L1 – linear symplectic category, composition of Lagr. corresp.

Note Title 2/7/2008

The linear symplectic category [Guillemin-Sternberg: The moment map revisited]

$(V_0,\omega_0), (V_1,\omega_1)$ symplectic vector spaces

(i.e. ω_j nondegenerate skew-symmetric bilinear form
$(V_j,\omega_j) \underset{isom.}{\cong} (\mathbb{R}^{2n_j}, \sum_{i=1}^{n_j} dx_i \wedge dy_i)$)

- Lagrangian subspace : $\Lambda \subset V_0$ subspace,

$$\omega_0|_\Lambda \equiv 0\ ,\ \dim\Lambda = \tfrac{1}{2}\dim V_0 \quad (\text{or } \omega_0(v,\Lambda)=\{0\} \Rightarrow v\in\Lambda)$$

- linear symplectomorphism: $\Phi : V_0 \xrightarrow{\sim} V_1$ isomorphism, $\Phi^*\omega_1 = \omega_0$

__Note__: $\text{graph}\,\Phi = \{(v_0,\Phi v_0)\,|\,v_0\in V_0\} \subset (V_0\times V_1\ ,(-\omega_0)\oplus\omega_1) =: V_0^- \times V_1$ is a Lagrangian subspace

[• $\dim \text{gr}\,\Phi = \dim V_0 = \frac{1}{2}(\dim V_0 + \underset{\dim V_0}{\underset{\shortparallel}{\dim V_1}})$

• $((-\omega_0)\oplus\omega_1)\,((v_0,\Phi v_0),(v_0',\Phi v_0')) = -\omega_0(v_0,v_0') + \omega_1(\Phi v_0, \Phi v_0')$
$= \text{———} + \underset{\omega_0}{\underset{\shortparallel}{\Phi^*\omega_1}}(v_0,v_0') = 0$]

__Def__$^{\underline{n}}$: A __canonical relation__ (linear Lagrangian correspondence) from V_0 to V_1 is a Lagrangian subspace $\Lambda_{01} \subset V_0^- \times V_1$.

We write $V_0 \xrightarrow{\Lambda_{01}} V_1$ since it generalizes symplectomorphisms.

<u>Def$^{\underline{n}}$</u>: The <u>composition</u> of $V_0 \xrightarrow{\Lambda_{01}} V_1$ and $V_1 \xrightarrow{\Lambda_{12}} V_2$ is

$$\Lambda_{01} \circ \Lambda_{12} := \{ (v_0, v_2) \mid \exists\, v_1 \in V_1 : (v_0, v_1) \in \Lambda_{01}, (v_1, v_2) \in \Lambda_{12} \}$$

$$= \pi_{02} (\underbrace{\Lambda_{01} \times_{\Delta_1} \Lambda_{12}}_{:= (\Lambda_{01} \times \Lambda_{12}) \cap (V_0 \times \Delta_1 \times V_2)}) \subset V_0 \times V_2$$

$\pi_{02} : V_0 \times V_1 \times V_1 \times V_2 \to V_0 \times V_2$ projection

$\Delta_1 = \{(v_1, v_1) \mid v_1 \in V_1\} \subset V_1 \times V_1$ diagonal

<u>Ex.</u>: $V_0 \xrightarrow{\Phi} V_1 \xrightarrow{\Psi} V_2$ symplectomorphisms, composite $\Psi \circ \Phi$

$V_0 \xrightarrow{gr\,\Phi} V_1 \xrightarrow{gr\,\Psi} V_2$

$$gr\,\Phi \circ gr\,\Psi = gr(\Psi \circ \Phi)$$

<u>Lemma</u>: $\Lambda_{01} \circ \Lambda_{12} \subset V_0^- \times V_2$ is a canonical relation

<u>Proof</u>: • $(v_0, v_2), (v_0', v_2') \in \Lambda_{01} \circ \Lambda_{12}$ with corresponding $v_1, v_1' \in V_1$

$$\big((-\omega_0) \oplus \omega_2\big)\big((v_0, v_2), (v_0', v_2')\big) = \overbrace{-\omega_0(v_0, v_0') + \omega_1(v_1, v_1')}^{} \overset{\Lambda_{01}\text{ Lagr.}}{=} 0$$
$$+ \omega_2(v_2, v_2') - \omega_1(v_1, v_1') \underset{\Lambda_{12}\text{ Lagr.}}{=} 0$$

