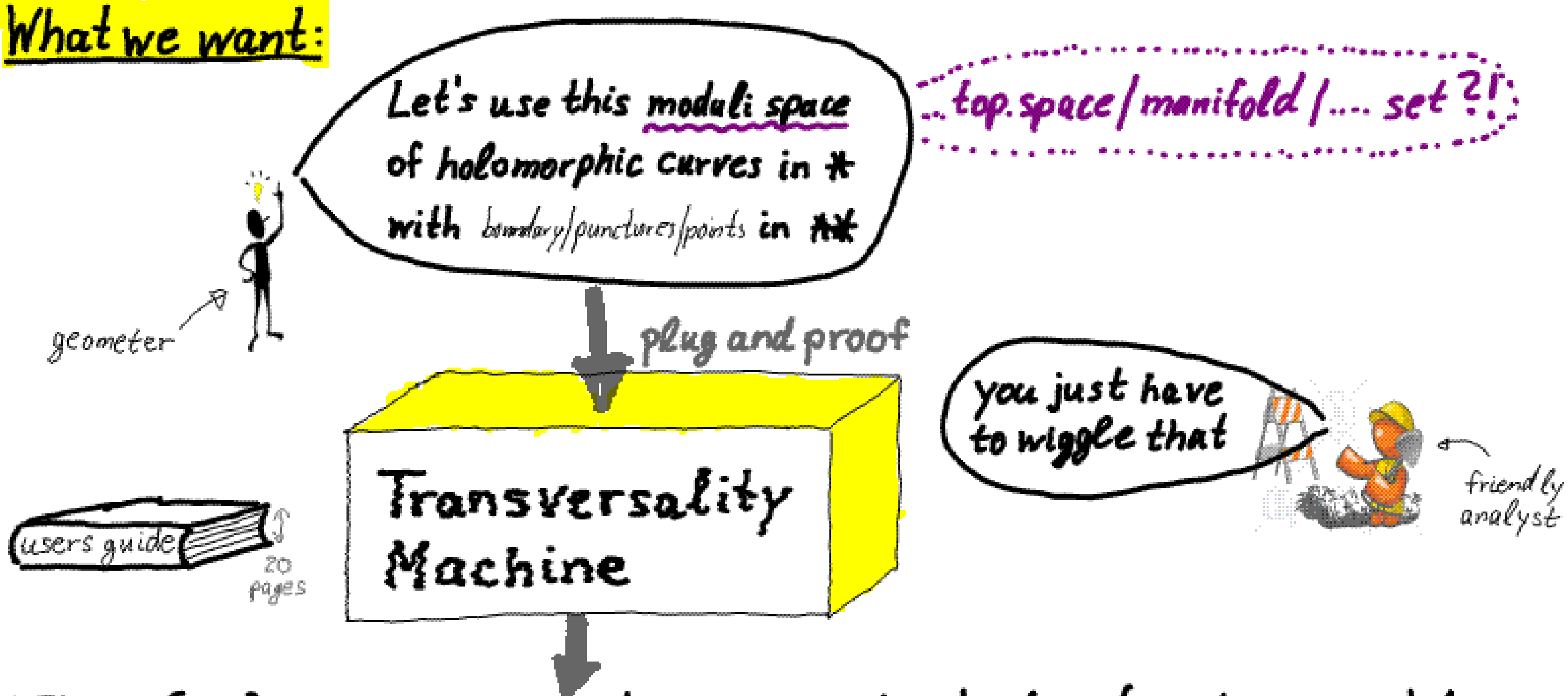
Analytic Foundations and holomorphic disks

See http://math.mit.edu/~katrin/ for

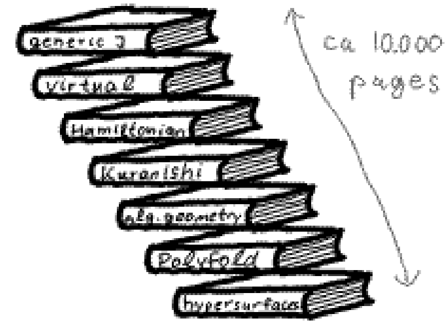
* Introduction to polyfolds (slides from Yashafest)
* my research

What we want:



Thm: {.....} is an invariant / manifold up to cobordism / can be integrated over.

