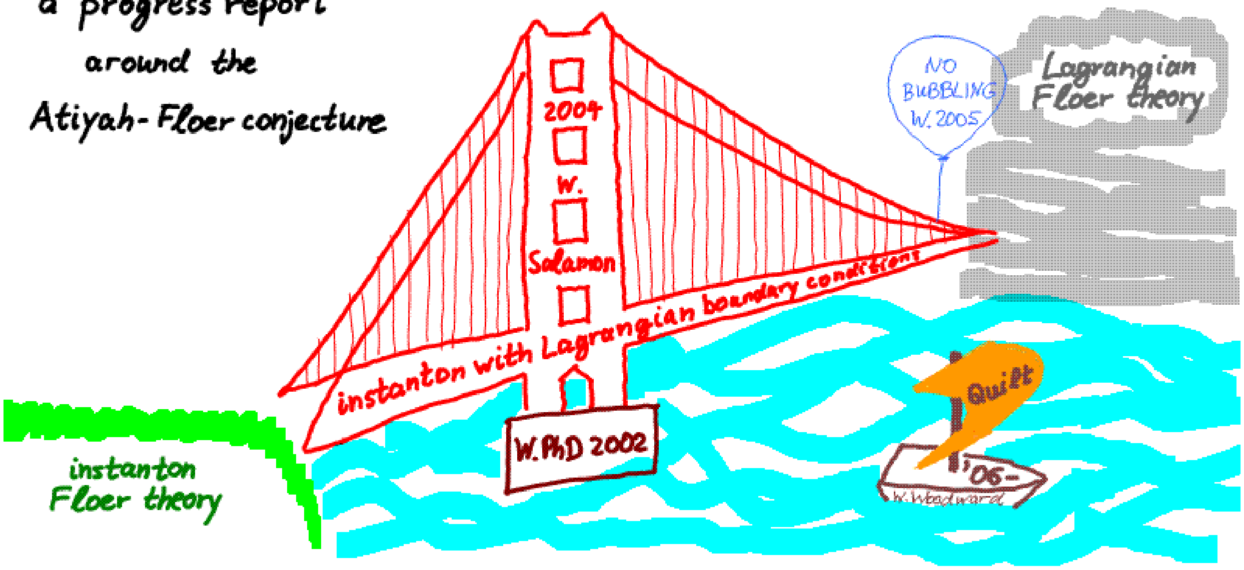
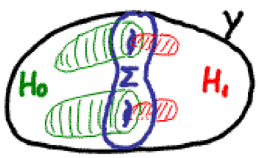
String diagrams in topology, geometry, and analysis

Katrin Wehrheim 2013

a progress report
around the
Atiyah-Floer conjecture



Atiyah-Floer conjecture: $HF(L_{H_0}, L_{H_1}) \simeq HF_{instanton}(Y)$

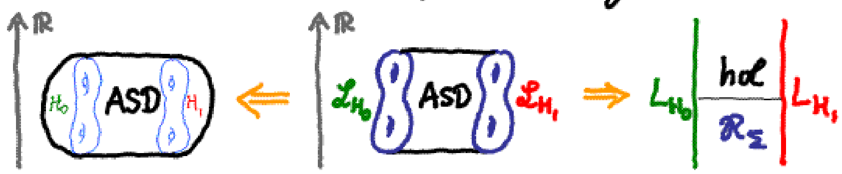


$\mathcal{R}_\Sigma = \mathcal{H}om(\pi_1(\Sigma), SU(2))/_{SU(2)} = \mathcal{A}_{flat}(\Sigma)/_{gauge} = \mathcal{A}(\Sigma)/\!\!/_{\mathcal{G}(\Sigma)}$ symplectic

$\cup \qquad \downarrow$

$L_{H_0}/L_{H_1} = \mathcal{H}om(\pi_1(H_0/H_1), SU(2))/_{SU(2)} = \mathcal{A}_{flat}(H_0/H_1)|_\Sigma / gauge = \mathcal{L}_{H_0/H_1}/_{\mathcal{G}(\Sigma)}$ Lagrangian

1994 Salamon approach by adiabatic limits



2005 Q: Why does decomposition & dim. reduced gauge theory yield topological invariants?

20... A [W-Woodward]: Because it uses functors $Top_{2+1} \to Symp/_{\text{embedded composition}}$ and Symp is a 2-category with isomorphisms for ↺

2012 [W] & if $Top_{2+1} \to Symp$ is "dualizable" it extends to Top_{2+1+1} ... WTF?

