Lagrangian correspondences & holomorphic quilts

- slides & papers on www-math.mit.edu/~katrin
- course notes on …. /teach/quilts

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based on joint work with C. Woodward, S. Mau

<u>Def^{n}</u> [Weinstein]: $M_i = (M_i, \omega_i)$ symplectic manifolds $i = 0,1$

A **Lagrangian correspondence** (relation) L_{01} from M_0 to M_1

is a Lagrangian submanifold $L_{01} \subset M_0^- \times M_1 := (M_0 \times M_1, (-\omega_0) \times \omega_1)$.

<u>Notation</u>: $M_0 \xrightarrow{L_{01}} M_1$

transposition: $L_{01}^T \subset M_1^- \times M_0$ $\quad M_0 \xleftarrow{L_{01}^T} M_1$

<u>Examples</u>:
- $\varphi : M_0 \to M_1$ symplectomorphism $\rightsquigarrow$ graph $\varphi \subset M_0^- \times M_1$
- $L_0 \subset M_0$ Lagrangian submanifold $\rightsquigarrow$ $pt \xrightarrow{L_0} M_0$, $M_0 \xrightarrow{L_0^T} pt$

Def$^{\underline{n}}$ [Weinstein]: The geometric composition of $M_0 \xrightarrow{L_{01}} M_1 \xrightarrow{L_{12}} M_2$

is $L_{01} \circ L_{12} := \pi_{M_0 \times M_2}(L_{01} \times L_{12} \cap M_0 \times \Delta_{M_1} \times M_2) \subset M_0^- \times M_2$.

generalizes $\operatorname{graph}\varphi \circ \operatorname{graph}\psi = \operatorname{graph}(\psi \circ \varphi)$ $\qquad M_0 \xrightarrow{\varphi} M_1 \xrightarrow{\psi} M_2$

"Generically" $L_{01} \times L_{12} \pitchfork M_0 \times \Delta_{M_1} \times M_2 \xrightarrow{\pi_{M_0 \times M_2}} M_0 \times M_2$ is an immersion.

Lemma: If $L_{01} \circ L_{12}$ is "embedded" ($:\Leftrightarrow L_{01} \times L_{12} \pitchfork M_0 \times \Delta_{M_1} \times M_2 \underset{\text{injective}}{\xrightarrow{\pi_{M_0 \times M_2}}} M_0 \times M_2$)

then it is a (smooth) Lagrangian correspondence from M_0 to M_2.

Example: $pt \xrightarrow{L'} M \xleftarrow{L} pt$ $\quad \underset{L' \circ L^T}{pt \longrightarrow pt}$ $\quad L' \circ L^T = \pi_{pt \times pt}(L' \cap L)$ embedded iff $\#(L \pitchfork L') \le 1$

Example: symplectic reduction

$G \circlearrowright M$ Hamiltonian group action w. moment map $\mu : M \to \mathfrak{g}$; 0 regular value

$\rightsquigarrow \mu^{-1}(0) \xhookrightarrow{\iota \times \pi} \Lambda_\mu \subset M^- \times \underbrace{M/\!/G}_{= \mu^{-1}(0)/G}$ $\qquad M \xrightarrow{\Lambda_\mu} M/\!/G$

- $M \xrightarrow{\Lambda_\mu} M/\!/G \xrightarrow{\ell^T} pt$ $\quad \Lambda_\mu \circ \ell^T = \pi^{-1}(\ell) \subset M$ $\quad$ always embedded
- $pt \xrightarrow{\pi^{-1}(\ell)} M \xrightarrow{\Lambda_\mu} M/\!/G$ $\quad \pi^{-1}(\ell) \circ \Lambda_\mu = \pi(\pi^{-1}(\ell) \cap \mu^{-1}(0)) = \ell$ $\quad$ never embedded
- $pt \xrightarrow{L} M \xrightarrow{\Lambda_\mu} M/\!/G$ $\quad L \circ \Lambda_\mu = \pi(L \cap \mu^{-1}(0))$ embedded iff

$$L \pitchfork \mu^{-1}(0) \;\&\; \forall x \in \mu^{-1}(0) \;\; \#(L \cap Gx) \le 1$$