What we have: many ideas, many pages, many doubts



- many geometric solutions in special cases

general approaches

- virtual moduli cycle *[1] [Liu-Tian,..]
- Kuranishi structures *[2] [Fukaya-Oh-Ohta-Ono, Joyce]
- Polyfold theory *[3] [Hofer-Wysocki-Zehnder]

informed users

Dusa McDuff
Mohammed Abouzaid
(Katrin Wehrheim)

and more ?!

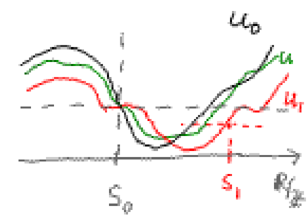
$*^{1}$ seems to be based on

questionable **FACT**: $W^{k,p}(S^2,M)^{*}/Aut(S^2,j_0)$ is a Banach manifold

$*$ not constant

closed manifold

wrong **Proof**: for simplicity $\mathcal{C}^1(\mathbb{R}/\mathbb{Z},\mathbb{R})^{*}/\text{shift} \ni [u_0]$ $\quad u_0'(s_0) \neq 0$



neighbourhood of $[u_0] \sim \{u : \mathbb{R}/\mathbb{Z} \to \mathbb{R} \mid u(s_0) = u_0(s_0)\}$ transition map

neighbourhood of $[u_1] \sim \{v : \mathbb{R}/\mathbb{Z} \to \mathbb{R} \mid v(s_1) = u_1(s_1)\}$ $\quad u \longmapsto v(s) = u(s+\tau_u)$

(shift so that $u(s_1+\tau_u) = u_1(s_1)$)

$$\text{shift}: \quad \sigma : \mathbb{R}/\mathbb{Z} \times \mathcal{C}^1(\mathbb{R}/\mathbb{Z},\mathbb{R}) \longrightarrow \mathcal{C}^1(\mathbb{R}/\mathbb{Z},\mathbb{R})$$

$$(\tau, v) \longmapsto (\tau * v)(s) = v(s+\tau)$$

$$d\sigma(\tau=0, v) : (\delta\tau, \delta v) \longmapsto \delta\tau \cdot v' + \delta v$$

maps $\mathbb{R}\times\mathcal{C}^1 \to \mathcal{C}^1$ only if $v \in \mathcal{C}^2$

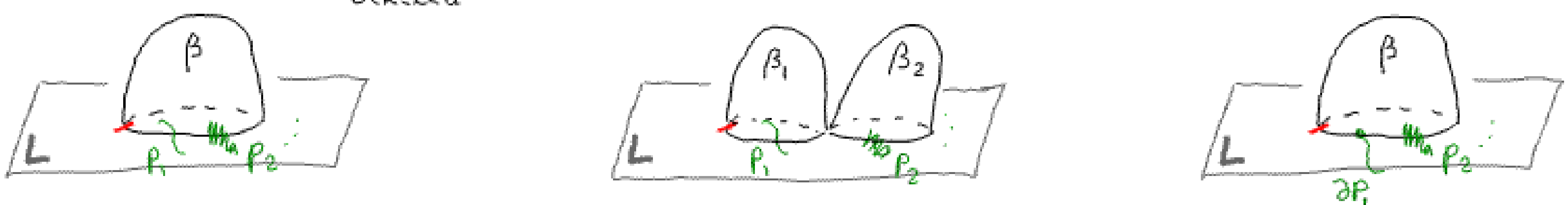
is not differentiable on any Banach space! QED

*² Kuranishi structures are constructed iteratively on

(noncompact) moduli spaces $\mathcal{M}_\beta(P_1,\dots,P_d)$ over • energy/index of $\beta\in H_2(M,L)$
• "complexity" of $P_i\in C_*(L)$

such that compactified spaces have boundary (in what sense?)

$$\partial\overline{\mathcal{M}}_\beta(P_1\dots P_d)=\bigcup_{\substack{\beta_1+\beta_2=\beta\\0\le k\le \ell\le d}}\mathcal{M}_{\beta_1}(P_1\dots P_k,P_{\ell+1},\dots P_d)\underset{L}{\times}\mathcal{M}_{\beta_2}(P_{k+1},\dots,P_\ell)\ \cup\ \sum_{1\le k\le d}\mathcal{M}_\beta(\dots\partial P_k\dots)$$



or at least we have enough fundamental-chain-like behaviour to conclude e.g.

$$\partial\mu_1^Q(P_1)\ =\ \mu_2(P,\mu_0)+\mu_1^Q(\mu_1^Q(P))+\mu_2(\mu_0,P)\ +\ \mu_1^Q(\partial P)$$

$(d=1)$ $(k,\ell)=(1,1)$ $(0,1)$ $(0,0)$

Note $\mu_1=\underset{\beta\neq0}{\mu_1^Q}+\underset{\text{"}\beta=0\text{"}}{\partial}$

*³ Who can wrap their head around polyfolds ???