What the F are you talking about	That's interesting - tell me more
Shut up and listen to me	I really want to understand that
That's complete BS	Brilliant idea, let's see the proof
We should have a party sometime	Invite the usual suspects, remind me on the day, and come early to chop garlic

Examples of 2-categories

	non-example	Cat	almost example Top_{2+1+1}
• **object**:	V space	category	closed, oriented 2-manifold
• **morphism**:	$V_1 \xrightarrow{A} V_2$ map	functor	3dim. cobordism
• **2-morphism**:	$\begin{tikzcd} V_1 \arrow[r, bend left, "A"] \arrow[r, bend right, "A'"'] \arrow[r, phantom, "\varphi_1 \Uparrow \varphi_2"] & V_2 \end{tikzcd}$ conjugacy $A' = \varphi_2^{-1} A \varphi_1$	natural transformation	4dim. cobordism of cobordism

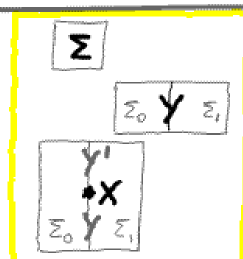
• **horizontal compositions**

$$\begin{tikzcd} V_1 \arrow[r, "A"] \arrow[rr, bend right, "A \overset{hor}{\circ} B"'] & V_2 \arrow[r, "B"] & V_3 \end{tikzcd}, \quad \begin{tikzcd} V_1 \arrow[r, bend left, "A"] \arrow[r, bend right, "A'"'] \arrow[r, phantom, "\Uparrow\varphi"] & V_2 \arrow[r, bend left, "B"] \arrow[r, bend right, "B'"'] \arrow[r, phantom, "\Uparrow\psi"] & V_3 \end{tikzcd} = \begin{tikzcd} V_1 \arrow[r, bend left, "A \overset{hor}{\circ} B"] \arrow[r, bend right, "A' \overset{hor}{\circ} B'"'] \arrow[r, phantom, "\Uparrow \varphi \overset{hor}{\circ} \psi"] & V_3 \end{tikzcd}$$

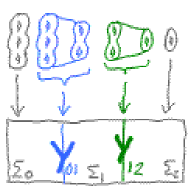
• **vertical composition**

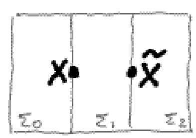
$$\begin{tikzcd} V_1 \arrow[r, bend left] \arrow[r] \arrow[r, bend right] \arrow[r, phantom, shift left=2, "\Uparrow\psi"] \arrow[r, phantom, shift right=2, "\Uparrow\varphi"] & V_2 \end{tikzcd} = \begin{tikzcd} V_1 \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Uparrow \varphi \overset{ver}{\circ} \psi"] & V_2 \end{tikzcd}$$

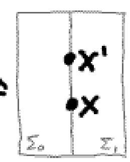
String diagrams for 2-categories

	Cat			Top_{2+1+1}
• **object**:	category	Σ		closed, oriented 2-manifold
• **morphism**:	functor	$\Sigma_0 \xrightarrow{Y} \Sigma_1$		3dim. cobordism
• **2-morphism**:	natural transformation	$\begin{tikzcd} \Sigma_0 \arrow[r, bend left, "Y'"] \arrow[r, bend right, "Y"'] \arrow[r, phantom, "\Uparrow X"] & \Sigma_1 \end{tikzcd}$		4dim. cobordism of cobordisms

• **compositions**:

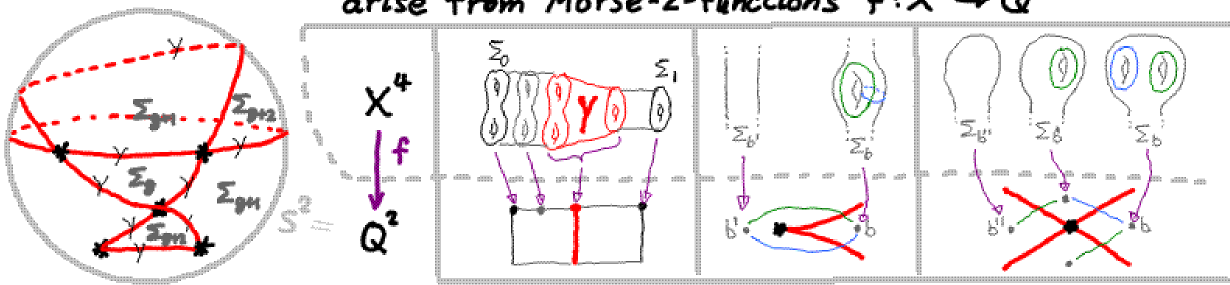
$Y_{01} \overset{hor}{\circ} Y_{12} \triangleq$ 

$X \overset{hor}{\circ} \tilde{X} \triangleq$ 

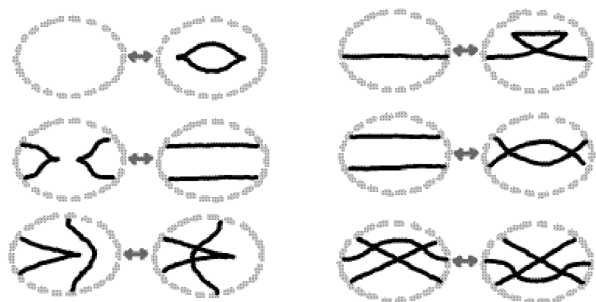
$X \overset{ver}{\circ} X' \triangleq$ 

• **2-category axioms**: *larger string diagrams represent well defined 2-morphisms*

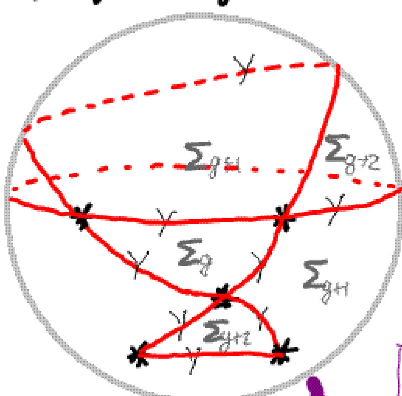
[Whitney,.. Gay-Kirby]: generalized string diagrams for $Top_{n+1|1}$
arise from Morse-2-functions $f: X^n \to Q^2$



all string diagrams for X
are related by "Cerf moves"



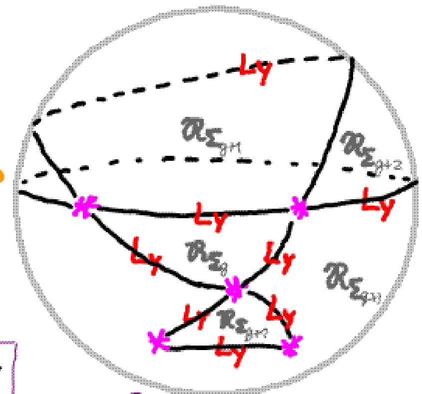
topological diagram for X^4



complete BS

Salamon-style
degeneration of
(Donaldson) ASD eq$^{\underline{n}}$
converges to
pseudoholomorphic
quilt eq$^{\underline{n}}$

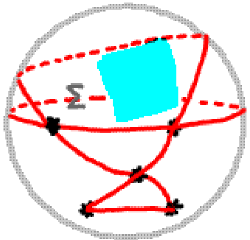
symplectic diagram



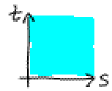
dim reduced gauge theory
functor $Top_{2+1} \to Symp$

- Heegard-type [Perutz]
- representations [W-Woodw]

