A polyfold proof of the weak Arnold conjecture

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work in progress

based on : "the right question" by Thomas Kragh

polyfold technology (in progress) by Hofer-Wysocki-Zehnder

weak Arnold conjecture : (M,ω) closed symplectic

$\phi \circlearrowright M$ time-1-flow of Hamiltonian vector field for $H : S^1 \times M \to \mathbb{R}$

nondegenerate : $\underbrace{\text{graph}\ \phi \pitchfork \Delta_M}$

$$\Rightarrow \boxed{\#\,\text{periodic orbits}} = \#\,\mathrm{Fix}\,\phi \boxed{\geq \sum_i \mathrm{rk}\, H_i(M,\mathbb{Q})}$$

PROOFS that Edi Zehnder and Katrin Wehrheim can understand

Eliashberg $(\dim M = 2)$

Conley-Zehnder (T^{2n})

Gromov (≥ 1 for $\pi_2(M) = 0$)

Floer $\left([\omega] = \tau\, c_2(TM, J) \quad \tau \geq 0 \right)$

Hofer Salamon

Ono

(M semipositive

for "generic" J $\nexists$ J-hol S^2, $c_2 < 0$)

Floer's proof

(& [H-S], [On] generalizations by
- simple bubble transversality
- Novikov ring Λ $\leadsto$ $\mathbb{Z}$ for M monotone)

$CF(\phi) := \sum_{q \in \mathrm{Fix}\,\phi} \mathbb{Z}\langle q \rangle$ ⊛(restrict to summand $\mathring{CF}$ generated by contractibles)

- construct Floer differential $\partial \circlearrowleft CF$, prove $\partial\circ\partial = 0$
- construct isomorphism $\mathring{HF}(\phi) \simeq HF(\phi_f)$ for C^2-small $f: M \to \mathbb{R}$ Morse

$CM := \sum_{p \in \mathrm{Crit}\,f} \mathbb{Z}\langle p \rangle = \mathring{CF}(\phi_f)$ since $\mathrm{Fix}\,\phi_f \overset{\Downarrow}{=} \mathrm{Crit}\,f$

- prove $\overset{\circlearrowleft}{\partial^{Morse}} = \overset{\circlearrowleft}{\partial^{Floer}}$ by S^1-action on Floer moduli space with fixed points ≙ Morse trajectory space

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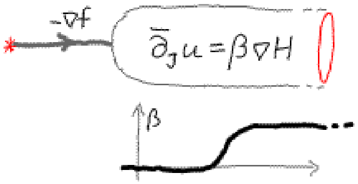
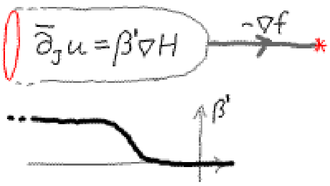
by regularizing 4 moduli spaces and **S^1-equivariant transversality**

Piunikhin - Salamon - Schwarz approach (outlined for M semipositive)

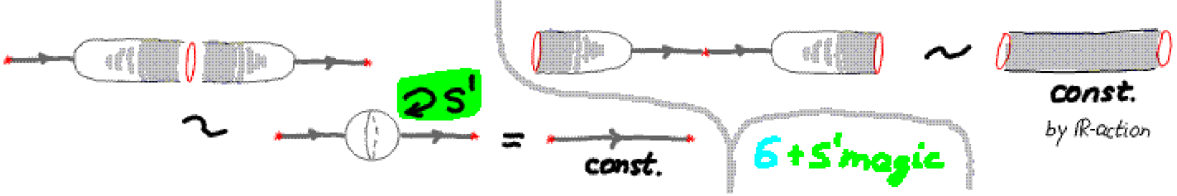
- construct **PSS** : $CM \to CF(\Phi)$ **SSP** : $CF(\Phi) \to CM$

"by counting"





- construct ∂^{Floer}, prove $\partial\circ\partial = 0$, and that PSS, SSP are chain maps
- prove $\boxed{PSS \circ SSP = id_{CM}}$, $\boxed{SSP \circ PSS = id_{\mathring{CF}}}$ by cobordisms



lazy PSS approach

$$\begin{tikzcd} CM \arrow[r, "PSS"] \arrow[rr, bend right, "h"] \arrow[rr, bend right=40, "\iota"'] & CF(\Phi) \arrow[r, "SSP"] & CM \end{tikzcd}$$