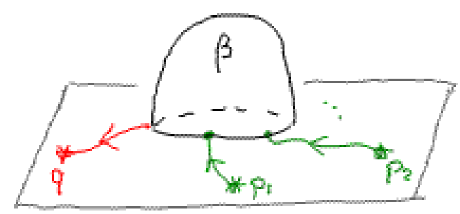
Let's test them on holomorphic disks .... but first ....

[Fukaya-Oh, Cornea-Lalonde, Seidel, ...]: A_∞-perturbation lemma suggests to

$C_*(L)$ $\xrightarrow{\text{replace}}$ $\bigoplus_{p \in Crit f} \mathbb{Z}\langle p \rangle$ for $f: L \to \mathbb{R}$ Morse

$ev_*[\overline{\mathcal{M}}_\beta(p_1 \dots p_d)] \in C_*(L)$

$\cong$ A_∞-equivalent



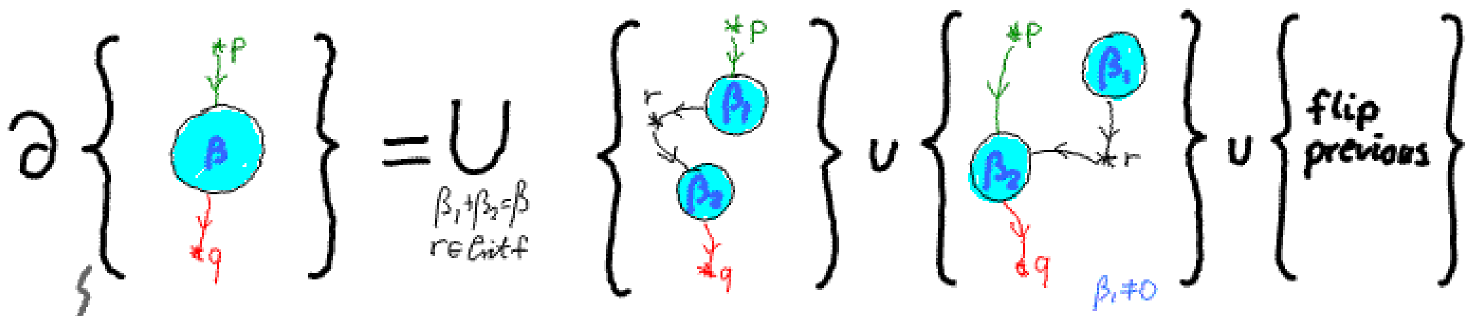
$\# \mathcal{M}_\beta(p_1 \dots p_d ; q) \in \mathbb{Q}$

Gain: finitely generated HF(L), A_∞-algebra, ...

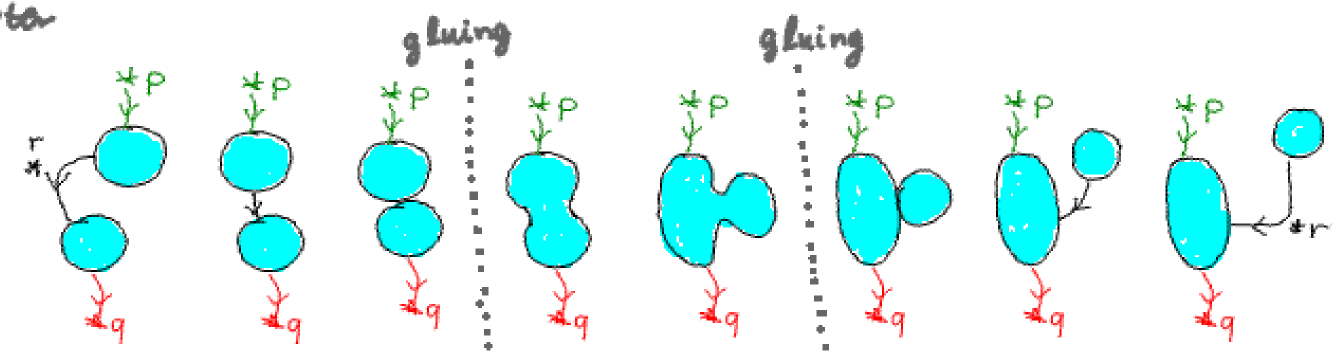
Loss: $\mathcal{M}_\beta(p_1..p_d; q) = \{$trees of holomorphic disks with Morse flow lines$\}$

Why trees? → Composition is gluing of Morse flow lines, so e.g.

dream proof of $\quad 0 = \mu_1 \circ \mu_1 + \mu_2(\cdot, \mu_0) + \mu_2(\mu_0, \cdot)$

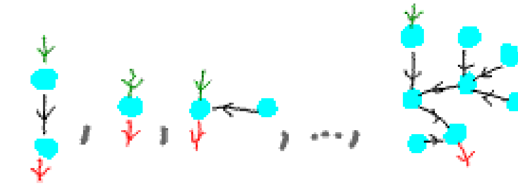


contains e.g. a 1-parameter family



So do we need to glue many moduli spaces ???

$$\mathcal{M}_\beta(p_i;q) = \bigsqcup_{\substack{\text{tree}\\ \text{type } T}} \cancel{\mathcal{M}_\beta(p_i,T,q)}_{\substack{\text{much}\\ \text{gluing}}}$$

e.g. $T \in \Big\{$  $\Big\}$

→ NO, we just ① "preglue" the spaces of maps to build a polyfold

Also allow broken $-\nabla f$-flow lines. (Goal: cut out $\overline{\mathcal{M}} = \bar{\partial}_J^{-1}(0) \subset \mathcal{B}$)

$$\boxed{\mathcal{B}_\beta(p_1..p_d;q)} = \bigcup_{\substack{T \text{ tree}\\ 1 \text{ root}\\ d \text{ leaves}}} \left\{ \left[(u_v)_{\substack{\text{vertices}\\ \text{of } T}}, (\gamma_e)_{\substack{\text{edges}\\ \text{of } T}} \right] \;\middle|\; \begin{array}{l} u_v : (D,\partial D) \to (M,L) \; W^{3,2}\text{-map} \\ \gamma_e \subset L \text{ broken flow line} \\ \gamma_{root}(+\infty) = q \;,\; \gamma_{leaf_j}(-\infty) = p_j \end{array} \right\}$$

matching conditions at marked points

(equivalence class under automorphisms of T, (D, j_0))

$$= \bigcup_{k=0}^{N} \mathcal{B}^k \quad ; \quad \mathcal{B}^k = \{\text{trees with } k \text{ breakings of flow lines}\}$$

$$= \bigcup_{\substack{\text{partitions}\\ \text{of } p_1..p_d\\ \text{and } \beta}} \underbrace{\mathcal{B}^0_\bullet(...) \times \ldots \times \mathcal{B}^0_\bullet(...)}_{k+1}$$

① build polyfold $\mathcal{B}_\beta(p_1..p_d;q)$ [Actually, include trees of spheres - see ③]

② prove that

$$\begin{tikzcd} \mathcal{E} \arrow[d] & {[(\bar\partial_J u_v)]} \\ \mathcal{B}_\beta(p_1..p_d;q) \arrow[u, bend right, "\bar\partial_J"'] & {[(u_v),(\gamma_e)]} \arrow[u, mapsto] \end{tikzcd}$$

is a Fredholm section

[This can likely be quoted / pieced together from standard cases + compact perturbations.]

③ check compactness of $\bar\partial_J^{-1}(0) \subset \mathcal{B}_\beta(p_1..p_d;q)$

[Here we need to choose $\mathcal{B}$ large enough to include all a priori possible bubbling phenomena. E.g. for Hamiltonian Floer theory must include trees of spheres unless $\omega|_{\pi_2(M)}=0$]

④ plug and proof

$\Downarrow$

$\overline{\mathcal{M}}_\beta(p_1..p_d;q)^{\text{perturbed}} \subset \mathcal{B}_\beta(p_1..p_d;q)$

compact branched suborbifold
unique up to cobordism

$\Big\} \Rightarrow$ well-defined $\#\overline{\mathcal{M}} \in \mathbb{Q}$ for 0-dimensional spaces

④ quote **Theorem** [HWZ General Fredholm Theory III]

$\mathcal{E} \to \mathcal{B}$ strong[⊛] polybundle

[⊛] ($\mathcal{E}$ has $W^{k,2}$ fibers over $u \in W^{k,2}$ although $\bar\partial u \in W^{k-1,2} \in \mathcal{E}_u$)

$f : \mathcal{B} \to \mathcal{E}$ Fredholm section, $f^{-1}(0) \subset \mathcal{B}$ compact

$\Rightarrow$ There exists a multisection $\lambda : \mathcal{E} \to \mathbb{Q}^+$ ($\leadsto S_\lambda(u) = \{\xi \in \mathcal{E}_u \,|\, \lambda(\xi) > 0\}$ finite weighted set)

such that the solution set $\{u \in \mathcal{B} \,|\, f(u) \in S_\lambda(u)\}$ inherits the structure of a compact branched orbifold with boundary and corners.