[W.2012] Cerf moves $\longrightarrow$ identities between counts of quilts



(ASD_ε) on



$\times\Sigma$

$$\begin{cases}(\partial_s A - d_A\Phi) + *_\Sigma(\partial_t A - d_A\Psi) = 0\\ (\partial_s\Psi - \partial_t\Phi + [\Phi,\Psi]) + \varepsilon^{-2} *_\Sigma F_A = 0\\ \int |\partial_s A - d_A\Phi|^2 + \varepsilon^{-2}|F_A|^2 \le \text{fixed energy}\end{cases}$$

[Dostoglou-Salamon '94]: up to finitely many bubbling points in $[0,1]^2$, (in a twisted bundle with smooth $\mathcal{R}_\Sigma$)

$$A_\varepsilon : [0,1]^2 \to \Omega^1(\Sigma; su(2)) \xrightarrow[\text{modulo gaug}]{\varepsilon\to 0} [A]_0 : [0,1]^2 \to \{F_A = 0\}/\text{gauge} = \mathcal{R}_\Sigma$$

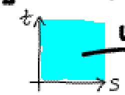
(ASD_ε) $\qquad\qquad$ $\partial_s[A]_0 + *_\Sigma \partial_t [A]_0 = 0$

Towards symplectic category:

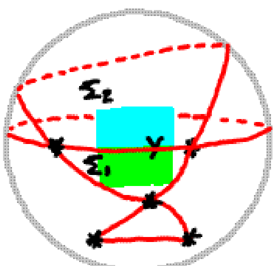
$\boxed{\Sigma} \longrightarrow \boxed{\mathcal{R}_\Sigma}$

$\partial_s u + J(u)\partial_t u = 0$ is a "good" elliptic PDE

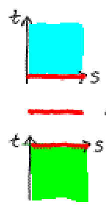
$\overline{\partial}_J u$



$\to \mathcal{R}_\Sigma$



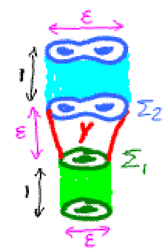
(ASD_ε) on



$\times\Sigma_2$ $\quad A_{2,\varepsilon}, \Phi_{2,\varepsilon}, \Psi_{2,\varepsilon}$

$\times Y$ $\quad B_\varepsilon$

$\times\Sigma_1$ $\quad A_{1,\varepsilon}, \Phi_{1,\varepsilon}, \Psi_{1,\varepsilon}$



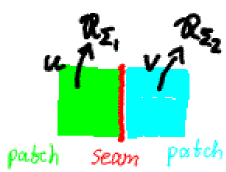
Conj: by convex span of [D.-S.'94] and [W.'05]

$$\xrightarrow[\text{mod gauge}]{\varepsilon\to 0} \begin{cases}\partial_s[A_2]_0 + *_\Sigma \partial_t [A_2]_0 = 0\\ A_{i,0}(s,0) = B_0(s)|_{\Sigma_i}, \quad F_{B_0(s)} = 0 \text{ on } Y\\ \partial_s[A_1]_0 + *_\Sigma \partial_t[A_1]_0 = 0\end{cases}$$

$$\Rightarrow ([A_1]_0(s), [A_2]_0(s)) \in L_Y = \left\{\begin{matrix}\text{representations of } \partial Y\\ \text{that extend to } Y\end{matrix}\right\} \subset \mathcal{R}_{\Sigma_1}\times\mathcal{R}_{\Sigma_2}$$

Towards symplectic category:

$\boxed{Y} \downarrow \boxed{L_Y}$

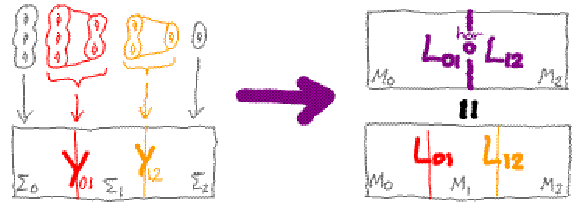


$$\begin{cases}\overline{\partial}_{J_{\mathcal{R}_{\Sigma_1}}} u = 0\\ \overline{\partial}_{J_{\mathcal{R}_{\Sigma_2}}} v = 0\end{cases} \quad u\times v(\text{seam}) \subset L_Y$$