The symplectic category – elementary version

- object : M symplectic manifold
- morphism $M_0 \to M_1$: Lagrangian correspondence $L_{01} \subset M_0^- \times M_1$
- geometric composition when embedded

naturally extends to a category Symp with same objects,

- $Mor(M_0, M_1) := \{$generalized Lagr. corresp. $\underline{L} = M_0 \xrightarrow{L_{01}} N_1 \xrightarrow{L_{12}} \ldots \longrightarrow N_{k-1} \xrightarrow{L_{(k-1)k}} M_1 \}/\sim$

$\sim$ generated by $\boxed{\ldots N_{j-1} \xrightarrow{L_\alpha} N_j \xrightarrow{L_\beta} N_{j+1} \ldots \sim \ldots N_{j-1} \xrightarrow{L_\alpha \circ L_\beta} N_{j+1} \ldots}$ if embedded

- composition $(M_0 \xrightarrow{\underline{L}_{01}} M_1) \# (M_1 \xrightarrow{\underline{L}_{12}} M_2) := M_0 \xrightarrow{\underline{L}_{01} \# \underline{L}_{12}} M_2$ by concatenation

Functoriality – goal: Construct a functor Symp $\to$ Cat extending

M symplectic $\longrightarrow$ Donaldson/Fukaya – category $\mathcal{C}(M)$

objects : $L \subset M$ Lagrangian submanifolds

$Mor(L, L')$ roughly : generated by $L \cap L'$

$M_0 \xrightarrow{L_{01}} M_1$ Lagr. corresp. $\longrightarrow$ functor $\mathcal{C}(L_{01}) : \mathcal{C}(M_0) \longrightarrow \mathcal{C}(M_1)$

on objects : $\boxed{L \longmapsto L \circ L_{01} \quad \text{if embedded}}$

$\mathcal{C}(L_{01}) : Mor(\underset{\subset M_0}{L, L'}) \overset{?}{\longrightarrow} Mor(\underset{\subset M_1 ?}{L \circ L_{01}, L' \circ L_{01}})$

minor detour: **The symplectic 2-category** (homology version) $Symp^\#$

goal: introduce 2-morphisms s.t. $L_\alpha \# L_\beta \underset{\text{2-isomorphic}}{\sim} L_\alpha \circ L_\beta$ if embedded

- object: M admissible symplectic (bounded geometry & ...)
- $Mor(M_0, M_1) :=$ Donaldson/Fukaya-category
  - ^{2}object: admissible gen. Lagr. corresp. (compact & ...) $\underline{L} = (M_0 \xrightarrow{L_{01}} N_1 \ldots N_k \xrightarrow{L_{(k-1)k}} M_1)$
  - ${}^2Mor(\underline{L}, \underline{L}') :=$ quilted Floer homology with ^{2}composition TBD
- composition functor $\#$: $Mor(M_0, M_1) \times Mor(M_1, M_2) \longrightarrow Mor(M_0, M_2)$
- identities $(M_0 \circlearrowleft^{\text{empty sequence}}) =: 1_{M_0} \in Mor(M_0, M_0)$, $1_{\underline{L}} \in {}^2Mor(\underline{L}, \underline{L})$

$$Mor(\underline{L}, \underline{L}') = Mor\begin{pmatrix} & \overset{L_{01}}{\nearrow} N_1 \;\ldots\; N_{k-1} \overset{L_{(k-1)k}}{\searrow} & \\ M_0 & & M_1 \\ & \underset{L'_{01}}{\searrow} N'_1 \;\ldots\; N'_{k'-1} \underset{L'_{(k'-1)k'}}{\nearrow} & \end{pmatrix}$$ is defined by

quilted Floer homology for cyclic sequences of Lagr. corresp.

