The symplectic category

• slides & papers on www-math.mit.edu/~katrin

• course notes on …. /teach/quilts

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based on joint work with C. Woodward, S. Mau

The symplectic 2-category (homology version) $Symp^{\#}$

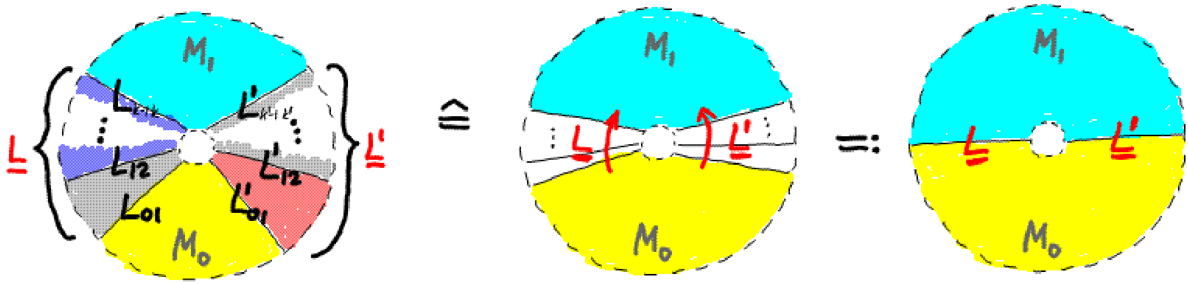
- object: M admissible symplectic
- $Mor(M_0, M_1)$:= Donaldson/Fukaya-category
  - ^{2}object: admissible gen. Lagr. corresp. $\underline{L} = (M_0 \xrightarrow{L_{01}} N_1 \ldots N_k \xrightarrow{L_{(k-1)k}} M_1)$
  - ${}^2Mor(\underline{L}, \underline{L}')$:= quilted Floer homology with ^{2}composition TBD
- composition functor $\#$: $Mor(M_0, M_1) \times Mor(M_1, M_2) \longrightarrow Mor(M_0, M_2)$

  on ^{2}objects: $(\underline{L}_{01}, \underline{L}_{12}) \longmapsto \underline{L}_{01} \# \underline{L}_{12}$ concatenation
- identities $(M_0 \circlearrowleft^{\text{empty sequence}}) =: 1_{M_0} \in Mor(M_0, M_0)$, $1_{\underline{L}} \in Mor(\underline{L}, \underline{L})$

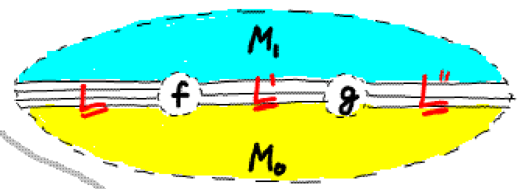
Recall : quilted Floer homology

$$Mor(\underline{L},\underline{L}') = Mor\left(\begin{matrix} & \xrightarrow{L_{01}} N_1 \; \ldots \; N_{k-1} \xrightarrow{L_{(k-1)k}} & \\ M_0 & & M_1 \\ & \xrightarrow{L'_{01}} N'_1 \; \ldots \; N'_{k'-1} \xrightarrow{L'_{(k'-1)k'}} & \end{matrix} \right) = \ker\partial / \operatorname{im}\partial$$

Floer differential ∂ counts quilted cylinders$/\mathbb{R}$



We define **²composition** ${}^2Mor(\underline{L},\underline{L}') \otimes {}^2Mor(\underline{L}',\underline{L}'') \longrightarrow {}^2Mor(\underline{L},\underline{L}'')$

as quilt invariant $f \circ g :=$ 

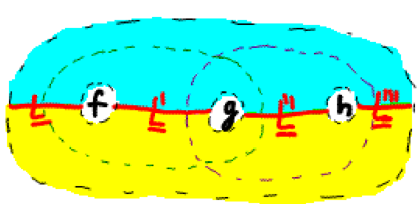
Lemma: ²composition is **associative**

with **2-identity** $1_{\underline{L}} :=$ 

$\in {}^2Mor(\underline{L},\underline{L})$

Proof:

$(f\circ g)\circ h \overset{\text{gluing}}{=}$

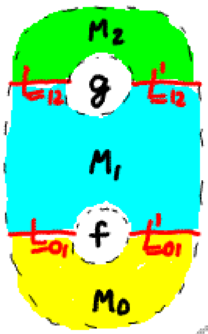


$f\circ(g\circ h) \overset{\text{gluing}}{=}$

$1_{\underline{L}} \circ g \;=\; g$

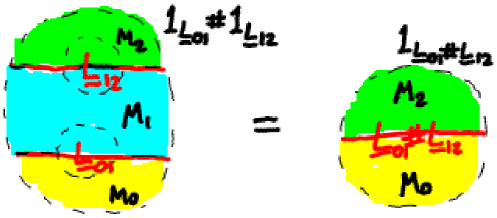
composition functor $\#: Mor(M_0,M_1) \times Mor(M_1,M_2) \longrightarrow Mor(M_0,M_2)$

$$\begin{array}{cccc} \underline{L}_{01} & \underline{L}_{12} & \longmapsto & \underline{L}_{01}\#\underline{L}_{12} \\ \underline{L}'_{01} & \underline{L}'_{12} & \longmapsto & \underline{L}'_{01}\#\underline{L}'_{12} \end{array}$$

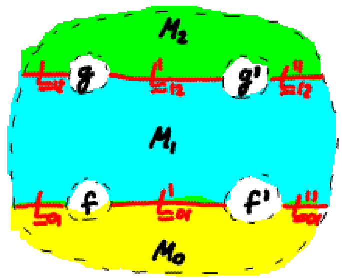


$=: f\#g$ $\qquad HF(\underline{L}_{01},\underline{L}'_{01}) \otimes HF(\underline{L}_{12},\underline{L}'_{12}) \rightarrow HF(\underline{L}_{01}\#\underline{L}_{12}, \underline{L}'_{01}\#\underline{L}'_{12})$

Lemma: # is Id-compatible, associative, "interchanges"



$1_{\underline{L}_{01}}\#1_{\underline{L}_{12}} = 1_{\underline{L}_{01}\#\underline{L}_{12}}$



$= (f\#g)\circ(f'\#g')$

$= (f\circ f')\#(g\circ g')$

summary: Thm1: There is a 2-category $Symp^{\#}$ with

- object : symplectic manifold
- 1-morphism : generalized Lagrangian correspondence

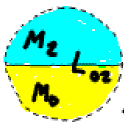
  1-composition : algebraic (concatenation)
- 2-morphism : quilted Floer homology class

  horizontal & vertical 2-composition given by relative quilt invariant

Thm2: $\boxed{L_{01}\#L_{12} \underset{\text{2-isomorphic}}{\sim} L_{01}\circ L_{12}}$ if geometric composition is embedded

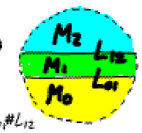
Thm2: $\boxed{L_{01}\#L_{12} \underset{\text{2-isomorphic}}{\sim} L_{01}\circ L_{12}} =: L_{02}$ if geometric composition is embedded

Proof: Define $L_{01}\#L_{12} \underset{\beta}{\overset{\alpha}{\rightleftarrows}} L_{02}$ from HF-isomorphisms:



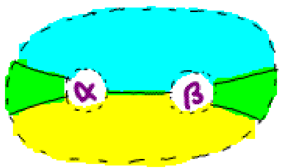
$\in HF(L_{02},L_{02})$ $1_{L_{02}}$ $\xrightarrow{\sim} HF(L_{01}\#L_{12}, L_{02}) \ni \alpha \xleftarrow{\sim} HF(L_{01}\#L_{12}, L_{01}\#L_{12}) \ni$ $1_{L_{01}\#L_{12}}$

$1_{L_{02}} \in HF(L_{02},L_{02}) \xrightarrow{\sim} HF(L_{02}, L_{01}\#L_{12}) \ni \beta \xleftarrow{\sim} HF(L_{01}\#L_{12}, L_{01}\#L_{12})$