is a "good" elliptic PDE

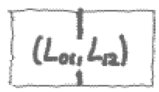
horizontal 1-composition

$$M_0 \xrightarrow{L_{01}} M_1 \xrightarrow{L_{12}} M_2, \qquad M_0 \xrightarrow{L_{01} \overset{hor}{\circ} L_{12}} M_2$$



geometric composition $\boxed{L_{01} \circ L_{12} := \pi_{M_0 \times M_2}\left(L_{01} \times L_{12} \pitchfork M_0 \times \Delta_{M_1} \times M_2\right)}$

- corresponds to topological composition $L(Y_{01} \cup Y_{12}) = L(Y_{01}) \circ L(Y_{12})$
- is an immersed Lagrangian if $\pitchfork$
- "embedded" if $\pitchfork$ and π injective

but we define $L_{01} \overset{hor}{\circ} L_{12} := (L_{01}, L_{12})$ and represent  as PDE 

Theorem [W-W]: There exists a 2-category $Symp_\mu$ (for $\mu \geq 0$ in $[\omega] = \mu c_1$) with

- **objects** closed μ-monotone ($\mu = 0$ geom. bounded) symplectic manifolds
- **morphisms** $Mor(M,N) \ni \underline{L} = \left(M = M_0 \xrightarrow{L_{01}} M_1 \to \dots \xrightarrow{L_{(k-1)k}} M_k = N\right)$ finite sequences of compact μ-monotone, min Maslov ≥ 3 Lagrangians $L_{ij} \subset M_i^- \times M_j$
- **[1]composition** $\left(M_0 \xrightarrow{L_{01}} \dots \to M_1\right) \overset{hor}{\circ} \left(M_1 \to \dots \xrightarrow{L_{12}} M_2\right) := M_0 \xrightarrow{L_{01}} \dots \to M_1 \to \dots \xrightarrow{L_{12}} M_2$

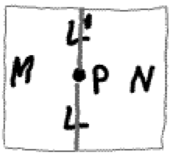
  isomorphisms $\boxed{L_{01} \circ L_{12} \simeq L_{01} \overset{hor}{\circ} L_{12}}$ for embedded $L_{01} \circ L_{12}$ given by

$$\begin{tikzcd} M_0 \arrow[r, bend left, "L_{01}\circ L_{12}"] \arrow[r, bend right, "(L_{01}{,}L_{12})"'] \arrow[r, phantom, "\Uparrow\alpha \; \Downarrow\beta"] & M_2 \end{tikzcd} \qquad \begin{aligned} \alpha \overset{ver}{\circ} \beta &= 1_{(L_{01},L_{12})} \\ \beta \overset{ver}{\circ} \alpha &= 1_{L_{01}\circ L_{12}} \end{aligned}$$

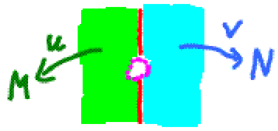
- **[2]morphisms** ${}^2Mor(\underline{L}, \underline{L}') = HF\left(M \underset{\underline{L}'}{\overset{\underline{L}}{\rightrightarrows}} N\right)$ quilted Floer homology classes

2-morphisms: Floer homology classes ≈ formal sums of intersection points

${}^2\mathrm{Mor}(L, L') = H_*(CF, \partial)$ $\qquad CF := \sum_{p \in L \cap L'} \mathbb{Z}\langle p\rangle$ if $L \pitchfork L'$, $\partial \circlearrowleft$

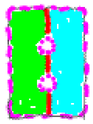


represented by "good" PDE



$$\begin{cases} \bar\partial_{J_M} u = 0 & u \times v(\text{upper seam}) \subset L' \\ \bar\partial_{J_N} v = 0 & u \times v(\text{lower seam}) \subset L \\ (u\times v)(\text{puncture}) \to p & \end{cases}$$