$$HF(N_0 \xrightarrow{L_{01}} N_1 \ldots N_{r-1} \xrightarrow{L_{(r-1)r}} N_r = N_0) := HF(\underbrace{L_{01} \times L_{12} \times \ldots \times L_{(r-1)r}}_{\cap}, \underbrace{\Delta_{N_0} \times \Delta_{N_1} \times \ldots \times \Delta_{N_{r-1}}}_{\supset})$$

$$N_0^- \times N_1 \times N_1^- \times N_2 \times \ldots \times N_{r-1}^- \times (N_r = N_0)$$

Thm: $HF(\ldots N_{j-1} \xrightarrow{L_\alpha} N_j \xrightarrow{L_\beta} N_{j+1} \ldots) \cong HF(\ldots N_{j-1} \xrightarrow{L_\alpha \circ L_\beta} N_{j+1} \ldots)$

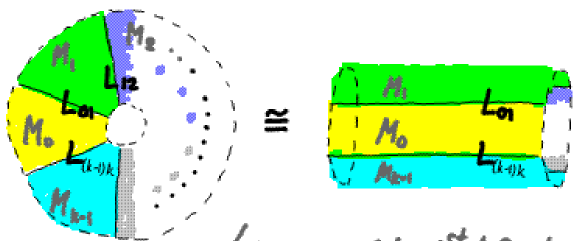
if $L_\alpha \circ L_\beta$ is embedded and "bubbling is apriori excluded"

Proof by strip shrinking in quilted definition of HF

<u>Def^{n}</u>: **quilted Floer homology** $\quad HF\left(\begin{array}{ccc} & M_1 & \\ \overset{L_{01}}{\nearrow} & & \overset{L_{12}}{\searrow} \\ M_0 & & \vdots \\ \underset{L_{(k-1)k}}{\nwarrow} & & \swarrow \\ & M_{k-1} & \end{array}\right)$

CF generated by "intersection points" $(p_i \in M_i)_{i=1..k-1}$ $(p_{i-1}, p_i) \in L_{(i-1)i}$

$\circlearrowleft$ ∂ defined by "counting" quilts of holomorphic strips (mod $\mathbb{R}$)

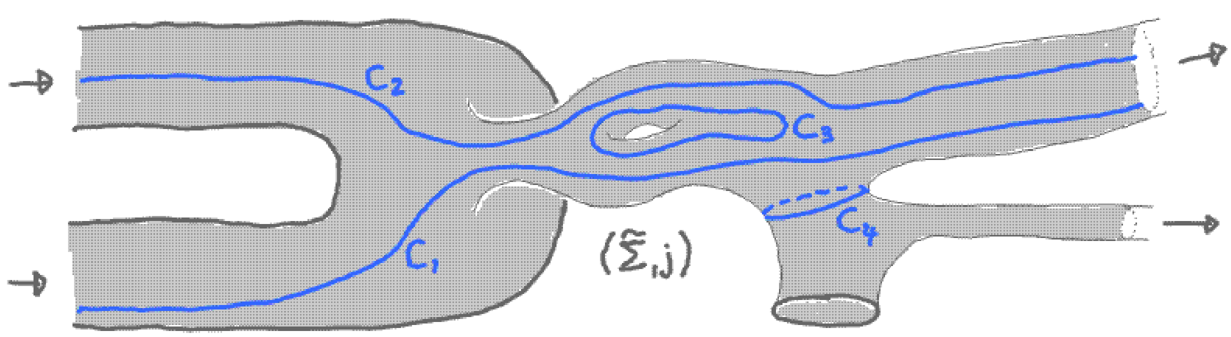


$\widehat{=} \quad \left\{ \left(u_i : \mathbb{R}\times[0,\Delta_i] \to M_i\right)_{i=0..k-1} \;\middle|\; \bar\partial_{J_i} u_i = 0, \left(u_i(s,\Delta_i), u_{i+1}(s,0)\right) \in L_{i(i+1)} \right\}$

(<u>Note</u>: unlike 1st definition we can use different widths $\Delta_i > 0$.)