$\Rightarrow$ For any two such $\lambda, \lambda' : \mathcal{E} \to \mathbb{Q}^+$ there is a cobordism of solution sets.

In the absence of symmetries (when the moduli space is the quotient of a free action : $u \circ \varphi = u \Rightarrow \varphi = id$)

we can use the manifold version:

(4') quote **Theorem** [HWZ General Fredholm Theory II]

$\mathcal{E} \to \mathcal{B}$ strong(*) M-polybundle

$f : \mathcal{B} \to \mathcal{E}$ Fredholm section, $f^{-1}(0) \subset \mathcal{B}$ compact

$\Rightarrow$ There exists a (arbitrarily small and supported near $f^{-1}(0)$) section $s : \mathcal{B} \to \mathcal{E}$

such that the solution set $(f+s)^{-1}(0) \subset \mathcal{B}$ inherits the structure

of a compact manifold with boundary and corners.

$\Rightarrow$ For any two such $s, s' : \mathcal{B} \to \mathcal{E}$ there is a cobordism $(f+s)^{-1}(0) \sim (f+s')^{-1}(0)$.

<u>Note</u>: We have cut out the compactified moduli space $\overline{\mathcal{M}}_\beta(p_1..p_k;q) = (\bar\partial_J + s)^{-1}(0)$

by an abstract perturbation of lower order $s : W^{k,2} \subset \mathcal{B} \to W^{k,2} \subset \mathcal{E}$.

This will be nonlocal ($s([(u_v)...])(z_0^{v_0} \in D_{v_0})$ depends on all maps u_v).

It will not be $\bar\partial_J + s = \bar\partial_{J'}$ for another almost complex structure J'.

However, if $\bar\partial_J$ is "classically transverse", then we can choose $s=0$.

⑤ The boundary master equation for moduli spaces

$$\partial \overline{\mathcal{M}}_\beta(p_1 \dots p_d ; q) = \bigcup_{\substack{\beta_1+\beta_2=\beta \\ 0\le k<\ell\le d \\ r\in \mathrm{Crit} f}} \mathcal{M}_{\beta_1}(p_1,\dots p_k, r, p_{\ell+1} \dots p_d ; q) \times \mathcal{M}_{\beta_2}(p_{k+1},\dots, p_\ell ; r)$$

$$\overset{(*)}{\|} \qquad\qquad \| \qquad\qquad \|$$

$$(\bar\partial_J + s_{\beta p_1 \dots p_d q})^{-1}(0) \cap \mathcal{B}^1_\beta(p_1 \dots p_d ; q) \qquad (\bar\partial_J + s_{\beta_1 p_1 \dots r \dots q})^{-1}(0) \cap \mathcal{B}^0_\beta(p_1 \dots r \dots ; q) \qquad (\bar\partial_J + s_{\beta_2 p_{k+1} \dots r})^{-1}(0) \cap \mathcal{B}^0_\beta(p_{k+1} \dots p_\ell ; r)$$

$\mathcal{B}^1_\beta$ = boundary of $\mathcal{B}_\beta$
= degeneration index 1
= {one broken Morse flow line}

$\mathcal{B}^0_\beta$ = interior of $\mathcal{B}_\beta$
= { unbroken Morse flow lines }

follows from Ⓐ boundary master equation for polyfolds $\mathcal{B}$

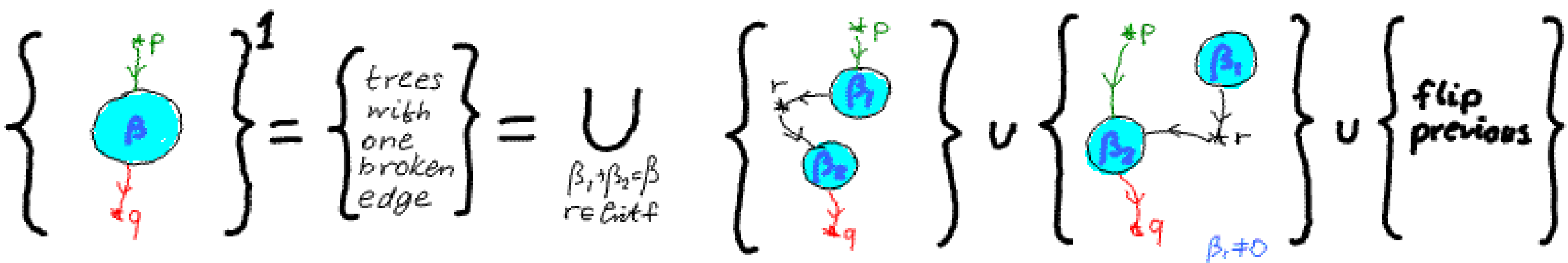
$$\mathcal{B}^1_\beta(p_1 \dots p_d ; q) = \bigcup_{\substack{\beta_1+\beta_2=\beta \\ 0\le k<\ell\le d \\ r\in \mathrm{Crit} f}} \mathcal{B}^0_{\beta_1}(p_1,\dots p_k, r, p_{\ell+1} \dots p_d ; q) \times \mathcal{B}^0_{\beta_2}(p_{k+1},\dots, p_\ell ; r)$$

Ⓑ$^{(*)}$ coherent choice of perturbations $\quad s|_{\mathcal{B}^0 \times \mathcal{B}^0 \subset \mathcal{B}^1} = s \times s$

A The boundary master equation for polyfolds $\mathcal{B}$ holds by construction

$$\mathcal{B}^1_\beta(p_1\ldots p_d;q) := \bigcup_{\substack{\beta_1+\beta_2=\beta\\ 0\le k\le \ell\le d\\ r\in \mathrm{Crit} f}} \mathcal{B}^0_{\beta_1}(p_1,\ldots p_k, r, p_{\ell+1}\ldots p_d; q)\times \mathcal{B}^0_{\beta_2}(p_{k+1},\ldots,p_\ell; r)$$

Example: $d=1$



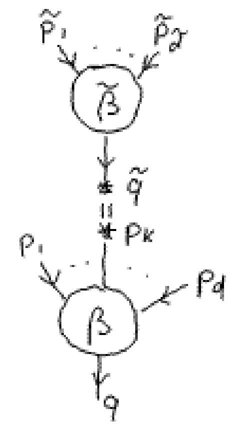
$\mathcal{B}^0_{\beta_1}(p;r) \times \mathcal{B}^0_{\beta_2}(r;q)$ $\cup$ $\mathcal{B}^0_{\beta_1}(\emptyset;r) \times \mathcal{B}^0_{\beta_2}(p,r;q)$ $\cup$ $\mathcal{B}^0_{\beta_1}(\emptyset;r) \times \mathcal{B}^0_{\beta_2}(r,p;q)$

⑤ The boundary master equation for moduli spaces follows from

Ⓐ boundary master equation for polyfolds $\mathcal{B}$

Ⓑ A prime decomposition on the set of symbols

$$\mathcal{S} = \{[\beta, p_1..p_d, q] \mid \beta \in \pi_2(M,L),\ p_i, q \in \mathrm{Crit} f\}$$

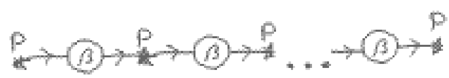


with operation $\mathcal{S} \times \mathcal{S} \longrightarrow$ subsets of $\mathcal{S}$

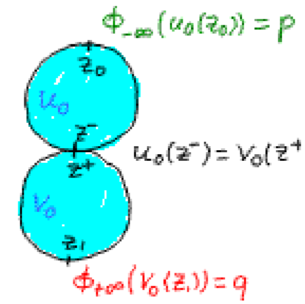
$$[\beta, p_1..p_d, q] \circ [\tilde\beta, \tilde p_1..\tilde p_{\tilde d}, \tilde q] = \{[\beta+\tilde\beta, p_1..p_{k-1}, \tilde p_1..\tilde p_{\tilde d}, p_{k+1}..p_d, q] \mid p_k = \tilde q\}$$

which ensures that the perturbations $(s_{\beta, p_1..p_d q})$ can be chosen coherently.