- string diagrams via outer puncture



define 2-morphisms $\displaystyle\sum_{q \in \bigcap \text{outer Lagrangians}} {}^{"}\#\left\{\begin{matrix}\text{pseudoholom. quilts} \\ \text{w. outer puncture} \to q\end{matrix}\right\}^{"} \langle q\rangle$

- gluing laws for pseudoholomorphic quilts $\Longrightarrow$ 2-category axioms
- strip shrinking

1990s Atiyah-Floer conjecture / Ozsvath-Szabo theorem } Splitting & symplectic Floer homology define a topological invariant (independent of choice of splitting)

3-manifold Y $\rightsquigarrow$ Heegård splitting

$\rightsquigarrow$ L_{Y_0} Lagrangian $\cap$ M_Σ symplectic $\cup$ L_{Y_1} Lagrangian $\rightsquigarrow$ Floer homology $HF(L_{Y_0}, L_{Y_1})$

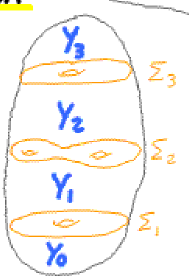
[AF]: SU(2)-representation spaces
[OS]: $\mathrm{Sym}^g(\Sigma)$ & attaching cycles

pseudoholomorphic curves / quilts

2010s explanation

$Y \rightsquigarrow Y = \bigcup_{i=1} Y_i$

decomposition of morphism in Top_{2+1}



$\rightsquigarrow$ functor $\mathrm{Top}_{2+1} \to \mathrm{Symp}$ $\quad \underline{L}(Y, \text{choice})$

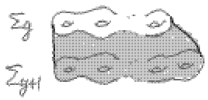
$$\mathrm{pt} \xrightarrow{L(Y_0)} M(\Sigma_1) \xrightarrow{L(Y_1)} M(\Sigma_2) \xrightarrow{L(Y_2)} M(\Sigma_3) \xrightarrow{L(Y_3)} \mathrm{pt}$$

$\rightsquigarrow$ 2-category structure on Symp $\quad HF(\underline{L}(Y, \text{choice})) = {}^2\mathrm{Mor}(\underline{L}, 1)$

[W-Woodward]: Any (partial)⊛ functor $\mathcal{T}op_{n+1} \to \mathcal{S}ymp$ gives rise to a "topological quantum field theory" $\mathcal{T}op_{n+1} \to \mathcal{C}at$.

⊛ • objects: n-manifolds (n=2) … Σ_g $\longrightarrow$ M_{Σ_g} symplectic

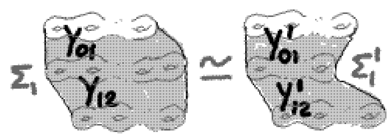
• elementary morphisms: handle attachment / trivial cobordism $\longrightarrow$ $L_Y \subset M_{\Sigma_g}^- \times M_{\Sigma_{g+1}}$ Lagrangian



$L_{\Sigma\times[0,1]}$ = diagonal $\subset M_\Sigma^- \times M_\Sigma$

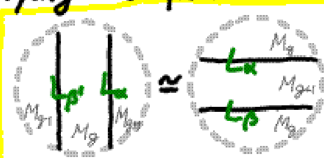
• Cerf moves between decompositions: $Y_{01} \cup_{\Sigma_1} Y_{12} \simeq Y'_{01} \cup_{\Sigma'_1} Y'_{12}$ $\longrightarrow$ $L_{Y_{01}} \circ L_{Y_{12}} = L_{Y'_{01}} \circ L_{Y'_{12}}$

- handle cancellation
- ~ switch
- (· ~ slide)
- diffeomorphism
- trivial cancellation



embedded geometric composition

[W.2012-...] Any partial functor $\mathcal{T}op_{n+1} \to \mathcal{S}ymp$ satisfying the quilt axiom



induces a 2-functor $\mathcal{T}op_{n+1+1} \xrightarrow{2} \mathcal{C}at$; in particular an (n+1+1)-manifold invariant.

conjectural **examples:** "dimensionally reduced gauge theory" e.g. [AF]: SU(2)-representation spaces; [OS]: $Sym^g(\Sigma)$ & attaching cycles — Conj.

Construction: