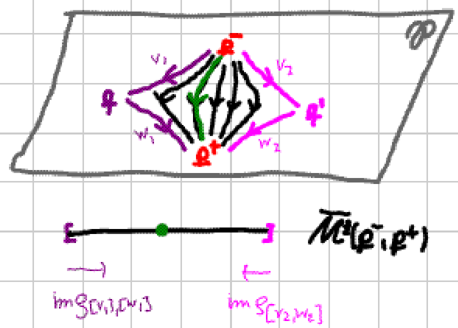
$\mathcal{E} = \tau \cdot \mathrm{Ind} + \mathrm{const}$, $\tau \geq 0$

(i) new index $\leq 1 - N_{\underline{L}} < 0 \Rightarrow \nexists u' \Rightarrow$ no bubbling

(ii) new index $\leq 2 - N_{L_{(j-1)j}} < 0 \Rightarrow \nexists u' \Rightarrow$ no bubbling "■"

To establish $\partial \bar{\mathcal{M}}^1(\underline{p}^-,\underline{p}^+) = \bigcup_{\underline{q} \in \cap_{\underline{L}}} \mathcal{M}^0(\underline{p}^-,\underline{q}) \times \mathcal{M}^0(\underline{q},\underline{p}^+)$

it remains to prove a



<u>Gluing theorem:</u>

There exist embeddings

$\rho_{[\underline{v}],[\underline{w}]} : (R_0,\infty) \hookrightarrow \mathcal{M}^1(\underline{p}^-,\underline{p}^+)$

for each $([\underline{v}],[\underline{w}]) \in \bigcup_{\underline{q} \in \cap_{\underline{L}}} \mathcal{M}^0(\underline{p}^-,\underline{q}) \times \mathcal{M}^0(\underline{q},\underline{p}^+)$

- with disjoint images
- such that $\mathcal{M}^1(\underline{p}^-,\underline{p}^+) \setminus \bigcup \mathrm{im}\, \rho_{[\underline{v}],[\underline{w}]}$ is compact

"<u>Proof</u>": some more in [Salamon, Lectures..., §3.3]

- pregluing: define $\underline{v} \#_R \underline{w} \in \mathcal{B}(\underline{L},\underline{p}^-,\underline{p}^+)$ by interpolating $\underline{v}(\cdot + R,\cdot)$ and $\underline{w}(\cdot - R,\cdot)$, then $\bar{\partial}\, \underline{v} \#_R \underline{w}$ = small
- implicit function theorem gives a nearby zero $\bar{\partial} \rho_{\underline{v},\underline{w}}(R) = 0$ (based on estimates for $D_{\underline{v} \#_R \underline{w}}$)

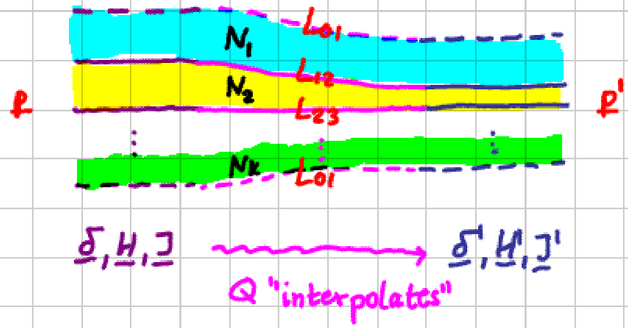
⋮ "■"

__Thm:__ Floer (co)homology groups $HF(\underline{L}) := \ker\partial / \operatorname{im}\partial$ are independent of the choice of $\underline{H}, \underline{\delta}, \underline{J}$; up to isomorphism.

"__Proof__": Construction of isomorphism $HF(\underline{L},\underline{H},\underline{\delta},\underline{J}) \xrightarrow{\sim} HF(\underline{L},\underline{H}',\underline{\delta}',\underline{J}')$

(i) Define $\Phi_Q : CF(\underline{L},\underline{H}) \to CF(\underline{L},\underline{H}')$, $\langle \underline{p} \rangle \mapsto \sum_{\underline{p}' \in \cap_{\underline{H}'} \underline{L}} \# \tilde{\mathcal{M}}^0(\underline{p},\underline{p}') \langle \underline{p}' \rangle$

from 0-dim. moduli space $\tilde{\mathcal{M}}^0(\underline{L}, Q, \underline{p}, \underline{p}')$ of holomorphic quilts



- index 0
- no $\mathbb{R}$-symmetry
- pick "regular" Q

(ii) Φ_Q is a chain homomorphism $\Phi_Q \partial = \partial' \Phi_Q$ "because"

we can exclude bubbling and compactify $\tilde{\mathcal{M}}^1(\underline{L}, Q, \underline{p}, \underline{p}')$ with boundary

$$\bigcup_{\underline{q} \in \cap_{\underline{H}} \underline{L}} \mathcal{M}^0(\underline{L},\underline{H},\underline{J},\underline{\delta},\underline{p},\underline{q}) \times \tilde{\mathcal{M}}^0(\underline{L},Q,\underline{q},\underline{p}') \;\cup\; \bigcup_{\underline{q}' \in \cap_{\underline{H}'} \underline{L}} \tilde{\mathcal{M}}^0(\underline{L},Q,\underline{p},\underline{q}') \times \mathcal{M}^0(\underline{L},\underline{H}',\underline{J}',\underline{\delta}',\underline{q}',\underline{p}')$$

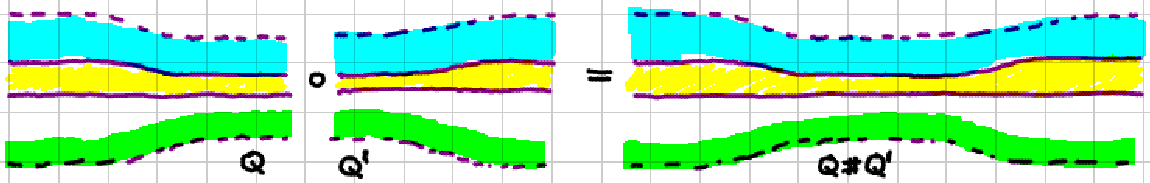
index $\quad 1 \; + \; 0 \quad$ or $\quad 0 \; + \; 1 \quad = 1$

Hence Φ_Q descends to a map $H\Phi_Q : HF(\underline{L},\underline{H},\underline{\delta},\underline{J}) \to HF(\underline{L},\underline{H}',\underline{\delta}',\underline{J}')$.

(iii) Similarly construct $H\Phi_{Q'}: HF(\underline{L}, \underline{H}', \underline{\delta}', \underline{J}') \to HF(\underline{L}, \underline{H}, \underline{\delta}, \underline{J})$, then

- $H\Phi_{Q'} \circ H\Phi_{Q} = H\Phi_{Q\#Q'} \circlearrowleft HF(\underline{L}, \underline{H}, \underline{\delta}, \underline{J})$

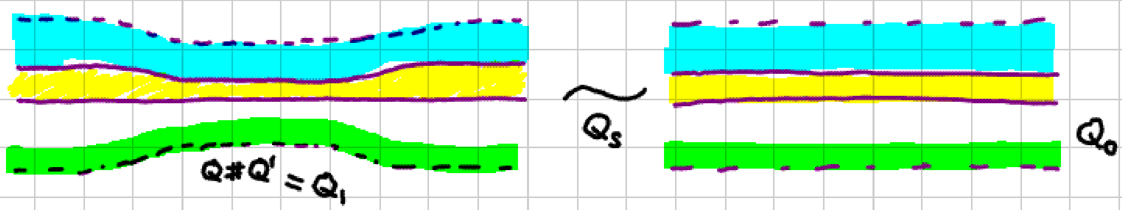
  "because composition is given by gluing"



$$\left(\tilde{\mathcal{M}}^0(Q, \underline{p}, \underline{p}') \times \tilde{\mathcal{M}}^0(Q', \underline{p}', \underline{q}) \xrightarrow{\sim} \tilde{\mathcal{M}}^0(Q \#_R Q', \underline{p}, \underline{q}) \right)$$

$(\underline{v}, \underline{w}) \xrightarrow{\text{pregluing}} \underline{v} \#_R \underline{w}$ — implicit function theorem for fixed large R

- $H\Phi_{Q\#Q'} = H\Phi_{Q_0}$ "because" (see (iv)) $Q\#Q'$ is homotopic to trivial quilt $Q_0 = (\underline{\delta}, \underline{H}, \underline{J})$



- $H\Phi_{Q_0} = Id_{CF(\underline{L}, \underline{H})}$ since $\tilde{\mathcal{M}}^0(\underline{L}, Q_0, \underline{p}, \underline{q})$ has an $\mathbb{R}$-action

  $\Rightarrow$ solutions have index ≥ 1 except for constant strips

  $\Rightarrow \# \tilde{\mathcal{M}}^0(\underline{p}, \underline{q}) = \delta_{\underline{p}, \underline{q}} \Rightarrow \Phi = Id$

(iv) A "homotopy $(Q_s)_{s\in[0,1]}$ of quilts" with fixed ends $(\underline{H}_i, \underline{\delta}_i, \underline{J}_i)$, $i=1,2$

defines a chain homotopy equivalence $T : CF(\underline{\mathcal{L}}, \underline{H}_1) \to CF(\underline{\mathcal{L}}, \underline{H}_2)$

$\Phi_{Q_1} - \Phi_{Q_2} = \partial_2 \circ T + T \circ \partial_1$, thus $HF_{Q_1} = HF_{Q_2} : HF(\underline{\mathcal{L}}, \underline{H}_1, \ldots) \to HF(\underline{\mathcal{L}}, \underline{H}_2, \ldots)$

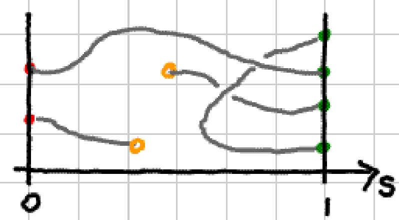
We construct $T\langle p_1 \rangle := \sum_{p_2} \# \hat{\mathcal{M}}^{-1}(\underline{\mathcal{L}}, \{Q_s\}, p_1, p_2) \langle p_2 \rangle$

from the index $k=-1$ moduli spaces

$$\hat{\mathcal{M}}^k(\underline{\mathcal{L}}, \{Q_s\}, p_1, p_2) := \{ (s, \underline{u}) \mid s \in [0,1],\ \underline{u} \in \underbrace{\tilde{\mathcal{M}}^k(\underline{\mathcal{L}}, Q_s, p_1, p_2)}_{} \}$$

(index $D_{\underline{u}} = k$)

The identity follows from $\hat{\mathcal{M}}^0$ having



- true boundary $\tilde{\mathcal{M}}^0(Q_0) \cup \tilde{\mathcal{M}}^0(Q_1)$
- ends ($\to$ compactified boundary)

$$\mathcal{M}^0(\underline{H}_1 \ldots p_1, q_1) \times \hat{\mathcal{M}}^{-1}(\{Q_s\}, q_1, p_2) \cup \hat{\mathcal{M}}^{-1}(\{Q_s\}, p_1, q_2) \times \mathcal{M}^0(\underline{H}_2 \ldots, q_2, p_2)$$

index $1 + -1$ or $-1 + 1 = 0$

"■"

for (a few) more details see [Salamon, Lectures on F.H., §3.4.]