Stanford 2012

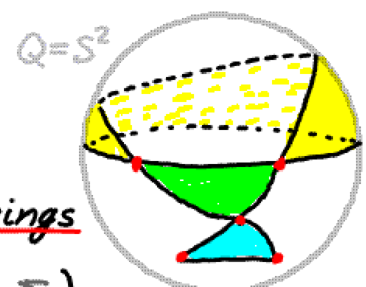
① From 4-manifolds to pseudoholomorphic quilts

② Transversality and strip shrinking

③ Construction recipe for 3-/4-manifold invariants

Morse 2-functions [Whitney... Gay-Kirby]

to view a 4-manifold X as "fibration" over a 2-manifold Q
take a "generic" $f: X \to Q$, i.e.

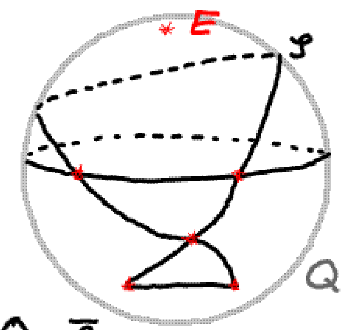


- $\operatorname{Sing} f := \{x \in X \mid df(x) \text{ not onto}\}$ 1-submanifold
- $f(\operatorname{Sing} f) \subset Q$ "submanifold" with cusps and crossings

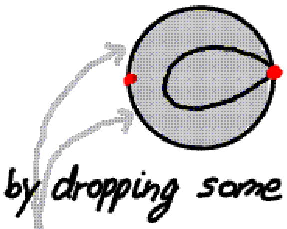
$\Rightarrow$ • "patches" $P \subset Q \setminus f(\operatorname{Sing} f)$ with fibration $(f^{-1}(p) \simeq \Sigma_P)_{p \in P}$

- "seams" $S \subset f(\operatorname{Sing} f) \setminus (\text{cusps} \cup \text{crossings})$ with attaching cycles
- "punctures" : cusp or crossing

Def: A (topological) **quilted surface** is a compact, oriented 2-manifold Q with $\mathcal{S} \subset Q$ 1-submanifold, $E \subset Q \setminus \mathcal{S}$ finite such that $\overline{\mathcal{S}} \setminus \mathcal{S} \subset E$.



Its **patches** are the connected components of $Q \setminus \overline{\mathcal{S}}$,
seams ——— " ——— $\mathcal{S}$,
ends are the points in E.



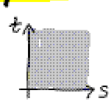
Note: We will obtain quilted surfaces with boundary by dropping some patches and designating the adjacent seams as boundary components.

From Donaldson invariants to pseudoholomorphic quilts

count (modulo gauge)

ASD connections on X w.r.t. G-bundle, metric on X

- **locally on patches** (for trivial $SU(2)$-bundle, product metric)



$\times \Sigma \hookrightarrow f^{-1}(\text{patch}) \subset X \qquad \Sigma \simeq f^{-1}(p)$ Riemann surface

$$\Xi = \Phi\, ds + \Psi\, dt + A(s,t) \in \Omega^1([0,1]^2 \times \Sigma; su(2))$$

$$F_\Xi + *_X F_\Xi = 0 \iff \boxed{\begin{cases} (\partial_s A - d_A \Phi) + *_\Sigma(\partial_t A - d_A \Psi) = 0 \\ (\partial_s \Psi - \partial_t \Phi + [\Phi, \Psi]) + *_\Sigma F_A = 0 \end{cases}}$$

$$\tfrac{1}{2}\int \langle F_\Xi \wedge F_\Xi \rangle = \int |\partial_s A - d_A \Phi|^2 + |F_A|^2 \qquad \text{globally fixed energy}$$

"Large structure limit" for ASD connections on $[0,1]^2\times\Sigma$

$\varepsilon\to 0$ in metric $ds^2+dt^2+\varepsilon^2 g_\Sigma$

$$(ASD_\varepsilon)\ \begin{cases} (\partial_s A - d_A\Phi) + *_\Sigma(\partial_t A - d_A\Psi) = 0 \\ (\partial_s\Psi - \partial_t\Phi + [\Phi,\Psi]) + \varepsilon^{-2} *_\Sigma F_A = 0 \end{cases}$$

$\int |\partial_s A - d_A\Phi|^2 + \varepsilon^{-2}|F_A|^2 \le$ globally fixed energy

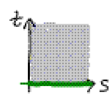
[Dostoglou-Salamon '94]: up to finitely many bubbling points in $[0,1]^2$,

$A_\varepsilon : [0,1]^2 \to \Omega^1(\Sigma; su(2))$ $\xrightarrow[\text{modulo gauge}]{\varepsilon\to 0}$ $[A]_0 : [0,1]^2 \to \{F_A=0\}/\text{gauge} = \mathcal{R}_\Sigma$ representation space corresp. to bundle over Σ; symplectic, J compatible

(ASD_ε) $\quad \partial_s[A]_0 + \underbrace{*_\Sigma}_{J}\partial_t[A]_0 = 0$

"Large structure limit" for ASD connections

- near a boundary



$\times\Sigma \hookrightarrow f^{-1}(\text{patch}) \subset X$

U —— $\times Y$

$\partial Y = \Sigma$

[W'05]: Energy Quantization for $(ASD_{\varepsilon\to 0})$ with Lagrangian boundary conditions

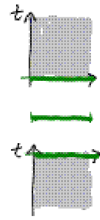
Corollary: degenerate metrics by $\begin{pmatrix} ds^2+dt^2+\varepsilon^2 g_\Sigma \\ ds^2+\varepsilon^2 g_Y \end{pmatrix}$ then

on compact subsets disjoint from finitely many bubbling points

- $\|\partial_s A_\varepsilon - d_{A_\varepsilon}\Phi_\varepsilon\|_{L^2(\Sigma)}(s,t) + \varepsilon^{-1}\|F_{A_\varepsilon}\|_{L^2(\Sigma)}(s,t) \le C(t+\varepsilon)^{-1}$
- $A_\varepsilon(s,0) = B_\varepsilon(s)|_{\partial Y}$, $\|F_{B_\varepsilon(s)}\|_{L^\infty(Y)} \to 0$

by convex span of [D.-S.'94] and [W.'05] expect [S.W.'16] / [D. Duncan]

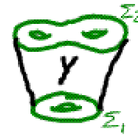
- near a seam



$\times\Sigma_2$ $\hookrightarrow f^{-1}(\text{patch}) \subset X$

$\cup$ $\times Y$

$\times\Sigma_1$



$\partial Y = \bar{\Sigma}_1 \cup \Sigma_2$

$$\left.\begin{array}{l} A_{2\varepsilon}, \Phi_{2\varepsilon}, \Psi_{2\varepsilon} \\ B_\varepsilon \\ A_{1\varepsilon}, \Phi_{1\varepsilon}, \Psi_{1\varepsilon} \end{array}\right\} \xrightarrow[\text{mod gauge}]{\varepsilon\to 0} \left\{\begin{array}{l} \partial_s[A_2] + *_\Sigma \partial_t[A_2] = 0 \\ A_i(s,0) = B(s)|_{\Sigma_i}, \quad F_{B(s)} = 0 \text{ on } Y \\ \partial_s[A_1] + *_\Sigma \partial_t[A_1] = 0 \end{array}\right.$$

$$([A_1](s), [A_2](s)) \in L_Y = \left\{\begin{array}{l}\text{representations} \\ \text{of } \partial Y \text{ that extend} \\ \text{to flat rep of } Y\end{array}\right\} \subset \mathcal{R}_{\Sigma_1} \times \mathcal{R}_{\Sigma_2}$$

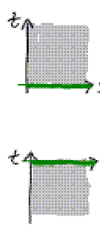
Lagrangian   symplectic

IDEA for bypassing (most) hard analysis [W.-Woodward // Perutz for Seiberg-Witten]

construct Donaldson-type (3-), 4-manifold invariants

by counting pseudoholomorphic quilts

near a seam



$$\xrightarrow{u_2} M_2 \quad (= [A_2],\ = \mathcal{R}_{\Sigma_2})$$

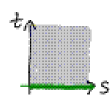
$$\xrightarrow{u_1} M_1$$

$$\left\{\begin{array}{l} \bar{\partial}_{J_2} u_2 = 0 \\ (u_1(s,0), u_2(s,0)) \in L_{12} \subset M_1 \times M_2 \\ \bar{\partial}_{J_1} u_1 = 0 \end{array}\right.$$

Lagrangian w.r.t. $(-\omega_1)\times\omega_2$

get same estimates as $(-J_1, J_2)$-holomorphic maps with L_{12} Lagrangian boundary condition

by "folding" :



$$(s,t) \longmapsto (u_1(s,-t), u_2(s,t)) \in M_1^- \times M_2$$
$$(s,0) \longmapsto (u_1(s,0), u_2(s,0)) \in L_{12}$$

Def.: Given a quilted surface $(Q, \mathcal{S}_{seams}, E_{punctures})$ with $\left(\begin{array}{l}\text{if } P_- = P_+ \text{ take} \\ \partial P_- \sqcup \partial P_+ \subset \overline{P \setminus P}\end{array}\right)$

symplectic labels: M_P symplectic for each patch $P \subset Q \setminus \mathcal{S}$

$L_S \subset M_{P_-}^- \times M_{P_+}$ Lagrangian for each seam $S \subset \partial P_\pm$

(almost) complex structures: j on Q s.t. $\mathcal{S}$ is real analytic and "straight"

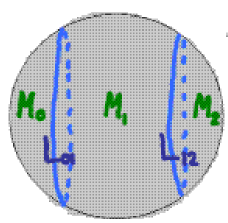
in $nbhd(E) \setminus E \overset{hol}{\cong} E \times (\mathbb{R}^+ \times S^1, i) \leftarrow \mathbb{R}^+ \times pt$

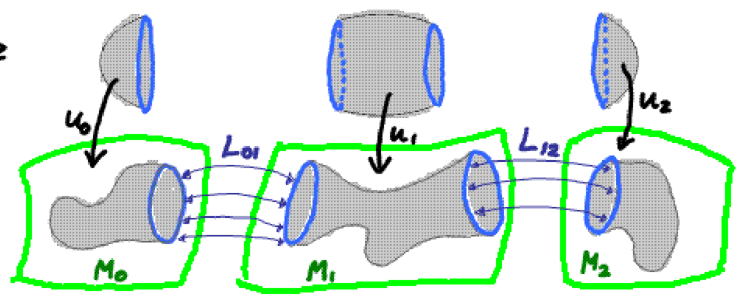
$\underline{J} = (J_P)_{P \text{ patches}}$ on M_P compatible almost complex structure

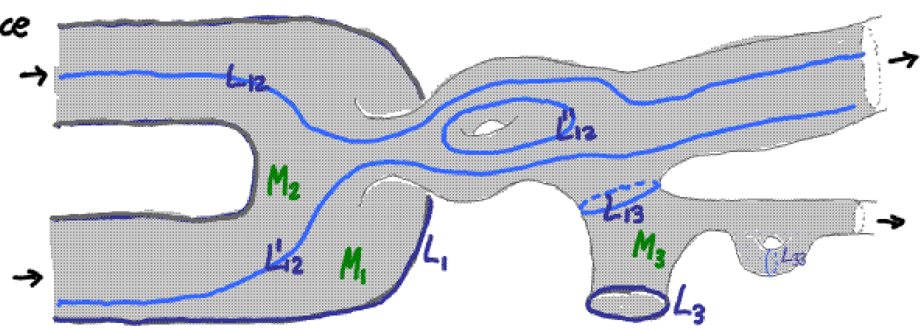
a **$(j, \underline{J})$-holomorphic quilt** $\underline{u} : Q \to (\underline{M} = (M_P), \underline{L} = (L_S))$ is a tuple of

$(j|_{\overline{P}}, J_P)$-holomorphic maps (of finite energy) $u_P : \overline{P} \to M_P$ for each patch P

with seam conditions $(u_{P_-}, u_{P_+})(S) \subset L_S$ for each seam $S \subset \partial P_\pm$

Ex.: quilts on 

$Q \simeq S^2$ are 

Ex: a quilted surface with boundary, in/outgoing ends, and symplectic labels 

[closed: Perutz ; with ends: W-Woodward] * exact/monotone or the like

Thm: Any quilted surface Q with symplectic labels* and ends $E = E_{in} \sqcup E_{out}$

defines a relative invariant $\boxed{\Phi_Q : \bigotimes_{e\in E_{in}} HF(\underline{L}_e) \longrightarrow \bigotimes_{e\in E_{out}} HF(\underline{L}_e)}$

depending only on – $(Q, \mathcal{S}, E)$ up to smooth isotopy

– $\underline{L}$ up to Hamiltonian* isotopy

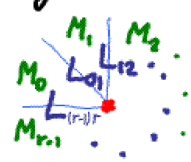
* "split" $\varphi_- \times \varphi_+ \circlearrowleft M_{p_-} \times M_{p_+}$ (general: ?use forms?)

and satisfying

- homotopy $(Q_t, j_t, \underline{J}_t) \Rightarrow \Phi_{Q_0} = \Phi_{Q_1}$ [WW]
- composition = gluing $\Phi_{Q_1} \underset{E^1_{in} \cong E^0_{out}}{\circ} \Phi_{Q_0} = \Phi_{Q_1 \# Q_0}$ [Mau]
- insertion of diagonal $\Phi_{(Q,\mathcal{S})} \simeq \Phi_{(Q,\mathcal{S} \sqcup S')}$ (up to potential shift in spin background classes) [WW]
  $L_{S'} = \Delta_{M_p}$

quilted Floer homology for cyclic generalized Lagrangian correspondences

$\underline{L}_e = (L_{01}, L_{12}, \dots, L_{(r-1)r})$ associated to end e



$$\boxed{HF(\underline{L}) := HF(\Delta_{M_0} \times \Delta_{M_1} \dots \times \Delta_{M_{r-1}},\ L_{01} \times L_{12} \times \dots \times L_{(r-1)r})}$$

$M_0 \times M_1 \times M_1 \times \dots \times M_{r-1} \times M_0$

- generated by $L_{01} \times L_{12} \times \dots \times L_{(r-1)r} \cap (\Delta_{M_0} \times \Delta_{M_1} \dots \times \Delta_{M_{r-1}})^T \cong \{(p_i \in M_i)_{i \in \mathbb{Z}_r} \mid (p_i, p_{i+1}) \in L_{i(i+1)}\}$

  ... if $\pitchfork$; otherwise perturb by "generic" $\phi \in Ham(M_0 \times \dots \times M_0)$

- ∂ counts quilted cylinders$/\mathbb{R}$