- construct Λ-linear maps from SFT polyfolds [HWZ ≈ 2013] & Morse spaces [W.2012]
- prove $\boxed{PSS \circ SSP - \iota = d \circ h + h \circ d}$ $d = d^{Morse}$

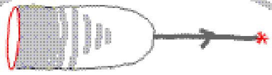
$$\iota \circ d = d \circ \iota$$

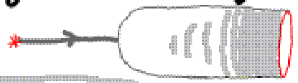
$$\boxed{\iota - id_{CM} = \sum_{\lambda > 0} t^{\lambda}(\dots)}$$ "upper triangular"

4 + algebra

- a little algebra : ι GHM isomorphism "factoring through CF" ⇒ rank CF ≥ rank HM

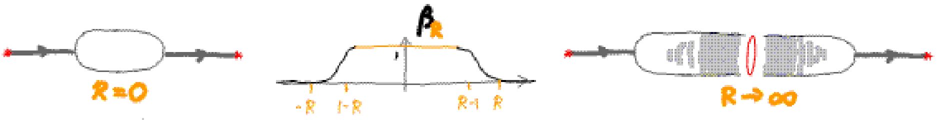
$$\mathcal{M}_k^{PSS} = \left\{ \left(u : \underset{-\infty \cup \mathbb{R}\times S^1 \simeq}{\mathbb{C}} \to M , \gamma : (-\infty, 0] \to M \right) \middle| \begin{array}{l} \bar\partial_J u = \beta \nabla H \\ \int |\partial_s u|^2 < \infty \end{array}, \dot\gamma = -\nabla f , u(\overset{-\infty}{\overset{\cong}{0}}) = \gamma(0) , ind\, D_{(u,\gamma)} = k \right\}$$





$$\mathcal{M}_k^{SSP} = \left\{ \left(u : \overset{\mathbb{R}\times S^1 \cup \infty \simeq}{\mathbb{C}^-} \to M , \gamma : [0, \infty) \to M \right) \middle| \begin{array}{l} \bar\partial_J u = \beta' \nabla H \\ \int |\partial_s u|^2 < \infty \end{array}, \dot\gamma = -\nabla f , u(\underset{\infty}{\underset{\cong}{0}}) = \gamma(0) , ind\, D_{(u,\gamma)} = k \right\}$$

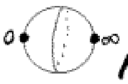
$$\mathcal{M}_k^h = \left\{ \left(\begin{array}{c} \gamma_- : (-\infty, 0] \to M \\ R \geq 0, u : \mathbb{CP}^1 \to M \\ \gamma_+ : [0, \infty) \to M \end{array} \right) \middle| \begin{array}{lll} \dot\gamma_- = -\nabla f & u(0) = \gamma_-(0) & \\ \bar\partial_J u = \beta_R \nabla H & \int |\partial_s u|^2 < \infty & ind\, D_{(\gamma_-, u, \gamma_+)} = k \\ \dot\gamma_+ = -\nabla f & u(\infty) = \gamma_+(0) & \end{array} \right\}$$



$$\mathbb{CP}^1 \simeq \underbrace{-\infty \cup \overbrace{\mathbb{R}\times S^1 \cup \infty}^{\mathbb{C}^-}}_{\mathbb{C}}$$

$$\mathcal{M}_k^{\iota} := \mathcal{M}_k^h \cap \{R=0\}$$

Thm [HWZ'12] $\exists\ \bar\partial : \widetilde{\mathcal{B}}_k^{GW} \to \widetilde{\mathcal{E}}_k^{GW}$ Fredholm section in polyfold bundle: $\partial \widetilde{\mathcal{B}}_k^{GW} = \emptyset$,

$\bar\partial^{-1}(0) = \overline{\mathcal{M}}_k^{GW}$ moduli space of holomorphic (nodal) spheres  / Aut

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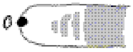
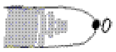
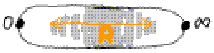
Assumption [HWZ'13 ?] $\exists\ \bar\partial : \widetilde{\mathcal{B}}_k^{\dots} \to \widetilde{\mathcal{E}}_k^{\dots}$ polyfold Fredholm:

→ $\bar\partial^{-1}(0) = \overline{\mathcal{M}}_k^{SFT}$ holomorphic buildings in $\mathbb{C}\times M$  / $\mathbb{C}^- \times M$ 

→ $\bar\partial^{-1}(0) = \overline{\mathcal{M}}_k^{stretch}$ holomorphic buildings for neck stretching

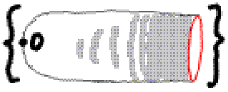
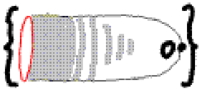
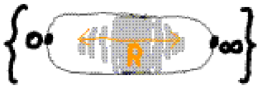
at $\mathbb{RP}^1 \times M \subset \mathbb{CP}^1 \times M$ with "Reeb field" $= (\partial_{\mathbb{RP}^1}, X_H)$ 

$$\widetilde{\mathcal{B}}_k^{\dots} = \begin{cases} \mathcal{B}_k^{\dots} & \text{1 level, smooth domain} \\ \cup\ \mathcal{B}_k^{nodal\,\dots} & \text{1 level, nodal domain} \\ \cup\ \partial\widetilde{\mathcal{B}}_k^{\dots} & \text{GW: } \emptyset\text{, SFT: multiple levels, stretch: } R=0 \text{ / multiple levels for } R\to\infty \end{cases}$$

"interior"

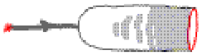
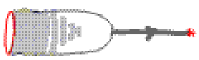
$$\boxed{ev_0, ev_\infty : \widetilde{\mathcal{B}}_k^{\dots} \to M \quad sc^\infty}$$

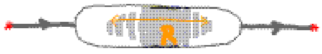
<u>Note</u>: The regular buildings in homotopy classes id × ... (with some marked points) correspond to holomorphic curves in M,

$\overline{\mathcal{M}}_k^{\dots} \cap \mathcal{B}_k^{\dots} \simeq$ {  } / {  } / {  }

$id_{\mathbb{C}} \times \dots$ $\quad$ $id_{\mathbb{C}^-} \times \dots$ $\quad$ $id_{\mathbb{CP}^1} \times \dots$

We need to couple these with half-infinite Morse trajectories:

$\mathcal{M}_k^{PSS} =$  $\quad \mathcal{M}_k^{SSP} =$  $\quad \mathcal{M}_k^{c} =$ 

$\mathcal{M}_k^{h} =$ 

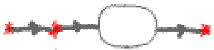
Claim: $\exists\, s : \widetilde{\mathfrak{B}}_k^{\cdots} \to \widetilde{\mathcal{E}}_k^{\cdots}$ Fredholm section in polyfold bundle :

$s^{-1}(0)$ compact

$\mathcal{M}_k^{\cdots} = \bar\partial^{-1}(0) \cap \mathfrak{B}_k^{\cdots}$

$$\left(\mathfrak{B}_k^{\cdots} = \widetilde{\mathfrak{B}}_k^{\cdots} \setminus \big(\underbrace{\partial \widetilde{\mathfrak{B}}_k^{\cdots}}_{\boxed{\text{boundary \& corners}}} \cup \underbrace{\overset{\text{nodal}}{\mathfrak{B}_k^{\cdots}}}_{\boxed{\text{interior "codim 2"}}} \big) ,\ \overline{\mathcal{M}}_k^{\cdots} := \bar\partial^{-1}(0) ,\ \partial \overline{\mathcal{M}}_k^{\cdots} := \overline{\mathcal{M}}_k^{\cdots} \cap \partial \widetilde{\mathfrak{B}}_k^{\cdots} \right)$$

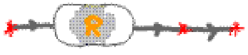
boundary \ corners

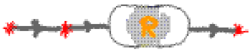


$$\partial' \widetilde{\mathfrak{B}}_1^{\iota} = \mathfrak{B}_0^{\iota} \,{}_{ev_{+\infty}}\!\times_{ev_{-\infty}} \mathcal{M}_1^{\text{Morse}} \ \cup\ \mathcal{M}_1^{\text{Morse}} \,{}_{ev_{+\infty}}\!\times_{ev_{-\infty}} \mathfrak{B}_0^{\iota}$$

up to fiber products in which one factor has negative index ⇒ ∅ after perturbation

$$\partial' \widetilde{\mathfrak{B}}_0^{h} = \underset{R\to\infty}{\mathfrak{B}_0^{PSS} \,{}_{ev_{+\infty}}\!\times_{ev_{-\infty}} \mathfrak{B}_0^{SSP}} \ \cup\ \underset{R=0}{\mathfrak{B}_0^{\iota}} \ \cup\ \mathfrak{B}_{-1}^{h} \,{}_{ev_{+\infty}}\!\times_{ev_{-\infty}} \mathcal{M}_1^{\text{Morse}} \ \cup\ \mathcal{M}_1^{\text{Morse}} \,{}_{ev_{+\infty}}\!\times_{ev_{-\infty}} \mathfrak{B}_{-1}^{h}$$



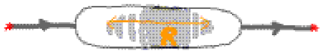


$\Downarrow$ [HWZ '07-'09]

Cor: Define PSS/SSP/ι/h by $\#\big(s + \text{polyfold perturbation}\big)^{-1}(0) \subset \widetilde{\mathfrak{B}}_0^{PSS} / \widetilde{\mathfrak{B}}_0^{SSP} / \widetilde{\mathfrak{B}}_0^{\iota} / \widetilde{\mathfrak{B}}_{-1}^{h}$,

then $0 = \iota \circ d + d \circ \iota$, $0 = PSS \circ SSP - \iota + h \circ d + d \circ h$.

Proof of Claim: E.g. $\mathcal{M}_*^{h} = s^{-1}(0)$ $\quad s : \widetilde{\mathfrak{B}}_*^{\text{stretch}} \times \overline{\mathcal{M}}_*^{\text{Morse},(-\infty,0]} \times \overline{\mathcal{M}}_*^{\text{Morse},[0,\infty)} \longrightarrow \widetilde{\mathcal{E}}^{\text{stretch}} \times TM \times TM$



$$(R, u, \underbrace{\gamma_-, \gamma_+}_{\text{finite dim.}}) \longmapsto \Big(\underbrace{\bar\partial_R u}_{\text{Fredholm}} , \underbrace{u(0) - \gamma_-(0) , u(\infty) - \gamma_+(0)}_{sc^\infty \text{ to finite dim.}} \Big)$$

$\overline{\mathcal{M}}_*^{\text{Morse},(-\infty,0]}$, $\overline{\mathcal{M}}_*^{\text{Morse},[0,\infty)}$: (broken) half-infinite Morse trajectories

[W'12] ⇒ compact manifolds with corners & "associative gluing"

(∇f "Euclidean" ⇒) $ev : \overline{\mathcal{M}}_*^{\text{Morse}} \to M$ $\quad \mathcal{C}^\infty$

main boundary stratum ("corner index" = 1) is

$$\partial' \big(\widetilde{\mathfrak{B}}_*^{\text{stretch}} \times \overline{\mathcal{M}}_* \times \overline{\mathcal{M}}_* \big) = \partial' \widetilde{\mathfrak{B}}_*^{\text{stretch}} \times \overline{\mathcal{M}}_* \times \overline{\mathcal{M}}_* \ \sqcup\ \widetilde{\mathfrak{B}}_*^{\text{stretch}} \times \partial' \overline{\mathcal{M}}_* \times \overline{\mathcal{M}}_*$$
$$\sqcup\ \widetilde{\mathfrak{B}}_*^{\text{stretch}} \times \overline{\mathcal{M}}_* \times \partial' \overline{\mathcal{M}}_*$$

- $R = 0 \ \simeq \widetilde{\mathfrak{B}}_*^{GW}$
- $\begin{cases} R = \infty \\ 2 \text{ levels} \end{cases} \simeq \widetilde{\mathfrak{B}}_*^{PSS} \times \widetilde{\mathfrak{B}}_*^{SSP}$

once broken

To prove $\iota = id_{CM} + \sum_{\lambda>0} t^\lambda(\dots)$ it suffices to consider the unperturbed moduli space $\widetilde{\mathcal{M}}_0^\iota(p,q,B) = \{$  $\mid u_*[S^2]=B\}$ for $B\in H_2(M)$.

