Introduction to Polyfolds

Note Title 2012

Polyfolds

a new technology
by Hofer-Wysocki-Zehnder

brought to you
by Katrin Wehrheim

"classical" transversality & gluing

$Aut(\Sigma) \circlearrowright$ $\mathcal{Y} = \bigcup_{u\in\mathcal{X}} \Omega^{0,1}(\Sigma, u^*TW)$ $\downarrow \uparrow s = \bar\partial$ $\mathcal{X} = \{u : \Sigma \to W\}$

→ describes moduli spaces as zero sets $\mathcal{M} = s^{-1}(0)/Aut(\Sigma)$

of Fredholm sections in Banach bundles modulo a Lie group

→ identifies ends of noncompact moduli spaces with

fibered products of lower dimensional moduli spaces

TO DO: * (geometric) construction of perturbations

$s \approx s' \pitchfork 0$-section ; $Aut(\Sigma)$-equivariant, compatible with gluing

* "the usual" analysis (100s of pages if done ~~properly~~ lovingly
little remains strictly quotable when setup changes)

Defect: smooth structure on compactified moduli space $\bar{\mathcal{M}}$

(with boundary & corners) requires additional constructions

based on: IMPLICIT FUNCTION THEOREM : $Ds|_{s^{-1}(0)}$ surjective
(i.e. $s \pitchfork 0$-sect.)
$\Rightarrow s^{-1}(0)$ manifold

actual idea/wish: generalize

TRANSVERSALITY: $Y \overset{\uparrow s}{\downarrow} X$ finite dim. vector bundle, $s^{-1}(0)$ compact
$\Rightarrow s+p \pitchfork 0$-section for generic p compactly supported
$(s+p_1)^{-1}(0) \sim (s+p_2)^{-1}(0)$ cobordant for p_1, p_2 as above

↳ to ∞-dim. function spaces

Polyfold Fredholm theory & operations

→ describes **compactified moduli spaces** as zero sets of **"Fredholm" sections in "polyfold" bundles** $\overline{\mathcal{M}} = s^{-1}(0)$ $\mathcal{Y} \underset{\downarrow}{\overset{\uparrow s}{}} \mathcal{X}$

→ identifies (codim. 1) **boundaries of moduli spaces** with fibered products of lower dimensional moduli spaces

AND encodes
- counts of 0-dim. moduli spaces
- relations from 1-dim. moduli spaces

in a general algebraic structure : "operations"

<u>TO DO</u>: understand compactified moduli space as a set, define appropriate ambient space & section, then quote

<u>TRANSVERSALITY & IMPLICIT FUNCTION THEOREM</u>

manifold version – for symmetries use orbifold version (without "M")

$\mathcal{X}$ **M-polyfold**, $\mathcal{Y} \to \mathcal{X}$ <u>strong</u> <u>M-polybundle</u>

with fibres for compact perturbations

$s : \mathcal{X} \to \mathcal{Y}$ **sc^∞-Fredholm** section ; $s^{-1}(0)$ compact

Fredholm after "filling"

$\Rightarrow \exists\, p : \mathcal{X} \to \mathcal{Y}$ <u>sc^+</u>-section (sufficiently small & supported near $s^{-1}(0)$)

compact operator

such that $(s+p)^{-1}(0)$ is a <u>smooth compact manifold with boundary with corners.</u>

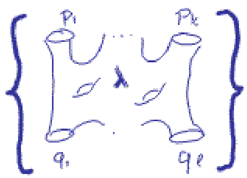
(And $(s+p_1)^{-1}(0) \sim (s+p_2)^{-1}(0)$ cobordant for different choices p_1, p_2)

Application to SFT

$$\mathcal{Y} \quad \mathcal{Y}_u = \Omega^{0,1}(u^* TW) \quad L^p$$

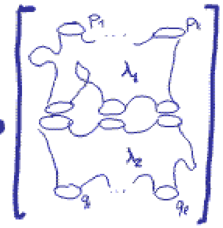
$$\mathcal{Y} \downarrow \; \big) \; s(u) = \bar\partial u$$

"classical": $\widehat{\mathcal{M}}_q^p(\lambda) = s^{-1}(0) \subset \mathcal{X}_q^p(\lambda) = \left\{ \begin{array}{l} u : \Sigma \to W \quad W^{1,p} \\ \text{ends} \to p, q \,,\; [u] = \lambda \end{array} \right\}$

$$\mathcal{M}_q^p(\lambda) = \widehat{\mathcal{M}}_q^p(\lambda) / \text{reparametrization} = \{$$  $$\}$$

$$\overline{\mathcal{M}}_q^p(\lambda) = \{\text{holomorphic buildings}\}$$

"new age": $\overline{\mathcal{M}}_q^p(\lambda) = \bar{s}^{-1}(0)$

$$\overline{\mathcal{X}}_q^p(\lambda) = \left\{ \begin{array}{l} (u_1, \dots u_k) \text{ not necc. holomorphic building} \quad W^{1,p} \\ \text{of } \underline{\text{any}} \text{ height } k;\ \text{ends } p, q\,;\ [u_1 \# \dots u_k] = \lambda \end{array} \right\} \Big/ \text{reparametrization} \ni$$ 

$$\overline{\mathcal{Y}}_{(u_1, \dots u_k)} = \Omega^{0,1}(u_1^* TW) \times \dots \times \Omega^{0,1}(u_k^* TW) \qquad \underset{\text{strong}}{\underline{W^{1,p}}}$$

$$\bar{s}(u_1, \dots, u_k) = (\bar\partial u_1, \dots, \bar\partial u_k)$$

<u>issues & inspirations</u>:

② dimension of charts for $\mathcal{X}$ & fibers of $\mathcal{Y}$ "jumps" at $\partial\mathcal{X}$

$\rightsquigarrow$ M-polyfolds & -bundles modelled on <u>"splicing cores"</u>

① action of reparametrization group is not smooth

E.g. $\quad \Phi : \mathbb{R} \times \mathcal{C}^1(\mathbb{R}) \longrightarrow \mathcal{C}^1(\mathbb{R}) \qquad$ time shift

$$(\tau, u) \longmapsto \tau * u\,(t) = u(\tau + t)$$

$$D_{(\tau_0, u_0)}\Phi : (T, V) \longmapsto \tau_0 * V + T \cdot (\tau_0 * \dot{u}_0) \;\in \mathcal{C}^1(\mathbb{R}) \text{ if } u_0 \in \mathcal{C}^2(\mathbb{R}) \;\Big/\!\!\Big/\; \mathcal{C}^k \;\; \mathcal{C}^{k+1}$$

↳ scale calculus

A sc-structure on Banach space E is a sequence $\mathbb{E}=(E_m)_{m\in\mathbb{N}_0}$ of subspaces $E_m\subset E$ with Banach norms $\|\cdot\|_{E_m}$ such that • $E_0=E$

$\forall m\geq 0$ • $E_{m+1}\hookrightarrow E_m$ compact, • $E_\infty:=\bigcap_{n\geq 0}E_n$ dense in E_m

Ex.: ~~$E=\mathcal{C}^1(\mathbb{R}), E_m=\mathcal{C}^{m+1}(\mathbb{R})$~~ $\rightsquigarrow E_\infty=\mathcal{C}^\infty(\mathbb{R})$ but $\mathcal{C}^{m+2}\hookrightarrow\mathcal{C}^{m+1}$ only bounded

$E=W^{1,2}(\mathbb{R})$, $E_m=W^{m+1,2}_{\delta_m}(\mathbb{R})=e^{-\delta_m|t|}\cdot W^{m+1,2}(\mathbb{R})$ $\quad 0=\delta_0<\delta_1<\ldots$

$\varphi:\mathbb{E}\to\mathbb{F}$ is sc^0 if $\varphi|_{E_m}:E_m\to F_m$ is $\mathcal{C}^0$ $\forall m\geq 0$

time shift $\Phi:\mathbb{E}\to\mathbb{F}$ is sc^0, i.e. $\Phi:\underbrace{\mathbb{R}\times W^{m+1,2}_{\delta_m}}_{E_m}\to\underbrace{W^{m+1,2}_{\delta_m}}_{F_m}$

$\varphi:\mathbb{E}\to\mathbb{F}$ is sc^1 if

- $\varphi:E_1\to F_0$ differentiable & induces $D\varphi(x_1):E_0\to F_0$ $\forall x_1\in E_1$
- $T\varphi:T\mathbb{E}\to T\mathbb{F}$, $(x,h)\mapsto(\varphi(x),D\varphi(x)h)$ is sc^0 $\left(\text{i.e. } E_{m+1}\times E_m\to F_m,\ (x,h)\mapsto D\varphi(x)h \quad \mathcal{C}^0\ \forall m\right)$

φ is sc^{k+1} if $T\varphi$ is sc^k ; φ is sc$^\infty$ if it is sc^k $\forall k$

Φ is sc$^\infty$, in particular $\Phi:\mathbb{R}\times W^{m+1,2}_{\delta_m}\to W^{1,2}$ is $\mathcal{C}^m$

e.g. $D\Phi:\mathbb{R}\times W^{m+2,2}_{\delta_{m+1}}\times\mathbb{R}\times W^{m+1,2}_{\delta_m}\longrightarrow W^{m+1,2}_{\delta_m}$, $(\tau,u,T,V)\mapsto\tau*V+T\cdot(\tau*\dot{u})$ is $\mathcal{C}^0$ $\forall m$

Chain rule: $\varphi:\mathbb{E}\to\mathbb{F}$, $\psi:\mathbb{F}\to\mathbb{G}$ sc^1 $\Rightarrow$ $\psi\circ\varphi$ sc^1, $T(\psi\circ\varphi)=T\psi\circ T\varphi$

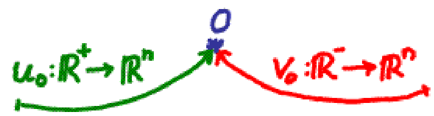
Proof crucially uses compactnes of $E_{m+1}\hookrightarrow E_m$

② ⤳ Gluing & Antigluing ⤳ Splicings & dimension jumps

Goal: charts for space $\mathcal{X}^{\varepsilon}_{\sharp}(N)$ of level 1,2,3,... buildings

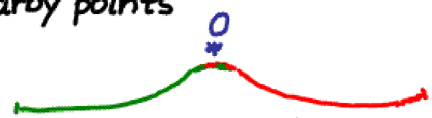
Goal': —"— $\mathcal{X}$ of broken & unbroken paths (in Morse theory)
... locally near crit. point $0 \in \mathbb{R}^n$ (and no $/\mathbb{R}$ yet)

$(u_0, v_0) \in \mathcal{X}$



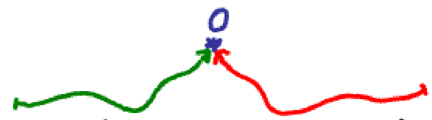
(length $R=\infty$ boundary point)

has nearby points



$w: [0,R] \to \mathbb{R}^n$

(length $1 \ll R < \infty$ interior point)



$(u: \mathbb{R}^+ \to \mathbb{R}^n,\ v: \mathbb{R}^- \to \mathbb{R}^n)$

(length $R=\infty$ boundary point)

parametrize a neighbourhood of (u_0, v_0) in $\mathcal{X}$:

$$[0,\tfrac{1}{2}) \times W^{1,2}(\mathbb{R}^+,\mathbb{R}^n) \times W^{1,2}(\mathbb{R}^-,\mathbb{R}^n) \xrightarrow[\text{not injective!}]{\Phi} \mathcal{X}$$

$\Downarrow r \mapsto R(r) = e^{1/r}$

$(e^2, \infty]$ "gluing profile"

$$(r,h,k) \longmapsto \begin{cases} (u_0+h,\ v_0+k) & ; r=0 \quad (R=\infty) \\ \oplus_{R(r)}(u_0+h,\ v_0+k) & ; r>0 \quad (R=e^{1/r}) \end{cases}$$

pregluing

fiber wise projection to complement of kernel $\Big\downarrow \pi$

$\swarrow\!\nearrow\ \varphi$ **M-polyfold chart**

$$\bigcup_{r\in[0,\frac{1}{2})} \{r\} \times \ker \ominus_{R(r)}$$ **splicing core**

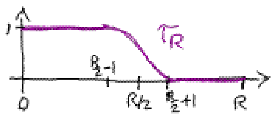
antigluing

Pregluing & Anti(pre)gluing define an isomorphism

$$\oplus_R \times \ominus_R : W^{1,2}(\mathbb{R}^+,\mathbb{R}^n) \times W^{1,2}(\mathbb{R}^-,\mathbb{R}^n) \xrightarrow{\sim} W^{1,2}([0,R],\mathbb{R}^n) \times W^{1,2}(\mathbb{R},\mathbb{R}^n)$$

hence $\ker \ominus_R$ is a complement of $\ker \oplus_R$.

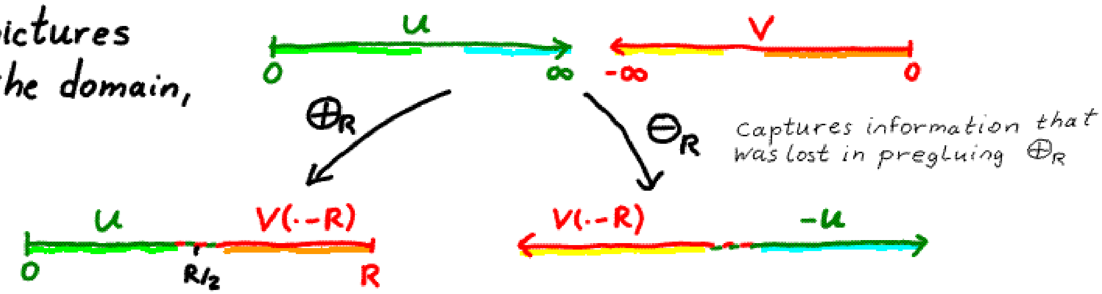
In formulas,



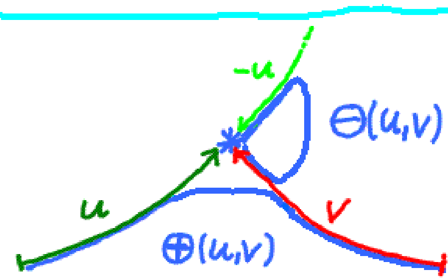
$$(\oplus_R \times \ominus_R)\begin{pmatrix} u \\ v \end{pmatrix} := \begin{pmatrix} \tau_R u + (1-\tau_R) v(\cdot - R) \\ -(1-\tau_R) u + \tau_R v(\cdot - R) \end{pmatrix}$$

is an isomorphism since the matrix $\begin{pmatrix} \tau & 1-\tau \\ -(1-\tau) & \tau \end{pmatrix}$ is

In pictures of the domain,



In picture of the image,



A splicing consists of

$V \subset \mathbb{R}^k \times [0,\infty)^{k'}$ open set (of gluing parameters)

$\mathbb{E}$ Banach space with scale structure

$\pi : V \times \mathbb{E} \to \mathbb{E}$ sc^∞ family of projections $(\pi_v^2 = \pi_v)$

$(v,e) \mapsto \pi_v e$

Gluing & Antigluing Example

$V = [0, \tfrac{1}{2})$

$E = W^{1,2}(\mathbb{R}^+, \mathbb{R}^n) \times W^{1,2}(\mathbb{R}^-, \mathbb{R}^n)$

$E_m = W^{m+1,2}_{\delta_m}(\ldots) \times W^{m+1,2}_{\delta_m}(\ldots)$

$$\pi_r(h,k) = (\oplus_R \times \ominus_R)^{-1}(\oplus_R(h,k), 0) \Big|_{R=e^{1/r}} = \left(\begin{cases} \frac{\tau_R^2}{\tau_R^2+(1-\tau_R)^2} h + \frac{\tau_R(1-\tau_R)}{\tau_R^2+(1-\tau_R)^2} k(\cdot - R) & ; r>0 \\ h & ; r=0 \end{cases}, \begin{Bmatrix} \sim \\ \sim \end{Bmatrix} \right)$$

Its splicing core is $K = \bigcup_{v \in V} \{v\} \times \operatorname{im} \pi_v \subset V \times \mathbb{E}$

The Gluing & Antigluing splicing core is the "fibration"

$$K = \bigcup_{r\in[0,\frac{1}{2})} \{r\} \times \ker \Theta_{R(r)} \qquad \ker \Theta_{R(r)} = \begin{cases} \overset{\oplus_R}{\cong} W^{1,2}([0,R]) & ; \binom{r>0}{R<\infty} \\ W^{1,2}(\mathbb{R}^+) \times W^{1,2}(\mathbb{R}^-) & ; \binom{r=0}{R=\infty} \end{cases}$$

over the set $[0,\frac{1}{2})$ of gluing parameters.

Now, the map $\varphi := \Phi|_K = \bigcup_{r\in[0,\frac{1}{2})} \oplus_{R(r)} : K \to \mathfrak{X}$ is injective, and will serve as chart for the M-polyfold $\mathfrak{X}$.

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finite dimensional example of a splicing core

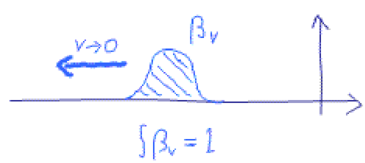
⚠ This example just illustrates the dimension jumps in splicing cores. In all applications, the fibres of splicing cores will be ∞-dimensional.

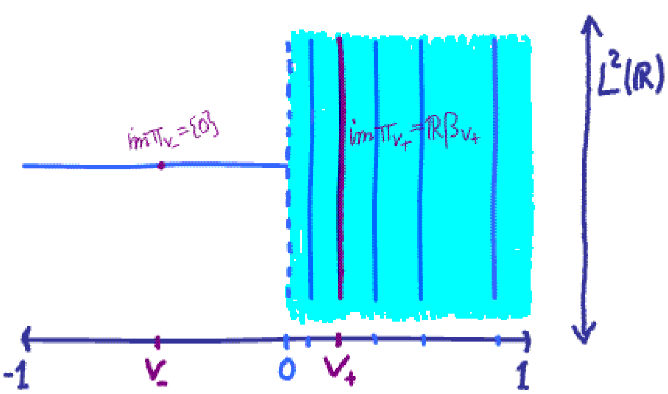
$V = (-1,1)$

$\mathbb{E} = L^2(\mathbb{R})$

$\pi_v f = \beta_v \left(\int f \cdot \beta_v \right)$

$$\beta_v = \begin{cases} \text{bump}(\cdot + e^{1/v}) & , v>0 \\ 0 & ; v \le 0 \end{cases}$$





Check that π is sc^0: Fix $f \in L^2(\mathbb{R})$, then $\int_{\mathbb{R}} f \cdot \beta_v \xrightarrow[v\to 0]{} 0$,

and hence $\lim_{v\to 0} \pi_v f = \lim_{v\to 0} \beta_v \cdot \left(\int f \cdot \beta_v\right) = 0 = \pi_0 f$.

$\mathcal{X}$ 2^{nd} countable Hausdorff space

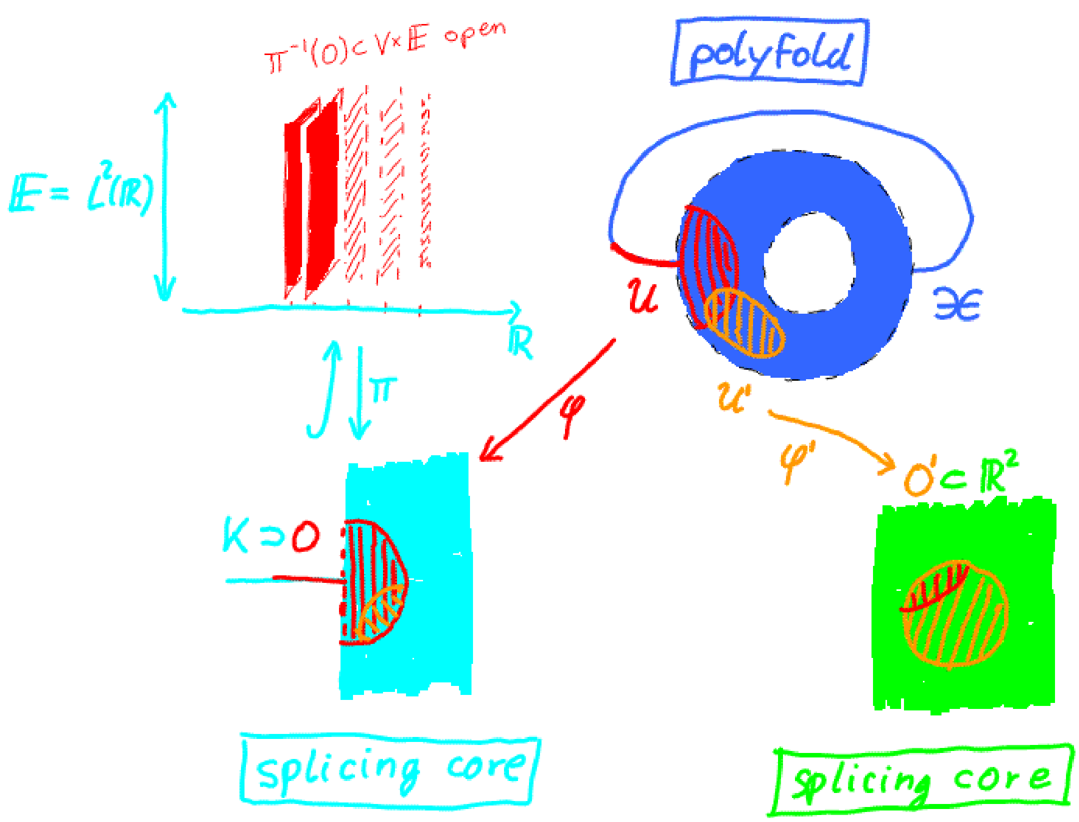
An M-polyfold chart is $\mathcal{X} \underset{\text{open}}{\supset} \mathcal{U} \underset{\text{homeom.}}{\xrightarrow{\varphi}} \mathcal{O} \underset{\text{open}}{\subset} K$ splicing core

Charts $\mathcal{U}_i \xrightarrow{\varphi_i} O_i \subset K_i \subset V_i \times \mathbb{E}_i$ are compatible if

$$\begin{array}{ccccccccc} \boxed{V_1 \times \mathbb{E}_1} & \xrightarrow{\pi} & K_1 & & \mathcal{X} & & & & \\ \cup & & \cup & & \cup & & & & \\ \pi^{-1}(\varphi_1(\mathcal{U}_1 \cap \mathcal{U}_2)) & \xrightarrow{\pi} & \varphi_1(\mathcal{U}_1 \cap \mathcal{U}_2) & \xrightarrow{\varphi_1^{-1}} & \mathcal{U}_1 \cap \mathcal{U}_2 & \xrightarrow{\varphi_2} & K_2 & \hookrightarrow & \boxed{V_2 \times \mathbb{E}_2} \end{array}$$

is sc^∞

An M-polyfold structure on $\mathcal{X}$ is a maximal collection of compatible M-polyfold charts covering $\mathcal{X}$.

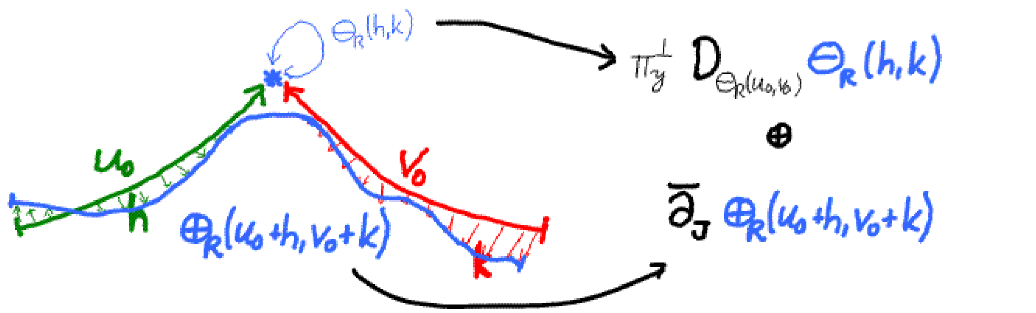


An **M-polybundle** $\mathcal{Y} \xrightarrow{\pi} \mathcal{X}$ is a sc^∞ surjection π between M-polyfolds $\mathcal{X}, \mathcal{Y}$ "with same gluing parameters" and linear fibers.

A **Fredholm section** $s: \mathcal{X} \to \mathcal{Y}$ is a sc^∞ map, $\pi \circ s = Id$, that "can locally be filled up to a Fredholm map $V \times \mathbb{E}_{\mathcal{X}} \to V \times \mathbb{E}_{\mathcal{Y}_{fiber}}$"

(usually by the linearized operator)

filled section



The zero set of a transverse Fredholm section in a M-polybundle is a smooth manifold.

⚠

In applications, the M-polyfold $\mathcal{X}$ and the bundle fibres $\mathcal{Y}_x$ will be ∞-dimensional

...but the zero set $s^{-1}(0)$ still is a finite dimensional manifold...

isn't that neat?!