<u>Def^{n}</u> [Perutz; W-Woodward]: A **quilted surface** $(\tilde\Sigma, C_1, \dots, C_k)$ consists of

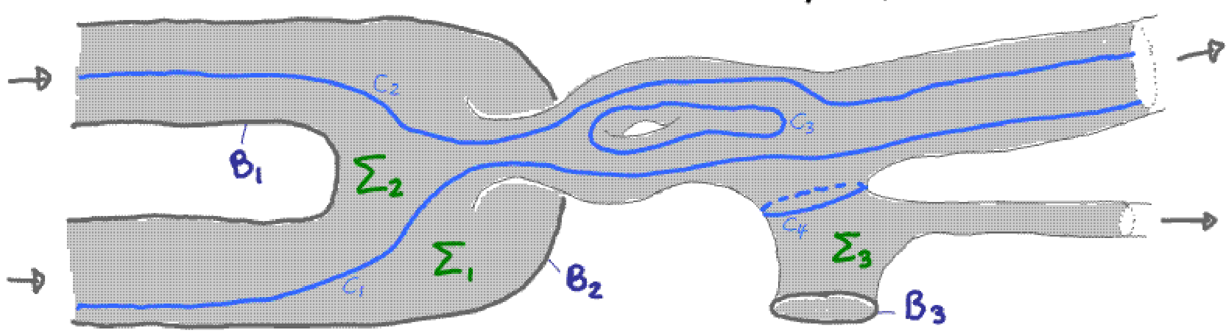
- Riemann surface $(\tilde\Sigma, j)$ with boundary, in-/outgoing strip-/cylindrical ends
- **"seams"** $C_1, \dots, C_k \subset \tilde\Sigma$ disjoint real analytic submanifolds; "straight in ends"



$\Leftrightarrow$ A **quilted surface** has

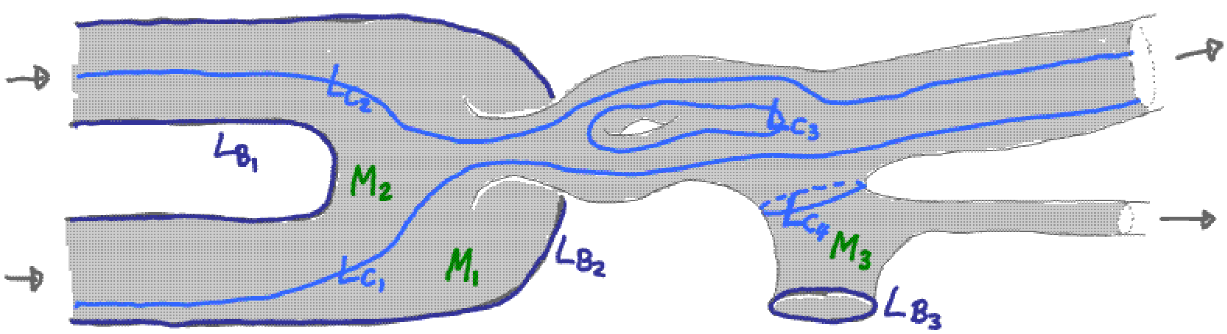
- "**patches**" $\Sigma_1 \sqcup \dots \sqcup \Sigma_\ell = \tilde{\Sigma} \setminus \bigcup C_i$
- "**boundary components**" $B_1 \sqcup \dots \sqcup B_m = \partial \tilde{\Sigma}$
- "**seams**" $C_1, \dots, C_k$; each identified with 2 boundary components of $\bar{\Sigma}_1 \sqcup \dots \sqcup \bar{\Sigma}_\ell$

$\tilde{\Sigma} = \bar{\Sigma}_1 \sqcup \dots \sqcup \bar{\Sigma}_\ell / C_1 .. C_k$



Def$^{\underline{n}}$: **symplectic labels** $((M_i),(L_B),(L_C))$ for a quilted surface associate

