How to construct 3- and 4-manifold invariants via pseudoholomorphic quilts

joint w. Chris Woodward (for quilts, 3-mfd invariants)

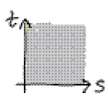
David Gay (for connected Cerf theory, 4-mfd topology)

inspired by Tim Perutz' Lagrangian matching invariant setup

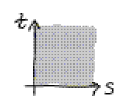
goal: framework for proving invariance

Idea: Symplectic version of gauge theoretic invariants by conjectured "large structure limit" (degeneration of metric) & some topological twists

---

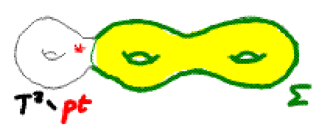
Recall: on  $\times \Sigma$ expect ASD wrt $ds^2+dt^2+\varepsilon^2_\Sigma$

$\downarrow \varepsilon \to 0$

$J = *_\Sigma$ - holomorphic maps  $\to M(\Sigma)$

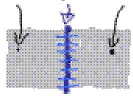
to get smooth representation spaces $M(\Sigma)$ use e.g.

Σ^2 closed oriented $\rightsquigarrow$ $M(\Sigma) := \left\{ \begin{array}{l} \rho : \pi_1(\overbrace{\Sigma \# T^2}^{\tilde{\Sigma}} \setminus \{pt\}) \to SU(r) \\ \rho(\circlearrowleft_{pt}) = -\mathbb{1} \end{array} \right\} \Big/ SU(r)$



Thm: $M(\Sigma)$ is smooth, symplectic, monotone, min $c_1 \geq 2$ for Σ connected.

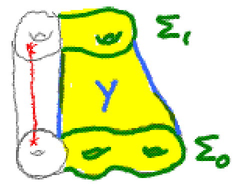
Recall: on Σ_0 singular fibers Σ_1 $\leadsto$ handle attachment $\Sigma_0 \xrightarrow{Y} \Sigma_1$



expect ASD wrt $ds^2 + \varepsilon^2 g_Y \xrightarrow[\varepsilon\to 0]{}$ Lagrangian seam condition

---

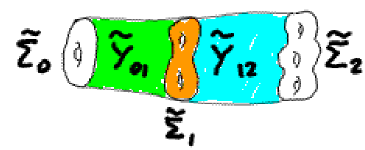
Y^3, $\partial Y = \Sigma_0^- \cup \Sigma_1$ cobordism $\leadsto$ $L(Y) := \left\{ (\rho_0, \rho_1) \in M(\Sigma_0)^- \times M(\Sigma_1) : \exists \text{ extension } \tilde{\rho} : \pi_1(\underbrace{Y \# (T^2 \setminus pt) \times [0,1]}_{\tilde{Y}}) \to SU(r) \right\}$



Thm: $L(Y)$ is smooth, Lagrangian for Y compression body
— " — spin, simply connected — " — elementary cobordism
- trivial
- handle attachment

Note: $L(Y_{01} \cup_{\Sigma_1} Y_{12}) = L(Y_{01}) \circ L(Y_{12})$

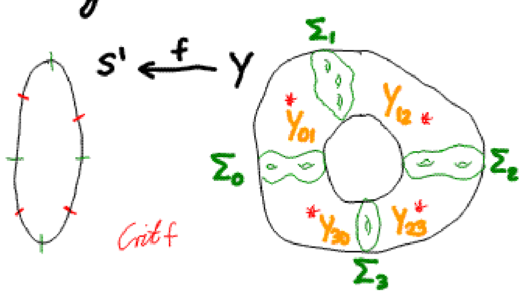
$= \pi_{M(\Sigma_0)\times M(\Sigma_2)} \left(L(Y_{01}) \times L(Y_{12}) \cap M(\Sigma_0) \times \Delta_{M(\Sigma_1)} \times M(\Sigma_2) \right)$



To construct an invariant for 3-manifolds Y from this, we need

- topology: decomposition of Y into elementary cobordisms w. connected dividing surfaces

  e.g. from S^1-valued Morse function (in fixed homotopy class)



above constructions $\leadsto$

$$\begin{tikzcd}
 & M(\Sigma_1) \arrow[dr, "L(Y_{12})"] & \\
M(\Sigma_0) \arrow[ur, "L(Y_{01})"] & & M(\Sigma_2) \arrow[dl, "L(Y_{23})"] \\
 & M(\Sigma_3) \arrow[ul, "L(Y_{30})"] &
\end{tikzcd}$$

- analysis: Floer homology

$$HF\left(\begin{tikzcd}
 & M(\Sigma_1) \arrow[dr, "L(Y_{12})"] & \\
M(\Sigma_0) \arrow[ur, "L(Y_{01})"] & & M(\Sigma_2) \arrow[dl, "L(Y_{23})"] \\
 & M(\Sigma_3) \arrow[ul, "L(Y_{30})"] &
\end{tikzcd} \right)$$

- topology — invariance of decompositions e.g. up to Cerf moves:
  - critical point switch
  - critical point cancellation
  - cylinder cancellation
  - diffeomorphism equivalence

  $\Big\}$ can all be viewed as (several) compositions of elem. cobordisms $\Sigma_0 \xrightarrow{Y_{01}} \Sigma_1 \xrightarrow{Y_{12}} \Sigma_2$, $\Sigma_0 \xrightarrow{Y_{01} \cup Y_{12}} \Sigma_2$

  Thm [Cerf, ..., Gay-Kirby, ..] Cerf moves with connected levels suffice.

- sympl. — Thm: $M(\Sigma_0) \xrightarrow{L(Y_{01})} M(\Sigma_1) \xrightarrow{L(Y_{12})} M(\Sigma_2)$, $\boxed{L(Y_{01} \cup Y_{12}) = L(Y_{01}) \circ L(Y_{12})}$ elementary cobordism compositions correspond to embedded composition of Lagrangian correspondences

- analysis — isomorphism of Floer homology under embedded composition

To construct an invariant for 4-manifolds X, we need:

- "string diagrams" for 4-manifolds, unique up to "Cerf moves"
- symplectification "topological string diagram" $\rightsquigarrow$ "symplectic string diagram"
  Cerf moves $\rightsquigarrow$ quilt moves
- quilt invariants "symplectic string diagram" $\rightsquigarrow$ $\mathbb{Z}_2 / \mathbb{Z} / \mathbb{Q} /$ Nov. ring
  invariant under quilt moves

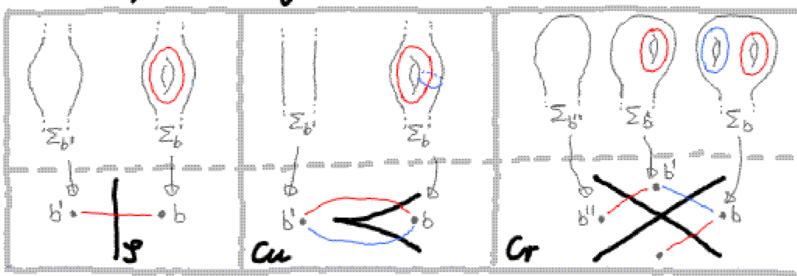
"Def^n": A "string diagram" in a 2-category is a quilted surface

with patches	labeled by	objects	e.g. 2-manifolds
seams	— " —	morphisms	3-cobordisms
ends	— " —	2-morphisms	4-cobordisms

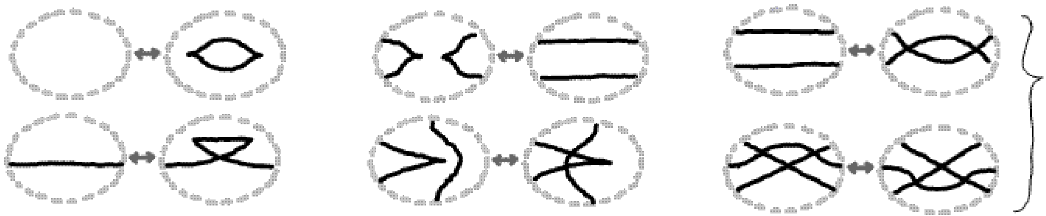
"string diagrams" for 4-manifolds [Gay-Kirby : Morse 2-functions]

X closed, connected 4-manifold; fix homotopy class $\Rightarrow \exists$ "generic" $f : X \to S^2$

- $f(\mathrm{Sing}\, f) \subset S^2$ "1-submanifold with cusps & crossings"
- $f^{-1}(b)$ connected $\forall b \in S^2 \setminus f(\mathrm{Sing})$
- local normal forms by attaching cycles $\quad X \xrightarrow{f} S^2$



- f unique up to

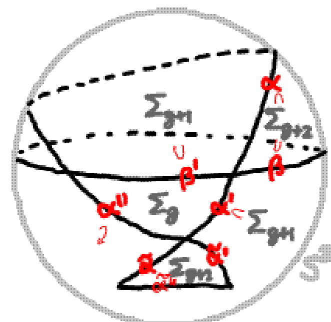


under certain conditions on attaching cycles

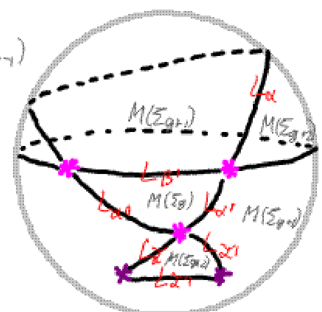
+isotopies

symplectification of "topological string diagrams"

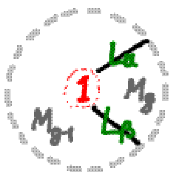
yields a quilted surface with symplectic labels :

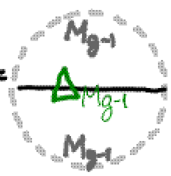


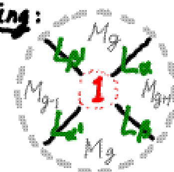
- **patch**: top. label $\Sigma_g \rightsquigarrow$ symp. label $M_g := M(\Sigma_g)$
  (fibration $(\Sigma_p)_{p\in P} \rightsquigarrow$ symp. fibration $(M(\Sigma_p))_{p\in P}$)
- **seam**: top. label $\alpha \subset \Sigma_g \rightsquigarrow$ symp. label $L_\alpha := L(Y_\alpha) \subset M(\Sigma_g)^- \times M(\Sigma_{g-1})$
  elementary cobordism $\Sigma_g \xrightarrow{Y_\alpha} \Sigma_{g-1}$
  (fibration $(\alpha_s)_{s\in S} \rightsquigarrow (L(Y_{\alpha_s}))_{s\in S}$)
- **puncture** (cusp or crossing): TBD

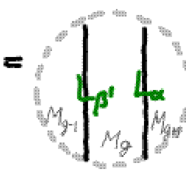


Thm: Suppose that the $\begin{pmatrix}\text{compact}\\ \text{monotone}\\ \text{oriented}\\ \text{rel. spin}\end{pmatrix}$ symplectic manifolds M_g and Lagrangian correspondences $L_\alpha \subset M_g^- \times M_{g-1}$ satisfy "3- and 4-manifold axioms". Then there exist canonical HF-classes for every cusp / crossing such that the quilt invariants $\in \mathbb{Z}$ associated to topological string diagrams are invariant under Cerf moves.

__cusp__: 

$1 \cong 1_{\Delta_{M_{g-1}}} =$ 

__crossing__: 

$1 := 1_{(L_\alpha^T \circ L_{\beta'})} =$ 

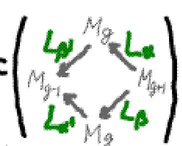
__3 manifold axioms__: $\underset{\text{cusp}}{\bullet}\ \alpha \pitchfork \beta$ in 1 point $\Rightarrow \underset{M_{g-1} \to M_g \to M_{g-1}}{L_\alpha^T \circ L_\beta} \overset{\text{Ham}}{\simeq} \Delta_{M_{g-1}}$ embedded

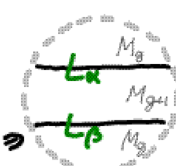
$\underset{\text{Crossing}}{\bullet}\ \alpha \cap \beta = \emptyset \Rightarrow \underset{M_g \to M_{g-1} \to M_{g-2}}{L_\alpha \circ L_{\beta'}} \overset{\text{Ham}}{\simeq} L_\beta \circ L_{\alpha'}$ embedded