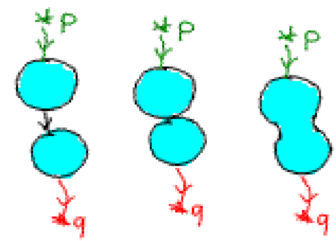
⊛ The diagonal relator axiom ($\sigma \circ \sigma = \emptyset\ \ \forall \sigma \in \mathcal{S}$) can be satisfied in most Floer theories (no self-connecting trajectories)
but not for SFT or trees of disks.

E.g. $\partial \mathcal{M}_{k\beta}(p;p)$ contains $\overset{k}{\otimes} \mathcal{M}_\beta(p;p)$:  $\Rightarrow$ higher order correction to master equation and algebra

Goal: polyfold chart near $[u_0, v_0, z_-, z_+, z_0, z_1] \in \mathcal{B}_\beta(p,q)$



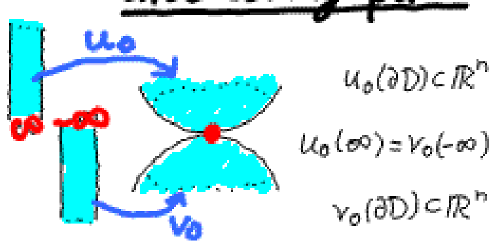
containing nearby points



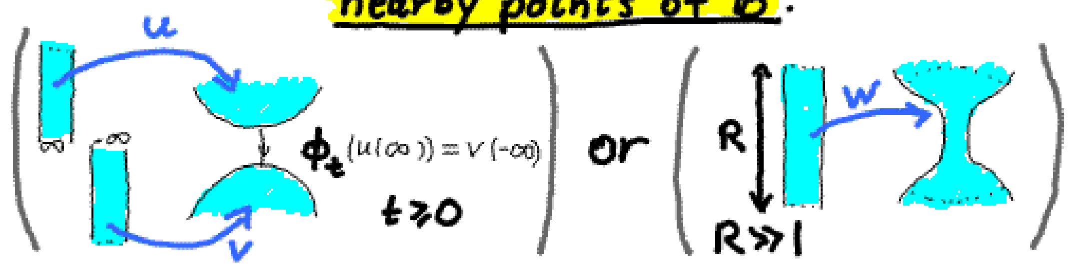
(i) Pick enough transverse hypersurfaces to fix parametrization ...
(...up to finite symmetry group, which becomes morphisms in the groupoid)

(ii) Restrict to local charts in domain (strip-like) and target $(M,L) \cong (\mathbb{C}^n, \mathbb{R}^n)$
(...actual polyfold chart will get patched together from these constructions)

interesting part:



nearby points of $\mathcal{B}$:



(iii) describe "points" near center by one set of data:

$u_0 : \mathbb{R}^+ \times [0,1] \to \mathbb{C}^n \quad \longleftrightarrow \quad v_0 : \mathbb{R}^- \times [0,1] \to \mathbb{C}^n$

$u_0(\infty) = v_0(-\infty) = 0$

• $t \geq 0$: "point" $\quad u = u_0 + \zeta + x \xrightarrow{-\nabla f} v = v_0 + \xi + \phi_t(x)$

$\phi_t(u(\infty,\cdot)) = v(-\infty,\cdot)$

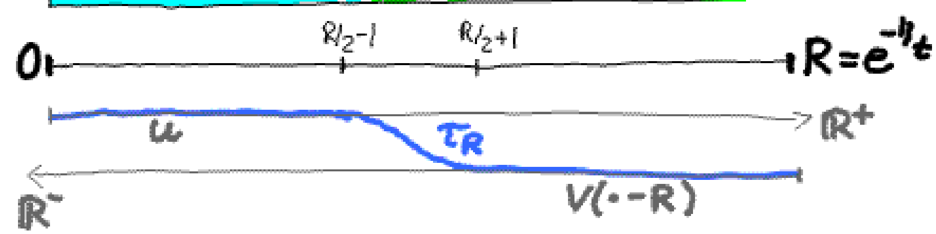
requires $\zeta \in W^{3,2}(\mathbb{R}^+ \times [0,1], \mathbb{C}^n)$, $\quad x \in \mathbb{R}^n$, $\quad \xi \in W^{3,2}(\mathbb{R}^- \times [0,1], \mathbb{C}^n)$

$\zeta|_{y=0}, \zeta|_{y=1} \in \mathbb{R}^n \qquad \xi|_{y=0}, \xi|_{y=1} \in \mathbb{R}^n$