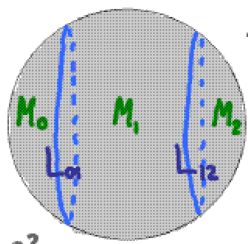
- **patch** Σ_i $\rightsquigarrow$ (M_i,ω_i) **symplectic**
- **boundary component** $B \subset \partial\Sigma_i$ $\rightsquigarrow$ $L_B \subset M_i$ **Lagrangian**
- **seam** $C \subset \partial\Sigma_i \cap \partial\Sigma_j$ $\rightsquigarrow$ $L_C \subset M_i^- \times M_j$ **Lagrangian correspondence**



Def^{n} [P;W-W]: Given a quilted surface with symplectic labels, a **holomorphic quilt** is a collection of maps $(u_i : \overline{\Sigma}_i \to M_i)_{i=1..\ell}$ satisfying

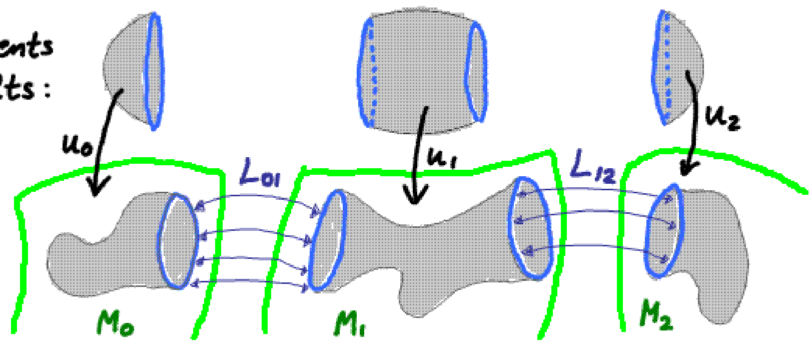
- $\forall$ patch $\boxed{\overline{\partial}_{(j_i, J_i)} u_i = 0}$
- $\forall$ boundary component $B \subset \partial\Sigma_i$ $\boxed{u_i(B) \subset L_B}$
- $\forall$ seam $C \subset \partial\Sigma_i \cap \partial\Sigma_j$ $\boxed{(u_i \times u_j)(C) \subset L_C}$

Ex.:



$\tilde{\Sigma} = S^2$

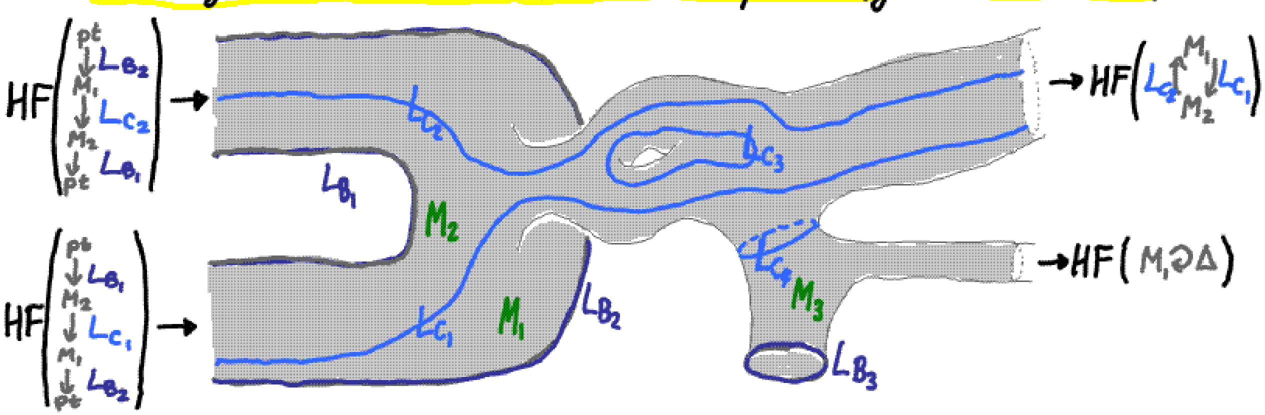
represents quilts:



Note: symplectic labels associate a cyclic gen. Lag. correspondence to each end

Prop^{n}: Holomorphic quilts of finite energy $\sum_{i=1}^{\ell} \int u_i^* \omega = \sum_{i=1}^{\ell} \frac{1}{2} \int |du_i|^2 < \infty$

converge at each end to "intersection points" (generators of HF)



[closed: Perutz ; with ends: W-Woodward]

Thm: Any quilted surface $(\tilde{\Sigma}, C_1..C_k)$ with symplectic labels $(M_i), (L_B), (L_C)$ defines a relative invariant $\boxed{\Phi_{\tilde{\Sigma}} : \bigotimes_{\text{incoming ends}} HF(....) \longrightarrow \bigotimes_{\text{outgoing ends}} HF(....)}$

depending only on
- $(\tilde{\Sigma}, C_1..C_k)$ up to smooth isotopy
- (M_i, L_B, L_C) up to Hamiltonian isotopy

and satisfying

- homotopy $(\tilde{\Sigma}_t)_{t\in[0,1]} \Rightarrow \Phi_{\tilde{\Sigma}_0} = \Phi_{\tilde{\Sigma}_1}$ [WW]
- composition = gluing $\Phi_{\tilde{\Sigma}_1} \circ \Phi_{\tilde{\Sigma}_2} = \Phi_{\tilde{\Sigma}_1 \# \tilde{\Sigma}_2}$ [Mau]
- insertion of diagonal $\Phi_{\tilde{\Sigma}} = \Phi_{\tilde{\Sigma}+(C,\Delta)}$ (up to potential shift in spin background classes) [WW]

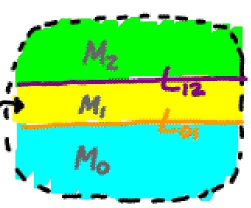
Thm (strip/annulus shrinking):

If $L_{01} \circ L_{12}$ is embedded and "bubbling is a priori excluded" then

$$\begin{tikzcd} \bigotimes_{\text{incoming ends}} HF(....) \arrow[r, "\Phi_{\tilde{\Sigma}}"] \arrow[d, "\text{isom.}"] & \bigotimes_{\text{outgoing ends}} HF(....) \arrow[d, "\text{isom.}"] \\ \bigotimes_{\text{incoming ends}} HF(....) \arrow[r, "\Phi_{\tilde{\Sigma}'}"] & \bigotimes_{\text{outgoing ends}} HF(....) \end{tikzcd}$$

induced by $(L_{01}, L_{12}) \cong L_{01} \circ L_{12}$ commutes

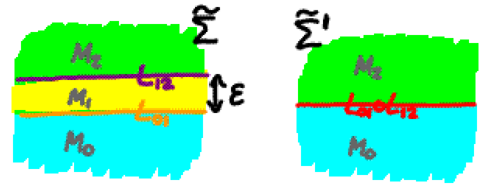
where $\tilde{\Sigma}$:



(Σ, strip or annulus)

$\tilde{\Sigma}'$:

Proof & generalization by adiabatic limit:



Suppose $L_{01} \times L_{12} \pitchfork M_0 \times \Delta_{M_1} \times M_2 \longrightarrow L_{01} \circ L_{12}$

is an N to 1 multiple cover (with smooth image), then

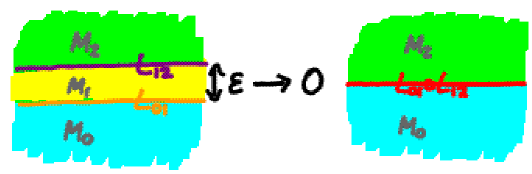
- $CF(\ldots L_{01}, L_{12} \ldots)$ generators $\xrightarrow{N \text{ to } 1}$ $CF(\ldots L_{01} \circ L_{12} \ldots)$ generators

  $= \{(\ldots p_0, p_1, p_2 \ldots) \mid \ldots \begin{smallmatrix} (p_0,p_1) \in L_{01} \\ (p_1,p_2) \in L_{12} \end{smallmatrix} \ldots \}$ $\qquad$ $= \{(\ldots p_0, p_2 \ldots) \mid \ldots (p_0,p_2) \in L_{01} \circ L_{12} \}$

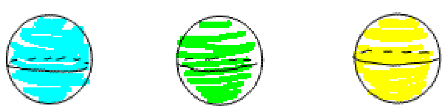
- for $\tilde{\Sigma}$ with strip/annulus width $\varepsilon > 0$ suff. small there is a bijection

0-dim. moduli space of hol. quilts for $\tilde{\Sigma}$ $\xleftrightarrow[\ast]{1 \text{ to } 1}$ 0-dim. moduli space of hol. quilts for $\tilde{\Sigma}'$ $\times$ lifts of seam $C \to L_{01} \circ L_{12}$ to $L_{01} \times_{\Delta_1} L_{12}$ ($\downarrow \pi_{02}$)

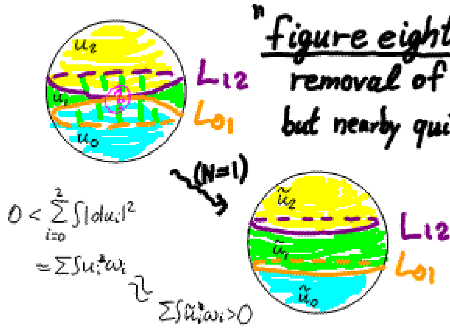
* if we can a priori exclude all possible bubbling in strip shrinking



$S^2 \to M_0 \qquad S^2 \to M_1 \qquad S^2 \to M_2$



$D^2 \to M_0^- \times M_1 \qquad D^2 \to M_1^- \times M_2$



"figure eight" removal of singularity unclear but nearby quilt "bubbles off some energy"

$D^2 \to M_0^- \times M_2$

Examples of a priori bubble exclusion / admissibility

• "all is exact": M admissible if $\omega = d\lambda$

OR $\qquad L \subset M^- \times N$ admissible if $(-\lambda_M) \times \lambda_N \big|_L = df$

• "all is monotone": M admissible if $[\omega] = c_1(TM) \in H_2(M;\mathbb{Z})$

$\Rightarrow \omega = \frac{i}{2\pi} F_\nabla \quad \nabla$ connection on $\mathcal{K}_M^{-1} \to M$

$L \subset M^- \times N$ admissible if oriented & Bohr-Sommerfeld monotone

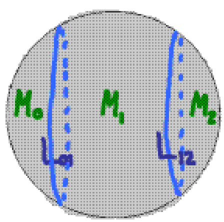
$\big((-\nabla_M) \times (\nabla_N) \big|_L$ is trivial$\big)$

$\Downarrow$

$$\left(\begin{array}{ll} \text{energy} \sim \text{index} & \text{for all quilts} \\ \sum u_i^* \omega_i = \sum c_1(..) + \sum \frac{1}{2} \mu_{Maslov}(..) & \end{array} \right)$$

OR

• "quilted aspherical": $\underline{u} = (u_0, u_1, u_2)$ smooth, satisfies seam conditions (but not necc. holomorphic)



$$E_{S^2}(\underline{u}) := \int u_0^* \omega_{M_0} + \int u_1^* \omega_{M_1} + \int u_2^* \omega_{M_2} = 0$$

OR

• "energy control": Fix energy $E_{\tilde{\Sigma}}$ and limit points of hol. quilts

s.t. $\quad E_{\tilde{\Sigma}} < \min\limits_{E(\underline{u})>0} E_{S^2}(\underline{u}) + E_{\tilde{\Sigma}'}^{min}$

minimal bubble energy

minimal energy of $\tilde{\Sigma}'$ quilt with corresp. limit points