__4 manifold axioms__: • $\alpha \cap \beta = \emptyset \Rightarrow \underset{M_{g-1} \to M_g \to M_{g-1}}{L_\alpha^T \circ L_\beta} \overset{\text{Ham}}{\simeq} \underset{M_{g-1} \to M_{g-2} \to M_{g-1}}{L_{\beta'} \circ L_{\alpha'}^T}$ embedded

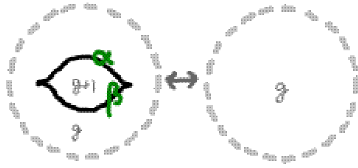
• $\alpha, \beta, \gamma \subset \Sigma_{g+1}$ disjoint $\Rightarrow$ all $L_{..} \circ L_{..} \circ L_{..}$ are embedded and equal
$M_{g+1} \to \dots \to M_{g-2}$ or $M_{g+1} \to \dots \to M_g$

• $\underset{\subset \Sigma_{g+1}}{\alpha \cap \beta = \emptyset} \Rightarrow 1_{L_\alpha L_{\beta'}} \simeq 1_{L_\alpha^T L_\beta}$

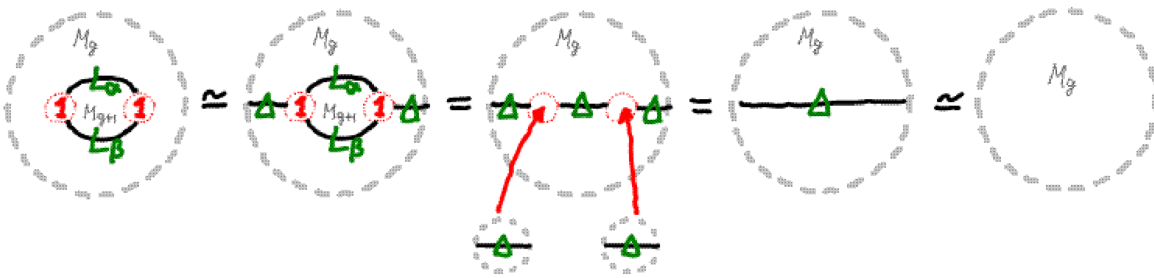
$\in$ HF(...) $\cong$ HF 

$\cong$ HF(...) $\ni$ 

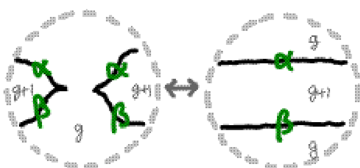
Proof:



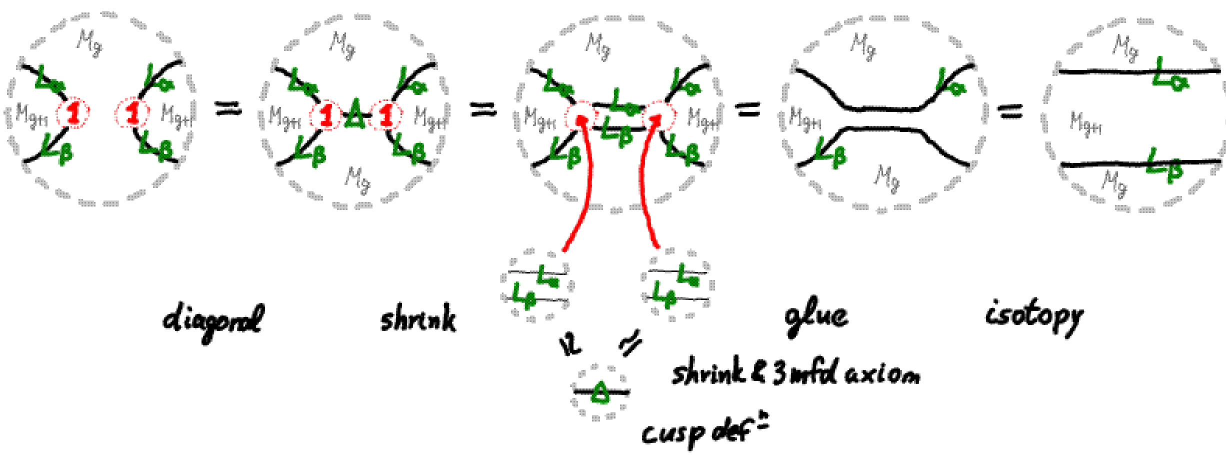
for $\alpha \pitchfork \beta = 1\text{pt}$

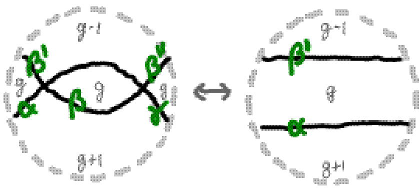


diagonal   shrink & 3mfd axiom   cusp def$^{\underline{n}}$ glue   diagonal

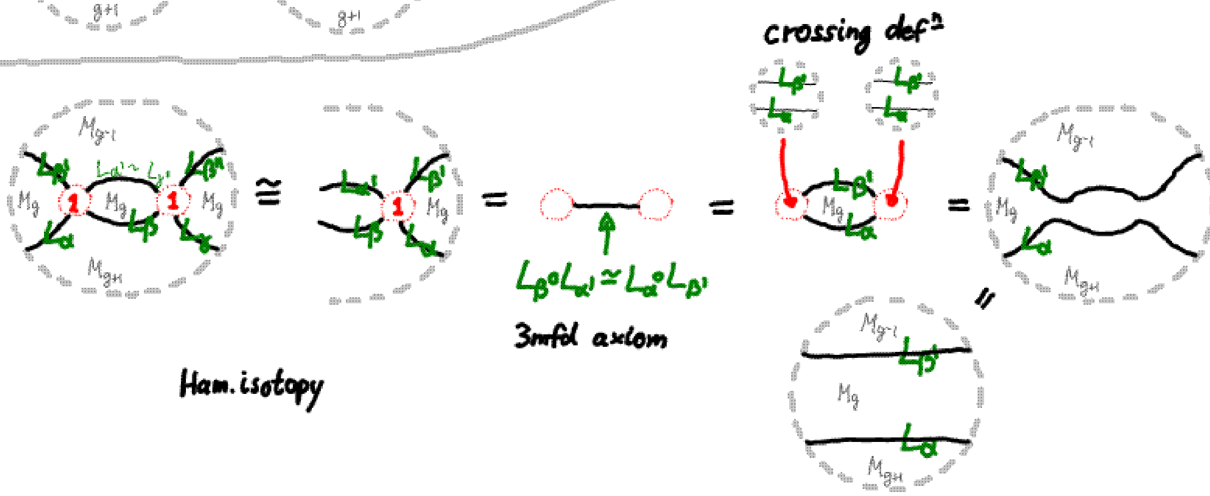


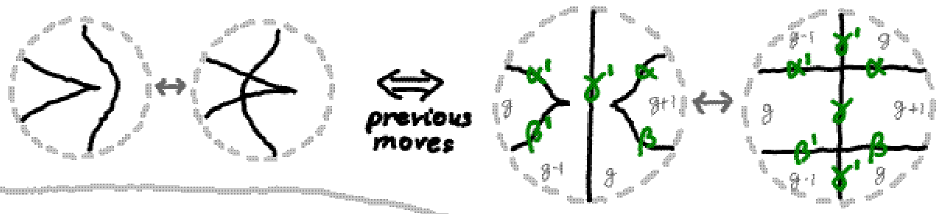
for $\alpha \pitchfork \beta = 1\text{pt}$





for $\alpha \simeq \gamma$, $\beta' \simeq \beta''$ $(\Rightarrow \alpha' \simeq \gamma')$

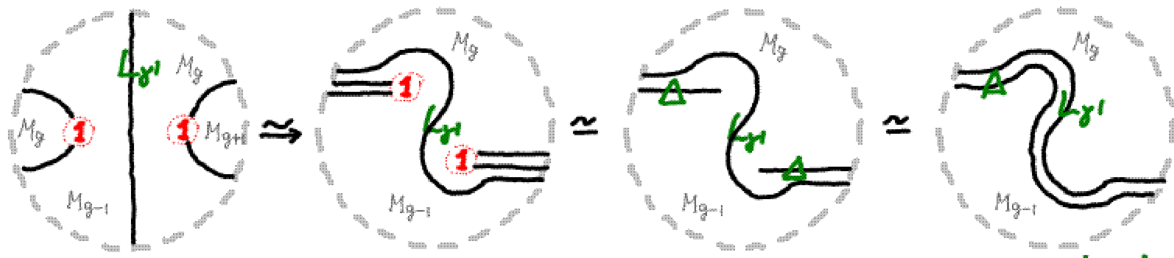




for $\alpha \pitchfork \beta = 1pt$

$\gamma \cap \alpha, \beta = \varnothing$

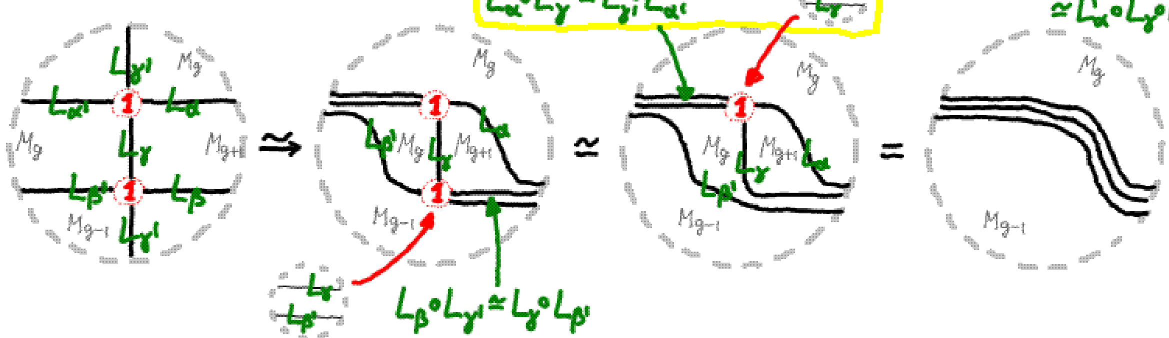
$(\Rightarrow \alpha' \pitchfork \beta' = 1pt)$



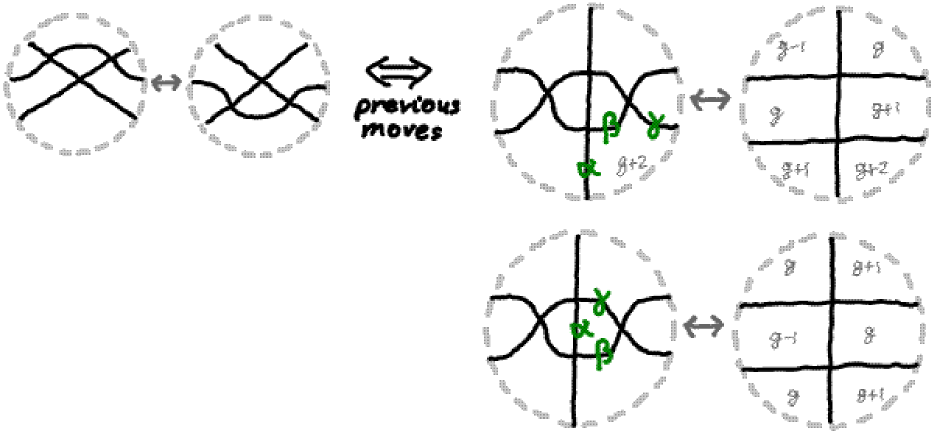
4mfd axiom

$L_\alpha^T \circ L_\gamma \simeq L_{\gamma'} \circ L_{\alpha'}^T$

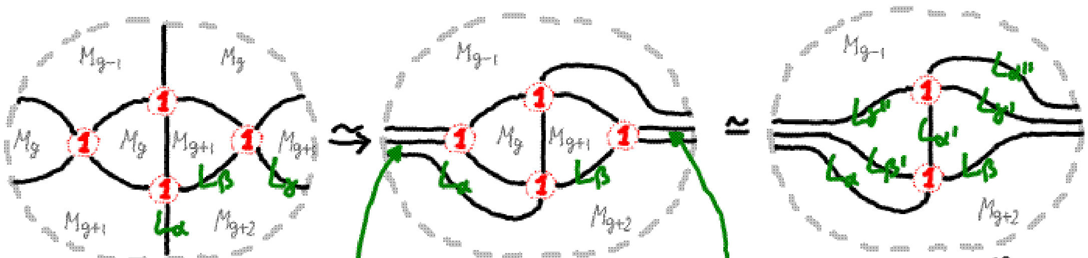
$L_{\gamma'} \circ \Delta$
$\cong$
$\simeq L_{\gamma'} \circ L_{\alpha'}^T \circ L_{\beta'}$
$\simeq L_\alpha^T \circ L_\gamma \circ L_{\beta'}$



$L_\beta \circ L_{\gamma'} \simeq L_\gamma \circ L_{\beta'}$

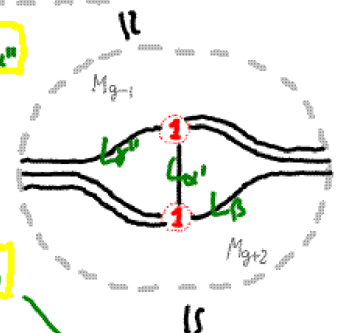


for $\alpha, \beta, \gamma \subset \Sigma_{g+2}$ disjoint



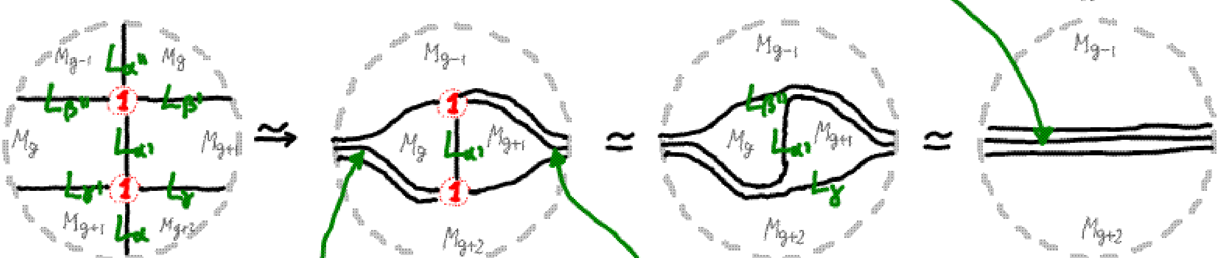
$L_\alpha \circ L_{\gamma'} \circ L_{\beta''} \simeq L_\alpha \circ L_{\beta'} \circ L_{\gamma''}$

$L_\gamma \circ L_\beta \circ L_{\alpha''} \simeq L_\beta \circ L_{\gamma'} \circ L_{\alpha''}$



4-mfd axiom

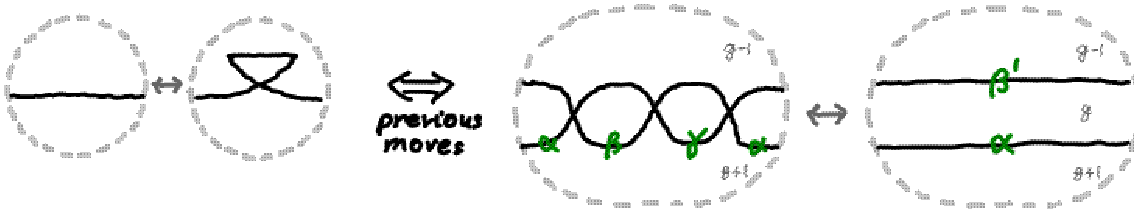
$L_\beta \circ L_{\alpha'} \circ L_{\gamma''} \simeq L_\gamma \circ L_{\alpha'} \circ L_{\beta''}$



$L_\alpha \circ L_\gamma \circ L_{\beta''} \simeq L_\gamma \circ L_{\alpha'} \circ L_{\beta''}$

$L_\gamma \circ L_{\alpha'} \circ L_{\beta''} \simeq L_\gamma \circ L_{\beta'} \circ L_{\alpha''}$

Exercise : prove 2^{nd} case similarly



for $\alpha, \beta, \gamma \subset \Sigma_{g+1}$ disjoint

Exercise – use only 3mfd axioms