• $t < 0$, ζ, x, ξ determine "point" $\quad w = \oplus_R(u_0 + \zeta + x, v_0 + \xi + x)$

by "pregluing"/interpolation

$\oplus_R(u,v) = \tau_R u + (1 - \tau_R) v(\cdot - R)$



(iv) project out ambiguity:

$$(-1,1)\times W^{3,2}(\mathbb{R}^+\times[0,1])\times\mathbb{R}^n\times W^{3,2}(\mathbb{R}^-\times[0,1]) \xrightarrow{\text{not injective}} \mathcal{B}_\beta(p;q) \text{ localized}$$

$$t$$

$$\Big\downarrow \pi$$

$$\bigcup_{t\in(-1,1)} \{t\}\times(\ker\oplus_R)^\perp \xrightarrow{\text{polyfold chart}} \mathcal{B}_\beta(p;q)$$

"splicing core"

$$(t,\zeta,x,\xi) \longmapsto \begin{cases} \begin{pmatrix} u_0+\zeta+x \\ v_0+\xi+\Phi_t(x) \end{pmatrix} & ;\ t\geq 0 \\[2ex] \oplus_{e^{-1/t}}\begin{pmatrix} u_0+\zeta+x \\ v_0+\xi+x \end{pmatrix} & ;\ t<0 \end{cases}$$

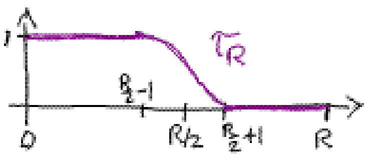
Note: No cone parameters $v\in[0,\infty)^k$, so this is interior chart.

All boundary and corners arise from breaking Morse trajectories.

(Need to use manifold with corner structure for Morse moduli spaces.)

Pregluing & Anti(pre)gluing define an isomorphism

$$\oplus_R \times \ominus_R : W^{3,2}(\mathbb{R}^+) \times W^{3,2}(\mathbb{R}^-) \xrightarrow{\sim} W^{3,2}([0,R]) \times W^{3,2}(\mathbb{R})$$

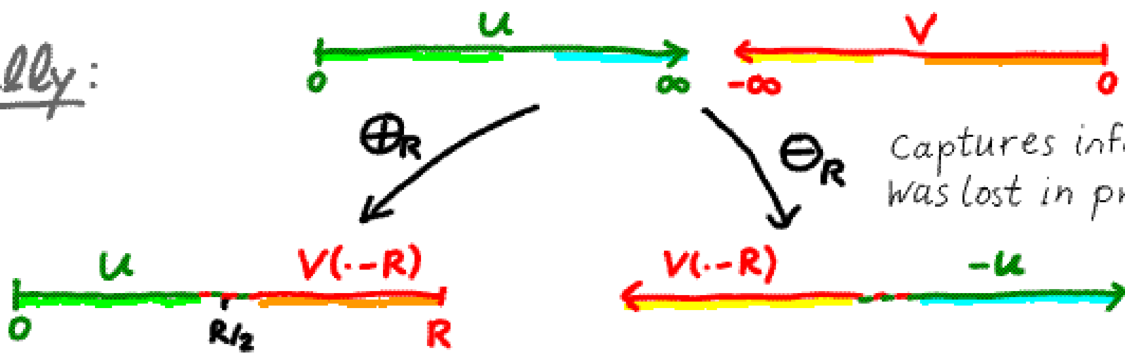


$$(u,v) \longmapsto \begin{pmatrix} \tau_R\, u + (1-\tau_R)\, v(\cdot - R) \\ -(1-\tau_R)\, u + \tau_R\, v(\cdot - R) \end{pmatrix}$$

invertible matrix

$$\Big(+ (1-2\tau_R)\underbrace{\big[\oplus_R(u,v)\big]^{\mathbb{R}^n}_{R/2}}_{\text{average } \mathbb{R}^n\text{-component}} \Big)$$

Graphically:



Captures information that was lost in pregluing $\oplus_R$

hence we can choose $(\ker \oplus_R)^{\perp} = \ker \ominus_R$ and define $\Pi_{t \geq 0} = Id$,

$$\Pi_{t<0}(\zeta, x, \xi) = (\hat{\zeta}, \hat{x}, \hat{\xi}) \quad \text{s.t.} \quad \oplus_R(\hat{\zeta} + \hat{x}, \hat{\xi} + \hat{x}) = \oplus_R(\zeta + x, \xi + x), \; \ominus_R(\hat{\zeta}, \hat{\xi}) = 0$$