By the energy identity $\int|\partial_s u|^2 = \int u^*\omega = \langle[\omega],B\rangle$ we have

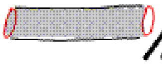
$\langle[\omega],B\rangle < 0 \Rightarrow \widetilde{\mathcal{M}}_0^\iota(p,q,B)=\emptyset \Rightarrow s \pitchfork 0$ over $\{\dots u_*[S^2]=B\dots\}\subset\tilde{\mathfrak{B}}_0^\iota$

$\langle[\omega],B\rangle = 0 \Rightarrow \widetilde{\mathcal{M}}_0^\iota(p,q,B) = \begin{cases}\emptyset & ;\ p\neq q \\ \begin{pmatrix}\gamma_-\equiv p\\ u\equiv p\\ \gamma_+\equiv p\end{pmatrix} & ;\ p=q\end{cases}$ $\left(u=const \Rightarrow s\mapsto\begin{cases}\gamma_-(s), & s\le 0\\ \gamma_+(s), & s\ge 0\end{cases} \text{ is a Morse trajectory with } |p|-|q|=0\right)$

$\Rightarrow s\pitchfork 0$ over $\{\gamma_-(-\infty)=p, \gamma_+(\infty)=q, u_*[S^2]=B\}\subset\tilde{\mathfrak{B}}_0^\iota$

Now polyfold perturbations can be chosen $=0$ on these components.
(using the gluing hierarchy $\tilde{\mathfrak{B}}^\iota, \tilde{\mathfrak{B}}^{PSS}, \tilde{\mathfrak{B}}^{SSP} \rightsquigarrow \tilde{\mathfrak{B}}^h$)

To construct Floer homology and $\boxed{HM \xrightarrow{PSS} HF \xrightarrow{SSP} HM}$ we need to compactify and regularize



$/\mathbb{R}$

$\mathcal{M}_k^F = \{u:\mathbb{R}\times S^1\to M \mid \bar\partial_J u = \nabla H, 0<\int|\partial_s u|^2<\infty, \mathrm{ind}\, D_u = k+1\}/\mathbb{R}$

so that $\partial\widetilde{\mathcal{M}}_1^F \simeq \mathcal{M}_0^F \,{}_{ev_{+\infty}}\!\times_{ev_{-\infty}} \mathcal{M}_0^F$, $\partial\widetilde{\mathcal{M}}_1^{PSS} \simeq \mathcal{M}_0^{PSS} \,{}_{ev_{+\infty}}\!\times_{ev_{-\infty}} \mathcal{M}_0^F \sqcup \mathcal{M}_1^{Morse} \,{}_{ev_{+\infty}}\!\times_{ev_{-\infty}} \mathcal{M}_0^{PSS}$, $\partial\widetilde{\mathcal{M}}_1^{PSS} \simeq \dots$

This can be achieved by "standard polyfold methods" based on the SFT assumptions

- $\bar\partial$ Fredholm section on $\tilde{\mathfrak{B}}_*^F$: SFT polyfold for $\mathbb{R}\times S^1\times M$ (in classes $id_{\mathbb{R}\times S^1}\times *$)
  with $\partial^1\tilde{\mathfrak{B}}_1^F = \bigcup_{k\in\mathbb{Z}} \tilde{\mathfrak{B}}_k^F \,{}_{ev_{+\infty}}\!\times_{ev_{-\infty}} \tilde{\mathfrak{B}}_{-k}^F$
- $\partial^1\tilde{\mathfrak{B}}_1^{SFT} = \bigcup_{k\in\mathbb{Z}} \tilde{\mathfrak{B}}_k^{SFT} \,{}_{ev_{+\infty}}\!\times_{ev_{-\infty}} \tilde{\mathfrak{B}}_{-k}^F$ for $\mathbb{C}\times M$; analogous for $\bar{\mathbb{C}}\times M$

Proving $\boxed{PSS \circ SSP = id_{CM}}$ would require a quotient theory for polyfolds.

Proving $\boxed{SSP \circ PSS = id_{CF}}$ would require "codim. 2 avoidance"

⚠ general cobordism of compactified moduli spaces fails: