Symplectic Category: correspondences, quilts & topological applications

Note Title 5/23/2008

- joint w. Chris Woodward $\rightarrow$ arXive
- A_∞-version in progress w. Sikimeti Ma'u

Motivations:

① Mirror Symmetry: Symplectic Geometry $\mathcal{F}(X^\vee)$ of $X^\vee$ $\longleftrightarrow$ Complex Geometry $\mathcal{C}(X)$ of X

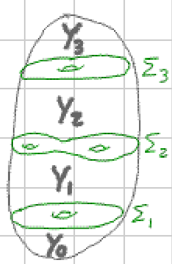
$\boxed{\text{will construct } \mathcal{F}(X^\vee \times X^\vee) \to \mathrm{Fun}(\mathcal{F}(X^\vee), \mathcal{F}(X^\vee)) \quad \text{"resolutions" of Lagrangians}}$

$\mathcal{C}(X \times X) \xrightarrow{\sim} \mathrm{Fun}(\mathcal{C}(X), \mathcal{C}(X))$ resolutions of sheaves

② Topology via Symplectic Geometry — $\boxed{\text{provide general framework}}$

Y 3-manifold $\xrightarrow[\text{into simple pieces}]{\text{decompose}}$ $Y = \bigcup_{i=1} Y_i$ $\xrightarrow{\text{go symplectic}}$ $\underline{L}(Y, \text{choice})$ $\xrightarrow[\text{holomorphic curves}]{\text{count}}$ $HF(\underline{L})$

! choice !



...

symmetric products & vanishing cycles

$\boxed{SU(n) \text{ representations}}$

$\uparrow L(Y_3)$
$M(\Sigma_3)$
$\uparrow L(Y_2)$
$M(\Sigma_2)$
$\uparrow L(Y_1)$
$M(\Sigma_1)$
$\uparrow L(Y_0)$

$\downarrow$

$\boxed{\text{topological invariant of } Y \quad \text{! independent of choice !}}$

category → objects e.g manifolds
→ morphisms e.g. maps $M_0 \xrightarrow{\varphi} M_1$
→ composition e.g. $M_0 \xrightarrow{\varphi} M_1 \xrightarrow{\psi} M_2 = M_0 \xrightarrow{\psi\circ\varphi} M_2$

WISH: define "transposed morphism" $M_0 \xleftarrow{\varphi^T} M_1$

→ $\varphi^T = \varphi^{-1}$ if φ is bijective

➡ allow **correspondences**: any subset $L_{01} \subset M_0 \times M_1$ is a morphism $M_0 \xrightarrow{L_{01}} M_1$

Ex: $\mathrm{graph}\,\varphi = \{(m_0, \varphi(m_0)) \mid m_0 \in M_0\} \subset M_0 \times M_1$

- transposition $L_{01}^T = \{(m_1, m_0) \mid (m_0, m_1) \in L_{01}\} \subset M_1 \times M_0$

$(\mathrm{graph}\,\varphi)^T = \{(\varphi(m_0), m_0)\} = \{(m_1, m_0) \mid m_0 \in \varphi^{-1}(m_1)\} = \text{"graph"}\,\varphi^{-1}$

- composition $\underset{M_0\times M_1}{L_{01}} \circ \underset{M_1 \times M_2}{L_{12}} = \{(m_0, m_2) \mid \exists\, m_1 \in M_1 : (m_0, m_1) \in L_{01}, (m_1, m_2) \in L_{12}\}$

$m_2 = \psi(\varphi(m_0))$

$\mathrm{graph}\,\varphi \circ \mathrm{graph}\,\psi = \{(m_0, m_2) \mid m_1 = \varphi(m_0), m_2 = \psi(m_1)\} = \mathrm{graph}(\psi\circ\varphi)$

Problem: $L_{01} \circ L_{12} = \Pi_{M_0 \times M_2}(L_{01} \times L_{12} \cap M_0 \times \Delta_{M_1} \times M_2)$

does not inherit (e.g. smooth) structure from L_{01}, L_{12}

good case: $\mathbb{R}^{2n_0} \xrightarrow{L_{01}} \mathbb{R}^{2n_1} \xrightarrow{L_{12}} \mathbb{R}^{2n_2}$ linear Lagrangian correspondences

$L \subset \mathbb{R}^{2m} \cong \mathbb{C}^m$ $\mathbb{R}$-subspace, $L \oplus_{\mathbb{R}^{2m}} iL = \mathbb{R}^{2m}$

$\Rightarrow \mathbb{R}^{2n_0} \xrightarrow{L_{01}\circ L_{12}} \mathbb{R}^{2n_2}$ linear Lagr. corresp.

symplectic category **Symp** defined by

objects: (M,ω) symplectic manifold

charts in $\left(\mathbb{C}^n = \{(x_k+iy_k)_{k=1..n}\},\ \omega_0 = \sum_{k=1}^n dx_k \wedge dy_k\right)$

($\rightsquigarrow$ compatible almost complex structures $J \subset TM$, $J^2=-\mathbb{1}$)

morphisms: - symplectomorphisms (for $M_0 \cong M_1$) $M_0 \xrightarrow{\varphi} M_1$, $\varphi^*\omega_1 = \omega_0$

- [Weinstein] Lagrangian correspondences $M_0 \xrightarrow{L_{01}} M_1$

$L_{01} \subset M_0^- \times M_1$ Lagrangian submanifold: $(-\omega_0)\oplus\omega_1|_{L_{01}} \equiv 0$
$\dim L_{01} = \frac{1}{2}\dim(M_0 \times M_1)$

Ex: $(-\omega_0)\oplus\omega_1|_{\mathrm{graph}\,\varphi} = (-\omega_0 + \varphi^*\omega_1)|_{M_0} \equiv 0$

- [WW] chains of Lagrangian correspondences $M_0 \xrightarrow{\underline{L}} M_1$

$\underline{L} = (N_0 \xrightarrow{L_{01}} N_1 \xrightarrow{L_{12}} \dots \xrightarrow{L_{(k-1)k}} N_k)$, $N_0 = M_0$, $N_k = M_1$
$= (L_{01}, L_{12}, \dots, L_{(k-1)k})$

composition: $Mor(M_0,M_1) \times Mor(M_1,M_2) \to Mor(M_0,M_2)$

⚠ to ensure $L_{01} \circ L_{12} = \Pi_{02}(L_{01} \times L_{12} \cap M_0 \times \Delta_{M_1} \times M_2)$ is Lagr. submfd
should assume injectivity & transversality

→ composition by concatenation: $M_0 \xrightarrow{L_{01}} M_1 \xrightarrow{L_{12}} M_2 = M_0 \xrightarrow{(L_{01},L_{12})} M_2$

⚠ should have isomorphism
if $L_\alpha \circ L_\beta$ injective & transverse

$$(\dots \to N_{j-1} \xrightarrow{L_\alpha} N_j \xrightarrow{L_\beta} N_{j+1} \to \dots) \cong (\dots \to N_{j-1} \xrightarrow{L_\alpha \circ L_\beta} N_{j+1} \to \dots)$$

(Restricting to monotone or exact manifolds)

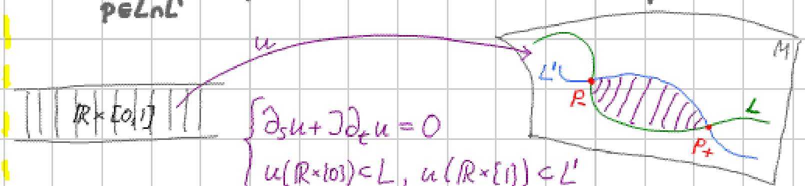
Thm: Symp can be extended to a 2-category in which injective & transverse $M_0 \xrightarrow{L_{01}\circ L_{12}} M_2$ is isomorphic to $M_0 \xrightarrow{L_{01}} M_1 \xrightarrow{L_{12}} M_2$.

[Donaldson/Floer/Fukaya]

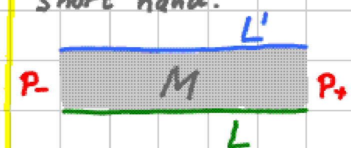
natural morphisms between Lagrangian submanifolds $L, L' \subset M$ are

Floer homology classes $f = \Big[\sum_{p \in L\cap L'} \overset{\#}{n_p} \langle p\rangle\Big] \in \ker\partial / \mathrm{im}\,\partial$

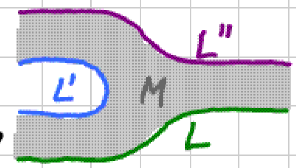
$\partial \circlearrowright \bigoplus_{p\in L\cap L'} \mathbb{Z}\langle p\rangle$ "counts" J-holomorphic curves



short hand:

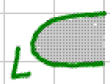


with composition $HF(L',L'') \otimes HF(L,L') \to$



$\to HF(L,L'')$

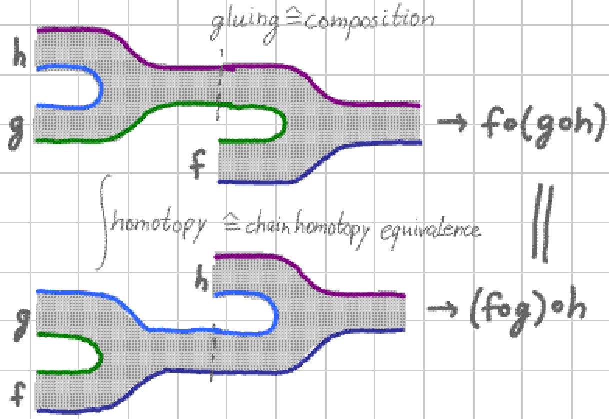
and identity morphisms



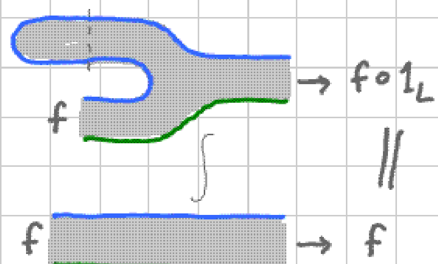
$\to 1_L \in HF(L,L)$

Proof of associativity

gluing ≙ composition



Proof of identity



similarly $1_L \circ g = g$

- Lagrangians $L \subset M$ are elliptic boundary conditions

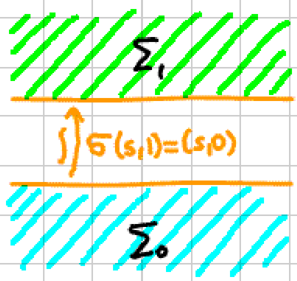
  for a J-holomorphic map $u : \Sigma \to M$ $\quad \boxed{\bar\partial_J u = 0, \ u(\partial\Sigma) \subset L}$

- Lagrangian correspondences $M_0 \xrightarrow{L_{01}} M_1$ are elliptic $\boxed{\text{seam}}$ conditions

  for pairs of J_i-holomorphic maps $u_i : \Sigma_i \to M_i$ $\quad ; \boxed{\sigma : \partial\Sigma_0 \xrightarrow{\sim} \partial\Sigma_1}$

$$\boxed{\bar\partial_{J_0} u_0 = 0 \ , \ \bar\partial_{J_1} u_1 = 0 \ , \ \boxed{(u_0 \times u_1 \circ \sigma)(\partial\Sigma_0) \subset L_{01}}}$$

because locally



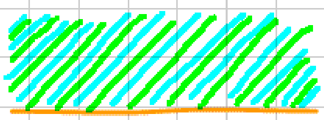
"patch" $\quad u_1 : \Sigma_1 \supset \mathbb{R}\times[0,1) \to M_1$

"seam"

"patch" $\quad u_0 : \Sigma_0 \supset \mathbb{R}\times(0,1] \to M_0$

$$\begin{cases} \bar\partial_{J_1} u_1 = 0 \\ \underline{(u_0(s,1), u_1(s,0)) \in L_{01}} \\ \bar\partial_{J_0} u_0 = 0 \end{cases}$$

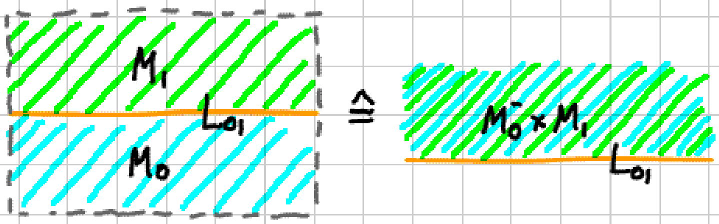
is equivalent to



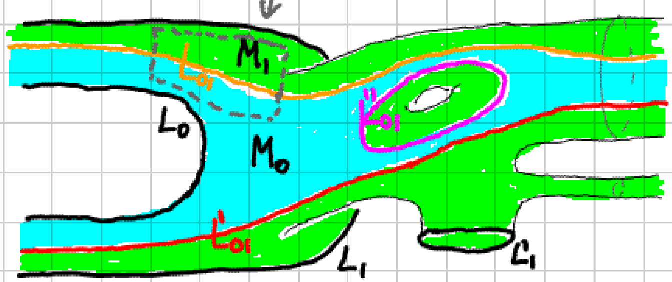
$\underbrace{(u_0(\cdot, -\cdot), u_1)}_{u_{01}} : \mathbb{R}\times[0,1) \to M_0 \times M_1$

$$\begin{cases} \bar\partial_{(-J_0, J_1)} u_{01} = 0 \\ \underline{u_{01}(s,0) \in L_{01}} \end{cases}$$

We picture patches and seams as parts of one surface and indicate target spaces M_i and seam conditions L_{ij}



This can be part of a more complicated **quilt** with genus, true boundary, and cylindrical / strip-like ends.



"Counting" holomorphic quilts $(u_i : \Sigma_i \to M_i \,;\ \bar\partial_{J_i} u_i = 0 \,;$ seam- & boundary conditions$)$

defines a **relative invariant** on quilted Floer homology, e.g.

$$\boxed{\Phi : HF(L_0, L_{01}, L_1) \otimes HF(L_1, L'_{01}, L_0) \longrightarrow HF(\Delta_{M_1}) \otimes HF(L''^{T}_{01}, L_{01})}$$

depends only on $(\Sigma_0, \Sigma_1, \text{seams})/\text{diffeom.}$ and L'_1, L''_{01}.

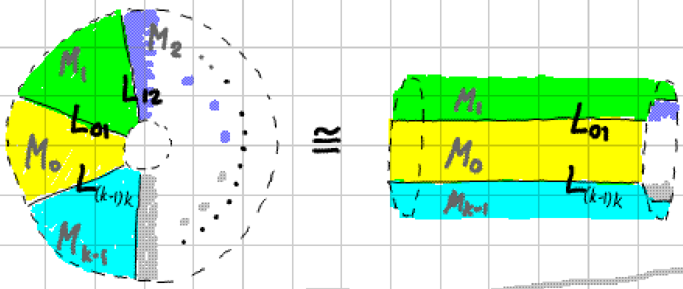
$\boxed{\text{Quilted Floer homology}}$ for cyclic chains of Lagrangian correspondences

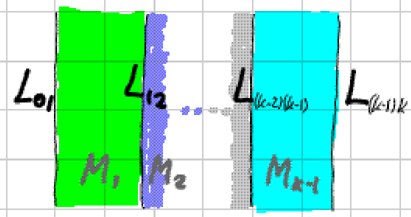
$$\mathrm{HF}\left(\begin{tikzcd} & M_1 \arrow[dr, "L_{12}"] & \\ M_0 \arrow[ur, "L_{01}"] & & \vdots \arrow[dl] \\ & M_{k-1} \arrow[ul, "L_{(k-1)k}"] & \end{tikzcd}\right)$$

special case: $M_0=\{\text{point}\}$

$$\mathrm{HF}\left(pt \xrightarrow{L_{01}} M_1 \xrightarrow{L_{12}} \dots \to M_{k-1} \xrightarrow{L_{(k-1)k}} pt\right)$$

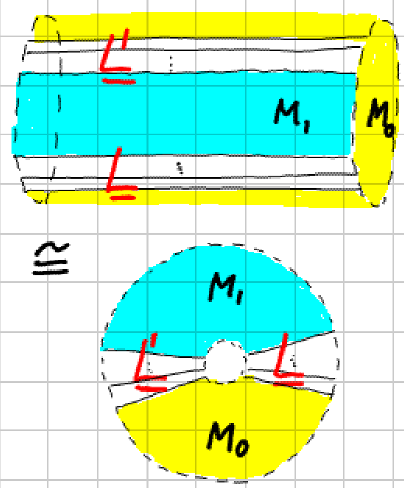
defined by "counting" quilts of holomorphic strips





$$\mathrm{HF}(\underline{L},\underline{L}') := \mathrm{HF}\left(M_0 \begin{array}{c} \overbrace{\xrightarrow{L_{01}} N_1 \to \dots \xrightarrow{L_{(k-1)k}}}^{\underline{L}} \\ \underbrace{\xrightarrow{L'_{01}} N'_1 \to \dots \xrightarrow{L'_{(k'-1)k'}}}_{\underline{L}'} \end{array} M_1 \right)$$

counts



generators: "intersection points" $\underline{L} \cap \underline{L}'$
= quilts of constant maps

The (monotone) symplectic 2-category Symp

objects: (M,ω) symplectic manifold

morphisms: $\mathrm{Mor}(M_0,M_1) = \{ M_0 \xrightarrow{\underline{L}} M_1$ chain of Lagrangian correspondences $\}$

composition (algebraic): $\mathrm{Mor}(M_0,M_1) \times \mathrm{Mor}(M_1,M_2) \longrightarrow \mathrm{Mor}(M_0,M_2)$

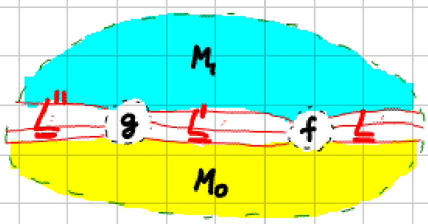
$$[M_0 \xrightarrow{\underline{L}_{01}} M_1] , [M_1 \xrightarrow{\underline{L}_{12}} M_2] \longmapsto [M_0 \overset{\underline{L}_{01}\#\underline{L}_{12}}{\longrightarrow M_1 \longrightarrow} M_2]$$

identity: $1_M := [\emptyset] \in \mathrm{Mor}(M,M)$ (and $\Delta_M \subset M^- \times M$ is a weak identity)

2-morphisms: ${}^2\mathrm{Mor}(M_0 \xrightarrow{\underline{L}} M_1 , M_0 \xrightarrow{\underline{L}'} M_1) = HF(\underline{L},\underline{L}')$

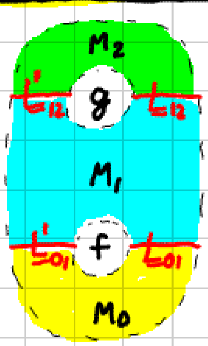
2-composition: ${}^2\mathrm{Mor}(\underline{L}, \underline{L}') \otimes {}^2\mathrm{Mor}(\underline{L}', \underline{L}'') \longrightarrow {}^2\mathrm{Mor}(\underline{L}, \underline{L}'')$

$$(f,g) \longmapsto f \circ g$$



2-identity: $1_{\underline{L}} \in {}^2\mathrm{Mor}(M_0 \xrightarrow{\underline{L}} M_1 , M_0 \xrightarrow{\underline{L}} M_1)$

composition functor: $\mathrm{Mor}(M_0,M_1) \times \mathrm{Mor}(M_1,M_2) \longrightarrow \mathrm{Mor}(M_0,M_2)$



$$\underline{L}_{01} \qquad \underline{L}_{12} \longmapsto \underline{L}_{01} \# \underline{L}_{12}$$

$$\underline{L}'_{01} \qquad \underline{L}'_{12} \longmapsto \underline{L}'_{01} \# \underline{L}'_{12}$$

$$HF(\underline{L}_{01},\underline{L}'_{01}) \otimes HF(\underline{L}_{12},\underline{L}'_{12}) \longrightarrow HF(\underline{L}_{01}\#\underline{L}_{12}, \underline{L}'_{01}\#\underline{L}'_{12})$$

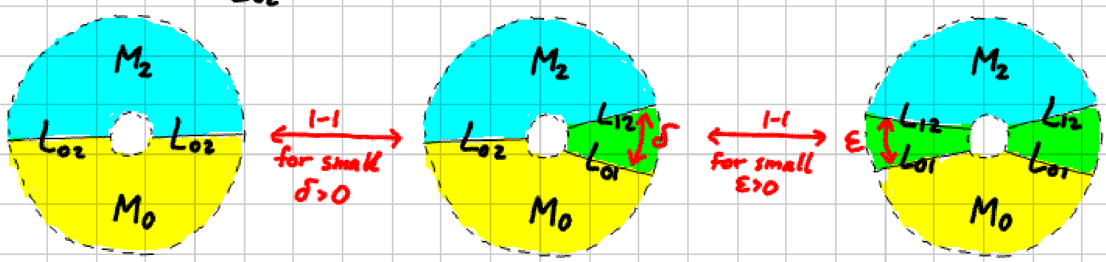
$$(f,g) \longmapsto f \# g$$

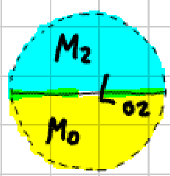
Proof of isomorphism $(L_{01},L_{12}) \simeq L_{01}\circ L_{12}$ in Symp

→ construct 2-morphisms $(L_{01},L_{12}) \underset{\beta}{\overset{\alpha}{\rightleftarrows}} L_{01}\circ L_{12}$ s.t. $\alpha\circ\beta = 1_{(L_{01},L_{12})}$
$\beta\circ\alpha = 1_{L_{01}\circ L_{12}}$

using isomorphism

$$HF(L_{01}\circ L_{12}, \underbrace{L_{01}\circ L_{12}}_{L_{02}}) \cong HF((L_{01},L_{12}), L_{01}\circ L_{12}) \cong HF((L_{01},L_{12}),(L_{01},L_{12}))$$

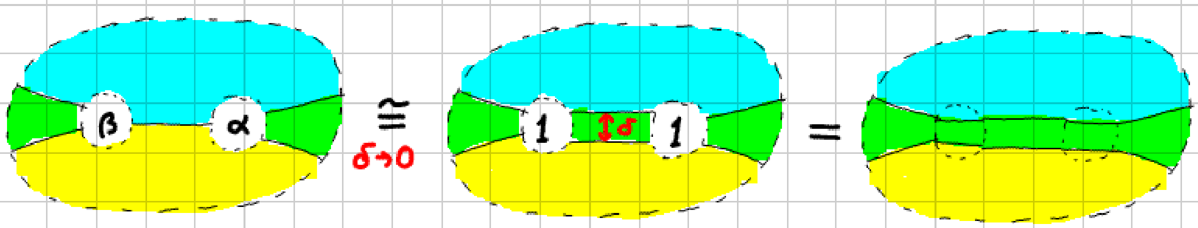




$1_{L_{02}} \cong \alpha \cong 1_{(L_{01},L_{12})}$

$\cong \quad \beta \quad \cong$ similarly

$\beta \in HF(L_{01}\circ L_{12},(L_{01},L_{12}))$



$\alpha\circ\beta = 1_{(L_{01},L_{12})}\circ 1_{(L_{01},L_{12})} = 1_{(L_{01},L_{12})}$

similarly $\beta\circ\alpha = 1_{L_{02}}$

Corollary (Categorification): There exists a functor

$$\mathrm{Symp} \longrightarrow \mathrm{Cat}$$

[Objects: categories
Morphisms: functors
– composition & identity functor]

Theorem: This extends to a 2-functor ${}^2\mathrm{Symp} \to {}^2\mathrm{Cat}$,
[HF-class $\longmapsto$ natural transformation]

in particular we have a functor

$$\mathfrak{S}(M_0, M_1) \longrightarrow \mathrm{Fun}(\mathfrak{S}(M_0), \mathfrak{S}(M_1))$$

where $\mathrm{Mor}(M_0, M_1)$ and $\mathrm{Mor}(pt, M_0)$ are enriched Donaldson-Fukaya categories

of Lagrangian corresp. $L_{01} \subset M_0^- \times M_1$ or $M_0 \to \dots \to M_1$ and Lagrangians $L \subset M_0$ or $pt \to \dots \to M_0$

$\longrightarrow$ Applications in Mirror Symmetry once established in A_∞-setting [$\to$ joint w. S. Ma'u]

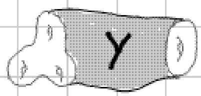
further
Applications

→ symplectic (e.g. nondisplaceable $S^3 \subset \mathbb{CP}^1 \times \mathbb{CP}^2$)

→ topological invariants & Topological Quantum Field Theories

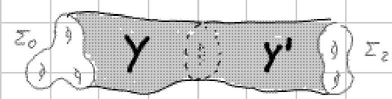
Top – a topological category, e.g. 2+1 cobordism

- objects : Σ 2-dim manifold
- morphisms : $\Sigma_0 \xrightarrow{Y} \Sigma_1$ 3-dim cobordism

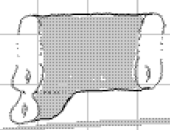


- composition by gluing

$$\Sigma_0 \xrightarrow{Y} \Sigma_1 \xrightarrow{Y'} \Sigma_2 = \Sigma_0 \xrightarrow{Y \cup_{\Sigma_1} Y'} \Sigma_1$$



Note: all morphisms are compositions of simple morphisms : handle attachments



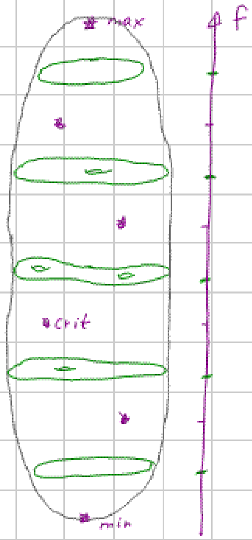
(just defined on simple morphisms)

Corollary: Any (partial) functor Top → Symp gives rise to a TQFT Top → Cat.

Corollary: Construct topological invariants by decomposition, e.g.

$$\underset{\text{3manifold}}{Y} \xrightarrow{\text{decomposition}} \underbrace{\bigcup_{j=1}^{N} Y_{(j-1)j}}_{\text{simple}} \xrightarrow{\text{partial functor}} \underline{L}_{Y,f} \xrightarrow{\text{Floer homology}} HF(\underline{L}_{Y,f})$$

independent of f up to isomorphism $\Longrightarrow$ well defined



$$\begin{array}{ccc}
\Sigma_4 & & M_{\Sigma_4} = pt \\
\uparrow Y_{34} & & \uparrow L_{Y_{34}} \\
\Sigma_3 & & M_{\Sigma_3} \\
\uparrow Y_{23} & & \uparrow L_{Y_{23}} \\
\Sigma_2 & & M_{\Sigma_2} \\
\uparrow Y_{12} & & \uparrow L_{Y_{12}} \\
\Sigma_1 & & M_{\Sigma_1} \\
\uparrow Y_{01} & & \uparrow L_{Y_{01}} \\
\Sigma_0 & \longrightarrow & M_{\Sigma_0} = pt
\end{array}$$

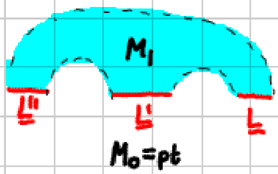
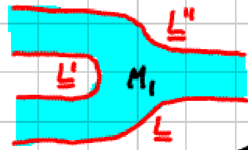
e.g. take moduli spaces of flat $SU(n)$-bundles (with fixed holonomy around punctures/lines)

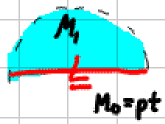
functor Symp → Cat

- M symplectic $\longmapsto$ $Mor(pt, M) =: Don^{\#}(M)$ Donaldson-Fukaya category

  Objects: gen. Lagrangians $pt \xrightarrow{\underline{L}} \ldots \to M$

  Morphisms: HF classes

  composition:  $\hat{=}$ $HF(\underline{L}', \underline{L}'') \otimes HF(\underline{L}, \underline{L}')$  $HF(\underline{L}, \underline{L}'')$ $\Phi_{\underline{S}}$

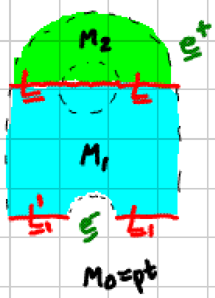
  identity:  $\hat{=}$  $\in HF(\underline{L}, \underline{L})$

- $M_1 \xrightarrow{\underline{L}} M_2$ gen. Lagr. corresp. $\longmapsto$ $\Phi_{\underline{L}} : Don^{\#}(M_1) \longrightarrow Don^{\#}(M_2)$ functor

  $\underline{L}_1$ ($pt \to \ldots \to M_1$) $\longmapsto$ $\underline{L}_1 \# \underline{L}$ ($pt \to \ldots \to M_1 \to \ldots \to M_2$)

  $f \longmapsto f \# 1_{\underline{L}}$

  $f \in HF(\underline{L}_1, \underline{L}_1')$, $f \# 1_{\underline{L}} \in HF(\underline{L}_1 \# \underline{L}, \underline{L}_1' \# \underline{L})$



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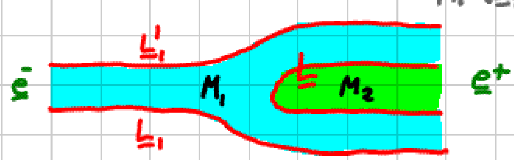


<u>Proof</u>: Given any 2-category $\mathcal{C}$ and distinguished object p_0 there is a functor of 1-categories $\mathcal{C} \to \mathcal{C}at$ given by

object $p \quad \longmapsto \quad Mor(p_0, p)$ is a category

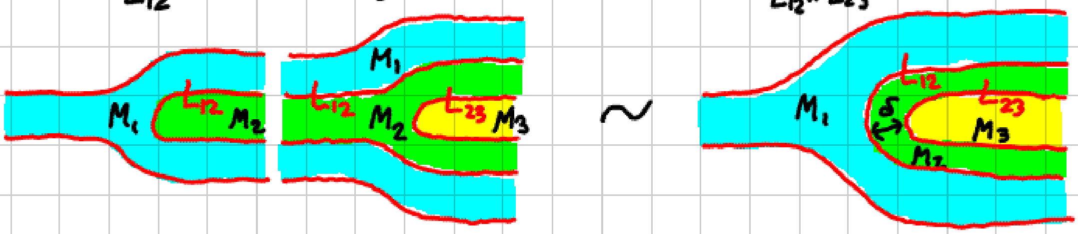
morphism $p \xrightarrow{h} q \quad \longmapsto \quad Mor(p_0,p) \longrightarrow Mor(p_0,q)$ is a functor

$Id \times (h, 1_h) \searrow \; Mor(p_0,p) \times Mor(p,q) \nearrow$ composition functor of $\mathcal{C}$

We picked $p_0 = pt$ as distinguished object in $\mathcal{S}ymp$.

Another proof of functoriality

$$\Phi_{L_{12}} \circ \Phi_{L_{23}} = \Phi_{L_{12} \# L_{23}}$$

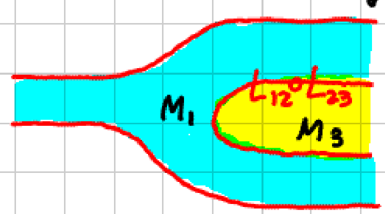


$\delta \to 0$

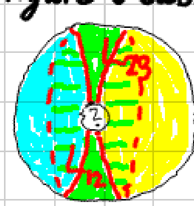
Thm: For good geometric composition $L_{12} \circ L_{23}$

$$\boxed{\Phi_{L_{12}} \circ \Phi_{L_{23}} = \Phi_{L_{12} \circ L_{23}}}$$

⚠ crucially requires monotonicity

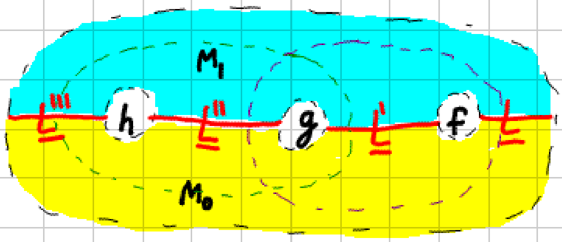


<u>obstructions</u>: hol. disks on $L_{12}, L_{23}, L_{12} \circ L_{23}$ and "figure 8 bubble"

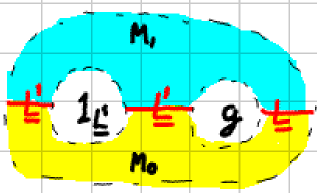


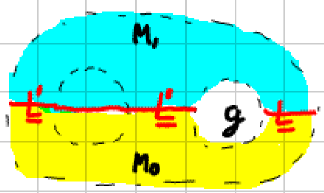
MORE PROOFS

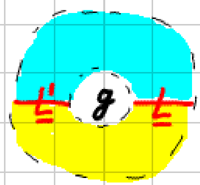
* associativity:

$(f \circ g) \circ h \underset{\text{gluing}}{=}$  $\underset{\text{gluing}}{=} f \circ (g \circ h)$

* $\forall g \in HF(\underline{L}, \underline{L}')$ $\quad g \circ 1_{\underline{L}'} = g \quad$ and $\quad 1_{\underline{L}} \circ g = g$

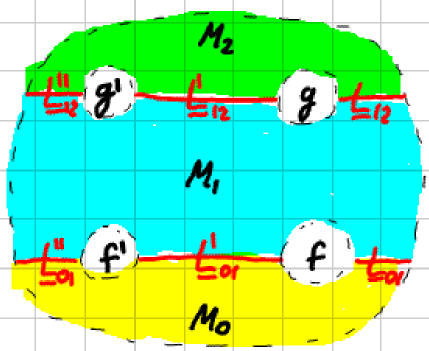


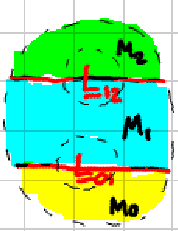
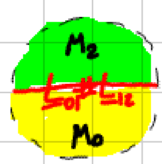
$=$ 

$\sim$ 

strips with $\mathbb{R}$-action

$\simeq Id(g)$

* $(f \circ f') \# (g \circ g') =$  $= (f \# g) \circ (f' \# g')$

* $1_{\underline{L}_{01}} \# 1_{\underline{L}_{12}} =$  $=$  $= 1_{\underline{L}_{01} \# \underline{L}_{12}}$