What is a polyfold and why would one care?

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©! [Eliashberg, Conley-Zehnder, Floer; Hofer-Salamon, Fukaya-Ono, Liu-Tian]

A polyfold[©] is the natural tool for generalizing Floers proof of the Arnold conjecture[©!] to all symplectic manifolds.

"Periodic Hamiltonian systems have at least as many periodic orbits as autonomous systems on the same configuration space."

$H: S^1 \times M \to \mathbb{R}$

$\mathcal{P}_H = \{\gamma : S^1 \to M \mid \dot\gamma(t) = X_{H_t}(\gamma(t))\}$

$h : M \to \mathbb{R} \Rightarrow \mathrm{Crit}\, h \subset \mathcal{P}_h$

(M,ω) compact

$[\omega] \in H^2(M) \quad \omega \wedge \ldots \wedge \omega \neq 0 \in H^{\dim M}(M)$

$\omega(X_H, \cdot) = dH$

i.e. $\underset{\parallel}{\#\mathcal{P}_H} \geq \min_h \#\mathrm{Crit}\, h$

Floer: $\#\mathrm{Fix}\,\varphi_H^1 \geq$ total rank of $H_*(M)$ if $\varphi_H^1 \pitchfork \Delta_M$ (time-1-flow $\pitchfork$ diagonal)

$\#\mathrm{Fix}\,\varphi_H^1$ = generators of the Floer complex

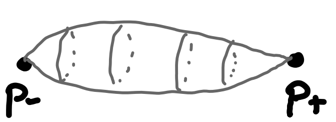
total rank of $H_*(M)$ = homology of Morse complex generated by $\mathrm{Crit}\, h$

Proof: ① Floer homology is well defined and independent of H (nontrivial could use polyfolds, too)

② Find $h: M \to \mathbb{R}$ Morse s.t. $\begin{cases} \mathrm{Fix}\,\varphi_h^1 = \mathrm{Crit}\, h & \Leftarrow h\ \mathcal{C}^2\text{-small} \\ \boxed{\text{Floer } \partial = \text{Morse } \partial} \end{cases}$

$\boxed{\text{Floer } \partial = \text{Morse } \partial}$ for $h: M \to \mathbb{R}$; $J \in End(TM)$, $J^2 = -id$, $\omega(\cdot, J\cdot) = g$ metric ; (h,g) Morse-Smale

$\forall p_\pm \in Crit\, h$ 0-dim parts of regularized trajectory spaces agree



$$\left(\left\{u: \mathbb{R}\times S^1 \to M \;\middle|\; \partial_s u + J\partial_t u = -\nabla h ,\; u(s,\cdot) \xrightarrow[s\to\pm\infty]{} p_\pm \right\}^{reg}\!\!/\mathbb{R}\right)_0$$

$$= \left(\left\{v: \mathbb{R} \to M \;\middle|\; \tfrac{d}{ds} v = -\nabla h ,\; v(s) \xrightarrow[s\to\pm\infty]{} p_\pm \right\}/\mathbb{R}\right)_0$$

because $\{v: \mathbb{R} \to M \ldots\}$ is the fixed point set of S^1-action

$$\boxed{S^1 \times \{u: \mathbb{R}\times S^1 \to M \ldots\} \to \{u: \mathbb{R}\times S^1 \to M \ldots\} \;,\; (\tau, u) \mapsto u(\tau + \cdot)}$$

and one can $\underbrace{\text{regularize}}_{\text{by perturbing a section}}$ $\underbrace{\{u: \mathbb{R}\times S^1 \to M \ldots\}/\mathbb{R} \setminus Fix_{S^1}}_{s^{-1}(0)}$ S^1-equivariantly

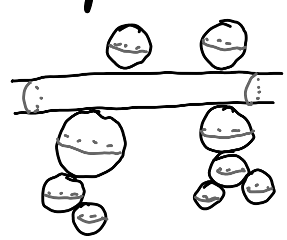
$, \quad \mathcal{E} \downarrow\uparrow s : u \mapsto \partial_s u + J\partial_t u + \nabla h$ over $\mathcal{B}$

"*technical issues*":

ⓐ S^1 does not act $\mathcal{C}^1$ on $\mathcal{B} = \mathcal{C}^\infty(\mathbb{R}\times S^1, M)$ in known Banach metrics

ⓑ $s^{-1}(0)$ must be compactified in $\widetilde{\mathcal{B}}$ = maps from nodal domains to M modulo automorphisms (e.g. $\mathbb{R}$ on $\mathbb{R}\times S^1$)

which is not a Banach manifold



solution of ⓑ: $\widetilde{\mathcal{B}}$ is a polyfold

"$u \mapsto \partial_s u + J\partial_t u + \nabla h$" is a Fredholm section over $\widetilde{\mathcal{B}}$

idea for ⓐ: S^1 acts sc-smoothly on $\widetilde{\mathcal{B}}$

so one could pull back perturbations from $\widetilde{\mathcal{B}}/S^1$