• $\Lambda_{01} \times_{\Delta_1} \Lambda_{12} \subset V_0 \times V_1 \times V_1 \times V_2$ is a subspace of dimension

$$\underbrace{\dim \Lambda_{01} + \dim \Lambda_{12} - \dim V_1}_{= \frac{1}{2} \dim V_0 \times V_2} + \dim \frac{V_0 \times V_1 \times V_1 \times V_2}{(\Lambda_{01} \times \Lambda_{12}) + (V_0 \times \Delta_1 \times V_2)}$$

• $\pi_{02}(\Lambda_{01} \times_{\Delta_1} \Lambda_{12})$ is a subspace of dimension

$$\dim(\Lambda_{01} \times_{\Delta_1} \Lambda_{12}) - \dim(\Lambda_{01} \times_{\Delta_1} \Lambda_{12} \cap \overset{= \ker \pi_{02}}{\{0\} \times V_1 \times V_1 \times \{0\}})$$

$$\frac{V_0\times V_1\times V_1\times V_2}{(\Lambda_{01}\times\Lambda_{12})+(V_0\times\Delta_1\times V_2)} \cong V_1 \big/ \pi_1(\Lambda_{01})+\pi_1(\Lambda_{12}) \cong \pi_1(\Lambda_{01})^{\perp\omega_1} \cap \pi_1(\Lambda_{12})^{\perp\omega_2}$$

$$/\!/ *$$

$$\Lambda_{01}\times_{\Delta_1}\Lambda_{12} \cap \{0\}\times V_1\times V_1\times\{0\} \cong \{v\in V_1 \mid (0,v)\in\Lambda_{01},\ (v,0)\in\Lambda_{12}\}$$

$$* \quad \pi_1(\Lambda_{01})^{\perp\omega_1} = \{v\in V_1 \mid \underbrace{\overset{-\omega_0(0,v_0)}{\omega_1(v,v_1)} = 0 \quad \forall (v_0,v_1)\in\Lambda_{01}}_{\Leftrightarrow \quad (0,v)\in\Lambda_{01}^{\perp_{-\omega_0+\omega_1}} = \Lambda_{01}} \}$$

$$= \{v\in V_1 \mid (0,v)\in\Lambda_{01}\} \qquad \blacksquare$$

We define the linear symplectic category <u>Symp</u> by

- objects : (V,ω) symplectic vector space
- morphisms $\mathrm{Mor}(V_0,V_1)$: canonical relations $V_0 \xrightarrow{\Lambda_{01}} V_1$
- composition – as above
- identity $\mathrm{Mor}(V,V) \ni 1_V := \Delta_V \subset V^- \times V$ diagonal

TO CHECK: composition is associative, $1_V \circ \Lambda = \Lambda = \Lambda \circ 1_V$

Lagrangian correspondences and geometric composition

$(M_0,\omega_0), (M_1,\omega_1)$ symplectic manifolds

$$\left[\begin{array}{l} \omega_j \text{ nondegenerate 2-form }, \quad d\omega_j = 0 \\ \text{Darboux : } \quad (M_i,\omega_i) \underset{\substack{\text{locally}\\ \text{diffeom.}}}{\cong} (\mathbb{R}^{2n_i}, \Sigma dx_i \wedge dy_i) \end{array}\right]$$

- Lagrangian submanifold : $L \subset M_0$ submanifold

  $\omega_0|_L \equiv 0$, $\dim L = \frac{1}{2}\dim M_0$ (or $\omega_0(v, T_xL) \equiv 0 \Rightarrow v \in T_xL \quad \forall x \in L$)

- symplectomorphism : $\varphi : M_0 \xrightarrow{\sim} M_1$ diffeom. , $\varphi^*\omega_1 = \omega_0$

Note: $\operatorname{graph}\varphi = \{(m_0,\varphi(m_0)) \mid m_0 \in M_0\} \subset M_0^- \times M_1 := (M_0 \times M_1, -\pi_0^*\omega_0 + \pi_1^*\omega_1)$ with $\pi_0 : M_0 \times M_1 \to M_0$, $\pi_1 : M_0 \times M_1 \to M_1$

is a Lagrangian submanifold

$\rightsquigarrow$ —— " —— correspondence from M_0 to M_1

Def$^{\underline{n}}$: A Lagrangian correspondence from M_0 to M_1 is a Lagrangian submanifold $L_{01} \subset M_0^- \times M_1$; short $M_0 \xrightarrow{L_{01}} M_1$.

Ex.: Lagr. corresp. $pt \to M$ $\hat{=}$ Lagr. submanifolds of M

Lagr. corresp. $M \to pt$ $\hat{=}$ Lagr. submanifolds of M^-

⚠ A Lagr. submfd $L \subset M_0^- \times M_1 \times M_2^-$ can be a correspondence

$$\begin{aligned} M_0 &\to M_1 \times M_2^- \\ M_0 \times M_1^- &\to M_2^- \\ pt &\to M_0^- \times M_1 \times M_2^- \\ M_0 \times M_1^- \times M_2 &\to pt \end{aligned}$$

<u>Def$^{\underline{n}}$</u>: The <u>dual</u> of $M_0 \xrightarrow{L_{01}} M_1$ is $M_1 \xrightarrow{L_{01}^t} M_0$

$$L_{01}^t = \{(v_1, v_0) \mid (v_0, v_1) \in L_{01}\} \subset M_1^- \times M_0 .$$

<u>Ex.</u>: $(gr\,\varphi)^t = gr(\varphi^{-1})$

Exercise: When is $L_{01} \circ L_{01}^t = \Delta_{M_0}$, $L_{01}^t \circ L_{01} = \Delta_{M_1}$, or both ?

<u>Def$^{\underline{n}}$</u>: The <u>geometric composition</u> of $M_0 \xrightarrow{L_{01}} M_1$ and $M_1 \xrightarrow{L_{12}} M_2$ is

$$L_{01} \circ L_{12} := \{(x_0, x_2) \mid \exists x_1 \in M_1 : (x_0, x_1) \in L_{01}, (x_1, x_2) \in L_{12}\}$$

$$= \pi_{02}(L_{01} \times_{\Delta_1} L_{12}) \quad \subset M_0^- \times M_2$$

<u>Ex</u>: $gr\,\varphi \circ gr\,\psi = gr(\psi \circ \varphi)$

$$gr\,\varphi_0 \circ L_{01} = (\varphi_0 \times \mathrm{Id}_{M_1})(L_{01}) \quad , \quad L_{01} \circ gr\,\varphi_1 = (\mathrm{Id}_{M_0} \times \varphi_1)(L_{01})$$

(IDENTITY) $\quad \Delta_{M_0} \circ L_{01} = L_{01} \quad , \quad L_{01} \circ \Delta_{M_1} = L_{01}$

(ASSOCIATIVITY) $\quad (L_{01} \circ L_{12}) \circ L_{23} = L_{01} \circ (L_{12} \circ L_{23})$

(DUALITY) $\quad (L_{01} \circ L_{12})^t = L_{12}^t \circ L_{01}^t$

<u>Def$^{\underline{n}}$</u>: $L_{01} \circ L_{12}$ is

- <u>transverse</u> if $(L_{01} \times L_{12}) \pitchfork (M_0 \times \Delta_{M_1} \times M_2)$

  i.e. $\left(T_{(x_0,x_1)}L_{01} \times T_{(x_1,x_2)}L_{12}\right) + \left(T_{x_0}M_0 \times T_{(x_1,x_1)}\Delta_{M_1} \times T_{x_2}M_2\right) = T_{(x_0,x_1,x_1,x_2)}M_0 \times M_1 \times M_1 \times M_2$

  for all $(x_0, x_1, x_1, x_2) \in (L_{01} \times L_{12}) \cap (M_0 \times \Delta_{M_1} \times M_2)$.

- <u>embedded</u> if $\forall (x_0, x_2) \in L_{01} \circ L_{12} \ \exists! x_1 \in M_1 : \begin{array}{l} (x_0, x_1) \in L_{01} \\ (x_1, x_2) \in L_{12} \end{array}$

<u>Lemma</u>: (i) transverse $\Rightarrow L_{01} \times_{M_1} L_{12} \subset M_0 \times M_1 \times M_1 \times M_2$ is a submanifold

and $\pi_{02} : L_{01} \times_{M_1} L_{12} \to M_0 \times M_2$ is an immersion

(ii) transverse & embedded $\Rightarrow L_{01} \circ L_{12} \subset M_0^- \times M_2$ is a Lagr. correspondence

(and π_{02} an embedding)

<u>Proof</u>: (i) implicit function theorem for $L_{01} \times L_{12} \longrightarrow M_1 \times M_1 \xrightarrow{\text{local coordinates}} \mathbb{R}^{2n_1}$

$(x_0, x_1, x_1', x_2) \mapsto (x_1, x_1') \longmapsto x_1 - x_1'$

$$\underbrace{\ker d\pi_{02} \cong}_{\text{as in linear Lemma}} T(M_0 \times M_1 \times M_1 \times M_2) \big/ (TL_{01} \times TL_{12}) + (TM_0 \times T\Delta_1 \times TM_2) \underbrace{=}_{\text{transversality}} \{0\}$$

(ii) $\pi_{02} : L_{01} \times_{M_1} L_{12} \longrightarrow M_0^- \times M_2$ is injective ∎