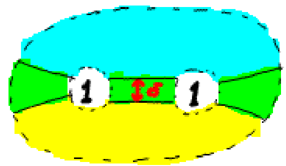


Then

$\alpha\circ\beta =$



$\underset{\delta\to 0}{\cong}$



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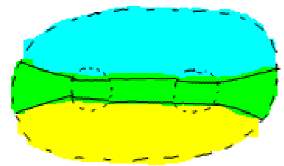
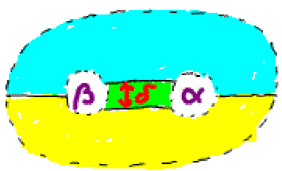


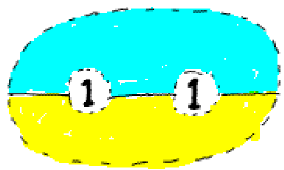
$= 1_{L_{01}\#L_{12}}$

and $\beta\circ\alpha = 1_{L_{02}}$ similarly.

$\beta\circ\alpha =$



$\cong$



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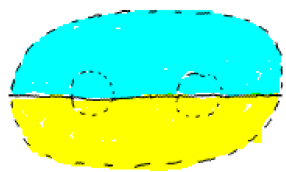
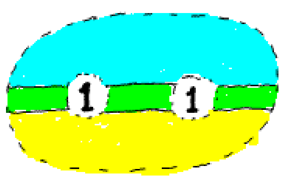


$= 1_{L_{02}}$

OR: $\cong$



$\circ I =$

$\circ I = I(1_{L_{01}\#L_{12}}) = 1_{L_{02}}$

$I : HF(L_{01}\#L_{12}, L_{01}\#L_{12}) \xrightarrow{\sim} HF(L_{02},L_{02})$

Corollary$^{\text{of Thm 1}}$: There is a **2-functor $\mathcal{Don} : \mathcal{Symp}^{\#} \longrightarrow \mathcal{Cat}$** $= \begin{pmatrix} \text{categories} \\ \text{functors} \\ \text{nat. transf's} \end{pmatrix}$

induced by the 2-category structure and choice of base object $M_{base} = pt$.

Corollary$^{\text{of Thm 1\&2}}$: There is a 2-functor **$\mathcal{Symp} \longrightarrow \mathcal{Cat}_{/\sim}$** $= \begin{pmatrix} \text{categories} \\ \text{functors/isom.} \\ \text{nat. transf/isom} \end{pmatrix}$

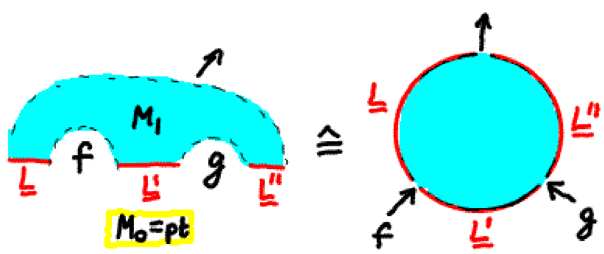
where $\mathcal{Symp}$ is $\begin{pmatrix} \text{admissible symplectic } M \\ \text{adm. gen. Lagr. corresp.}/\sim \text{ by embedded composition} \\ \text{HF class} \end{pmatrix}$.

concretely, $\mathcal{Symp}^{\#} \rightarrow \mathcal{Cat}$ is given as follows:

- M symplectic $_{\&\text{ admissible}}$ $\searrow$

  extended Donaldson-category $\mathcal{Don}(M) := \mathcal{Mor}(pt, M)$

  - object: $pt \rightarrow \overset{\underline{L}}{\cdots} \rightarrow M$ generalized $_{\text{admissible}}$ Lagrangian
  - $\mathcal{Mor}(\underline{L}, \underline{L}') := {}^{2}\mathcal{Mor}(\underline{L}, \underline{L}') = HF(\underline{L}, \underline{L}')$
  - composition



• $M_0 \xrightarrow{\underline{L}_{01}} M_1$ generalized **Lagrangian corresp.** & admissible

$\hookrightarrow$ **functor** $\quad \mathcal{D}on(\underline{L}_{01}) : \mathcal{D}on(M_0) \longrightarrow \mathcal{D}on(M_1)$

$$\# : \underbrace{Mor(pt, M_0)}_{\mathcal{D}on(M_0)} \otimes \overbrace{\begin{bmatrix} \underline{L}_{01} \\ 1_{\underline{L}_{01}} \end{bmatrix}}^{Mor(M_0,M_1)} \longrightarrow \underbrace{Mor(pt, M_1)}_{\mathcal{D}on(M_1)}$$

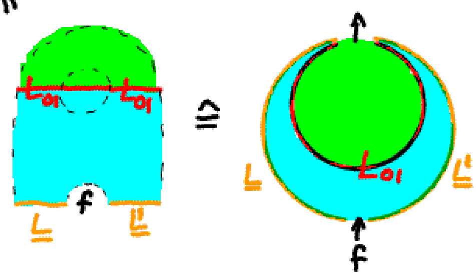
on objects: $\underset{pt \to \dots \to M_0}{\underline{L}} \longmapsto \underset{pt \to \dots \to M_0 \to M_1}{\underline{L} \# L_{01}}$

on morphisms: $HF(\underline{L}, \underline{L}') \longrightarrow HF(\underline{L} \# L_{01}, \underline{L}' \# L_{01})$

$f \longmapsto f \# 1_{L_{01}}$

$\mathcal{D}on(\underline{L}_{01}) f$

$\parallel$



• $HF(\underline{L}_{01}, \underline{L}'_{01}) \ni \alpha \longrightarrow$ **natural transformation**

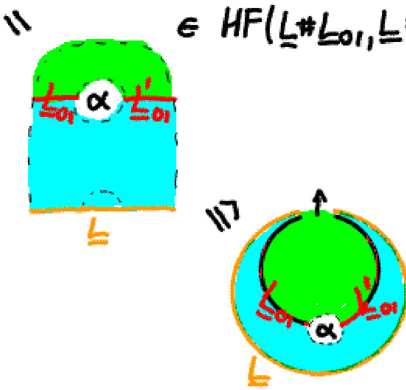
$$\mathcal{D}on(\underline{L}_{01}) \xrightarrow{\mathcal{D}on(\alpha) =: T_\alpha} \mathcal{D}on(\underline{L}'_{01})$$

To $\underline{L} \in \mathcal{D}on(M_0)$ associate $T_\alpha(\underline{L}) := 1_{\underline{L}} \# \alpha \in HF(\underline{L} \# \underline{L}_{01}, \underline{L} \# \underline{L}'_{01})$

s.t. $\forall (\underline{L} \xrightarrow{f} \underline{L}')$

$$\boxed{\begin{array}{ccc} \underline{L} \# \underline{L}_{01} & \xrightarrow{\mathcal{D}on(\underline{L}_{01}) f} & \underline{L}' \# \underline{L}_{01} \\ T_\alpha(\underline{L}) \downarrow & & \downarrow T_\alpha(\underline{L}') \\ \underline{L} \# \underline{L}'_{01} & \xrightarrow{\mathcal{D}on(\underline{L}'_{01}) f} & \underline{L}' \# \underline{L}'_{01} \end{array}}$$

by the interchange axiom for $\mathcal{S}ymp^{\#}$.



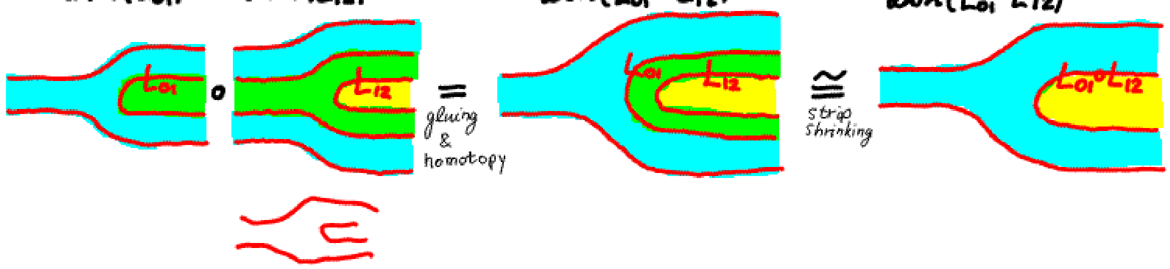
Corollary: Functors and geometric composition

$$\mathrm{Don}(L_{01}) \circ \mathrm{Don}(L_{12}) = \mathrm{Don}(L_{01} \# L_{12}) \cong \mathrm{Don}(L_{01} \circ L_{12})$$

always by functoriality — by $L_{01} \# L_{12} \sim L_{01} \circ L_{12}$ if embedded

direct proof:

$\mathrm{Don}(L_{01}) \circ \mathrm{Don}(L_{12})$ $\qquad$ $\mathrm{Don}(L_{01} \# L_{12})$ $\qquad$ $\mathrm{Don}(L_{01} \circ L_{12})$



So far (homology version)

- conformal structure of quilted surface, perturbations, ... fixed
- given symplectic labels, count isolated holomorphic quilts

$\Rightarrow$ map on Floer homology independent of choices

- all disk/sphere/figure 8 bubbling excluded by admissibility

Next: A_∞-/ chain level version [Mau - WW]

- construct polytopes $Q \subset \{\tilde{\Sigma}$ quilted surface$\}$
- given symplectic labels $(\underline{M}, \underline{L})$, count isolated pairs

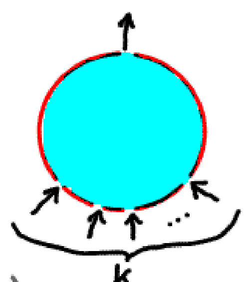
$$\{(\underline{u}, \tilde{\Sigma}) \mid \tilde{\Sigma} \in Q,\ \underline{u} : \tilde{\Sigma} \to \underline{M} \text{ hol. quilt}\}$$

$\Rightarrow$ map on chain level satisfies relations given by boundary facets of Q

- still need admissibility to exclude disk/sphere/figure 8 bubbling

(including them should yield more general polytopes and algebra)

extended Fukaya - A_∞ category $\mathfrak{Fuk}^\#(M)$

- object: $pt \to \underset{\cdots}{\overset{\underline{L}}{=}} \to M$ generalized Lagrangian (admissible)
- $Mor(\underline{L}, \underline{L}') := CF(\underline{L}, \underline{L}')$
- compositions $\hat{m} = (m_k)_{k=1,2,\ldots}$ defined by associahedra $Q_k^{ass} \cong \{$  $\}$

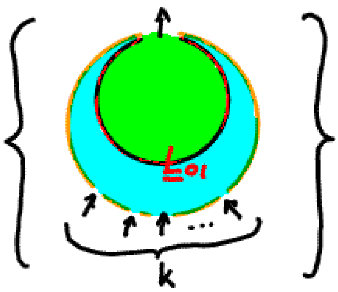
(boundary arcs $\hat{=}$ tuples of strips)

• $M_0 \xrightarrow{\underline{L}_{01}} M_1$ generalized Lagrangian corresp. & admissible

↳ A_∞-functor $\mathcal{F}(\underline{L}_{01}) : Fuk^\#(M_0) \longrightarrow Fuk^\#(M_1)$

on objects: $\underset{pt\to\dots\to M_0}{\underline{L}} \longmapsto \underset{pt\to\dots\to M_0\to M_1}{\underline{L} \; \# \; L_{01}}$

on morphisms: $\bigotimes_{i=1}^{k} CF(\underline{L}^{(i-1)}, \underline{L}^{(i)}) \longrightarrow CF(\underline{L}^{(0)} \# \underline{L}_{01}, \underline{L}^{(k)} \# \underline{L}_{01})$



$\cong Q_k^{mult}$

defined by multiplihedra

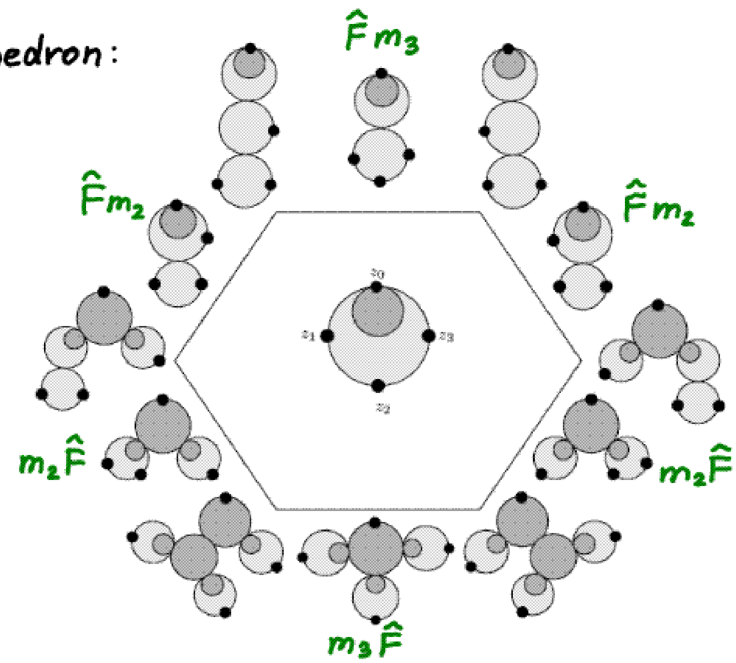
construct strip-like conformal structure at outgoing end

facets of multiplihedron:

$\Downarrow$

A_∞-functor relation

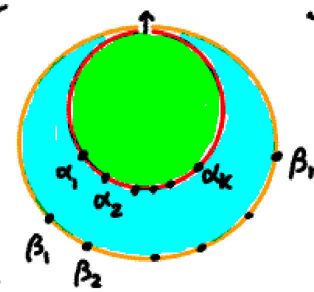
"$F\hat{m} + \hat{m}F = 0$"



work in (slow) progress : Extend $\underline{L}_{01} \longmapsto \mathcal{F}(\underline{L}_{01})$ to an

A_∞-functor $\boxed{\mathrm{Fuk}^\#(M_0,M_1) \longrightarrow A_\infty\mathrm{Fun}(\mathrm{Fuk}^\#(M_0),\mathrm{Fuk}^\#(M_1))}$

• on morphisms $\underset{\ni(\alpha_i)}{\bigotimes_{i=1}^{k} CF(\underline{L}_{01}^{(i-1)},\underline{L}_{01}^{(i)})} \longrightarrow \left(\bigotimes_{j=1}^{r} CF(\underline{L}_{0}^{(j-1)},\underline{L}_{0}^{(j)}) \ni (\beta_j) \;\longrightarrow\; CF(\underline{L}_0^{(0)}\#\underline{L}_{01}^{(0)}, \underline{L}_0^{(r)}\#\underline{L}_{01}^{(k)}) \right)$

$\Big\{$  $\Big\} \cong Q^{mult}_{k,r}$

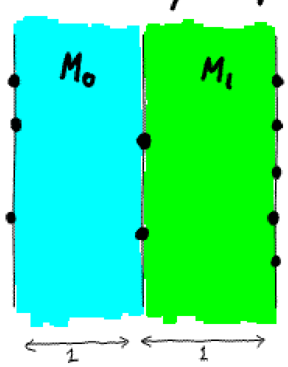
requires realization of generalized multiplihedra by quilted surfaces

Moreover: $L_{01}\#L_{12} \sim \underset{\text{embedded}}{L_{01}\circ L_{12}}$ and $\mathcal{F}(L_{01})\circ\mathcal{F}(L_{12}) = \mathcal{F}(L_{01}\#L_{12}) \cong \mathcal{F}(L_{01}\circ L_{12})$

work in (rapid) progress [Ma'u] : There exists an

A_∞-functor $\boxed{\mathcal{B} : \underset{\text{or } \mathrm{Fuk}^\#(M_0,M_1)}{\mathrm{Fuk}(M_0^-\times M_1)} \longrightarrow \mathrm{Bimodules}(\underset{\text{or } \mathrm{Fuk}^\#}{\mathrm{Fuk}(M_0)},\underset{\text{or } \mathrm{Fuk}^\#}{\mathrm{Fuk}(M_1)})}$

defined by "simpler" polyhedra of quilted surfaces



Moreover $\mathcal{B}(L_{01}\#L_{12}) \cong \mathcal{B}(\underset{\text{embedded}}{L_{01}\circ L_{12}})$,

but relation to $\mathcal{B}(L_{01})\otimes\mathcal{B}(L_{12})$ is unclear.