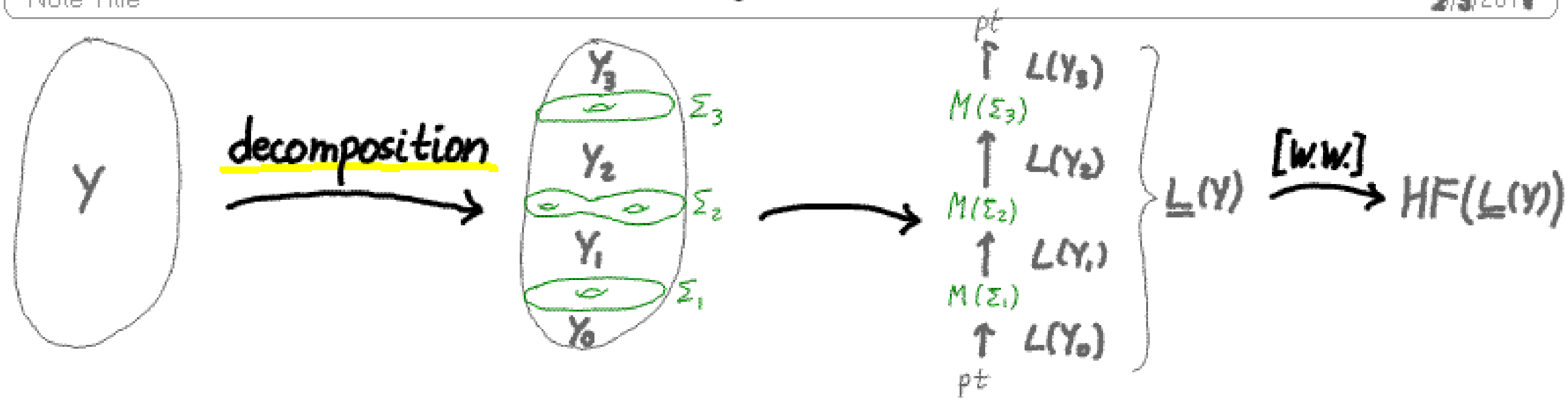
How to construct topological invariants via

Note Title



and the symplectic category

[Katrin Wehrheim – joint work with Chris Woodward & partially with Sikimeti Ma'u]

1980s Weinstein : "Everything is Lagrangian" in symplectic topology

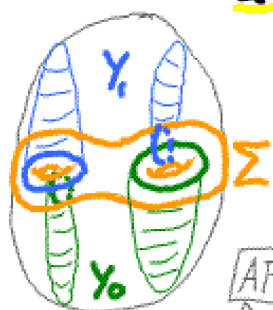
Gromov : pseudoholomorphic curves as tools ——— " ———

1990s Atiyah-Floer; Ozsvath-Szabo : topological 3-manifold invariants via "magic":

(Heegård) splitting

$Y = Y_0 \cup_\Sigma Y_1$

3-manifold



& "representation" as pair of

L_{Y_0} Lagrangian

$\cap$

M_Σ symplectic

$\cup$

L_{Y_1} Lagrangian

$\rightsquigarrow$ Floer homology $HF(L_{Y_0}, L_{Y_1})$

pseudoholomorphic curves

AF

$M_\Sigma = \{\tilde{\rho} : \pi_1(\Sigma) \to SU(2) \text{ group hom}\}/_{SU(2)}$

$\cup \qquad \downarrow$

$L_Y = \{\rho : \pi_1(Y) \to SU(2) \text{ group hom}\}/_{SU(2)}$

OS

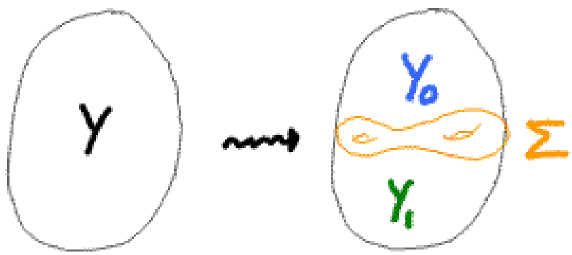
$M_\Sigma = Sym^g \Sigma = \{(z_1, \dots z_g) \in \Sigma^g\}/\text{permutation}$

$\cup$

$L_Y = \{(z_1, \dots z_g) \mid z_k \in k\text{-th attaching circle}\}/\text{permutation}$

"magic" Atiyah-Floer conjecture : This defines a topological invariant,

Ozsvath-Szabo theorem : ie. is independent of the choice of splitting



$\rightsquigarrow$ L_{Y_0} Lagrangian $\cap$ M_Σ symplectic $\cup$ L_{Y_1} Lagrangian $\rightsquigarrow$ Floer homology $HF(L_{Y_0}, L_{Y_1})$

[AF]: SU(2)-representation spaces

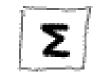
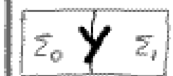
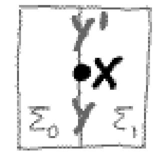
[OS]: $Sym^g(\Sigma)$ & attaching cycles

pseudoholomorphic curves/quilts

2010s **categorical interpretation**:

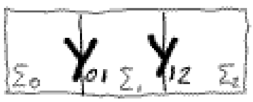
$(S^2 \xrightarrow{Y} S^2) = (S^2 \xrightarrow{Y_0} \Sigma \xrightarrow{Y_1} S^2)$ $\overset{\text{functor}}{\rightsquigarrow}$ $(\underset{\substack{\shortparallel \\ \text{point}}}{M_{S^2}} \xrightarrow{L_{Y_0}} M_\Sigma \xrightarrow{L_{Y_1}} \underset{\substack{\shortparallel \\ \text{point}}}{M_{S^2}})$ $\xrightarrow{\text{2-category structure}}$ ${}^2\mathrm{Mor}(L_{Y_0}, L_{Y_1}^T)$

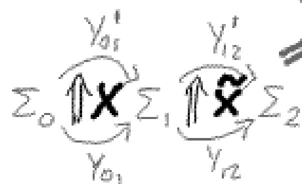
Examples of 2-categories & string diagram notation

	Cat	notation	Top_{2+1+1}	string notation
• object:	category	Σ	closed, oriented 2-manifold	
• morphism:	functor	$\Sigma_0 \xrightarrow{Y} \Sigma_1$	3dim. cobordism	
• 2-morphism:	natural transformation	$\Sigma_0 \underset{Y}{\overset{Y'}{\Uparrow X}} \Sigma_1$	4dim. cobordism of cobordism	

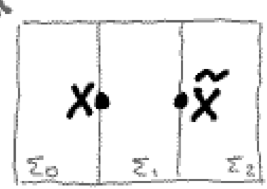
• horizontal composition

$Y_{01} \overset{hor}{\circ} Y_{12} = \Sigma_0 \xrightarrow{Y_{01}} \Sigma_1 \xrightarrow{Y_{12}} \Sigma_2 =$

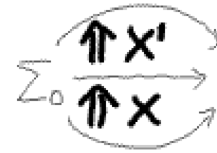




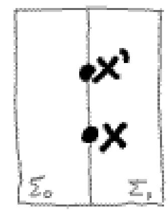
$= X \overset{hor}{\circ} \tilde{X} =$



• vertical composition $X \overset{ver}{\circ} X'$

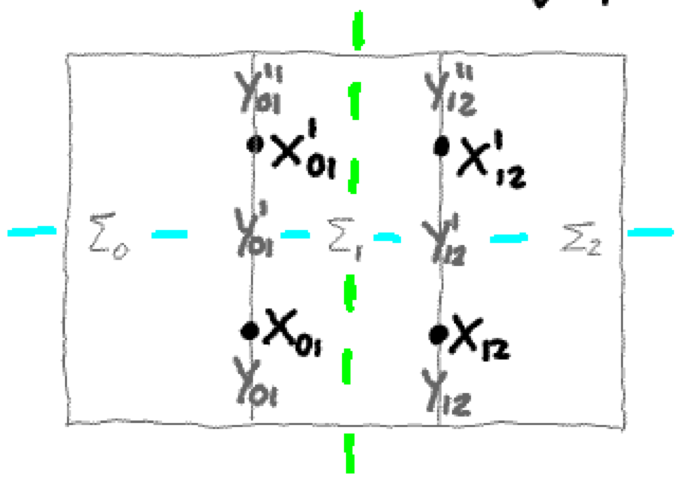


$=$



• 2-category axioms : larger string diagrams make sense

Main 2-category axiom:



defines a 2-morphism

$$\begin{tikzcd}
\Sigma_0 \arrow[rr, bend left, "Y''_{01} \overset{hor}{\circ} Y''_{12}"] \arrow[rr, bend right, "Y_{01} \overset{hor}{\circ} Y_{12}"'] \arrow[rr, phantom, "\Uparrow X"] & & \Sigma_2
\end{tikzcd}$$

that is

$$X = \left(X_{01} \overset{ver}{\circ} X'_{01}\right) \overset{hor}{\circ} \left(X_{12} \overset{ver}{\circ} X'_{12}\right) = \left(X_{01} \overset{hor}{\circ} X_{12}\right) \overset{ver}{\circ} \left(X'_{01} \overset{hor}{\circ} X'_{12}\right)$$

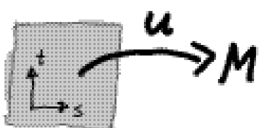
Def^{n} of symplectic category & realization of string diagrams as quilts

- **objects**: (M,ω) symplectic manifold

  is locally $(\mathbb{C}^n, \underbrace{\Sigma dz_j \wedge d\bar{z}_j}_{\text{preserved by transition maps}})$ $\rightsquigarrow$ carry compatible almost complex structures $J: M \to \mathrm{End}(TM)$, $J^2 = -\mathbb{1}$ (but $\nabla J \neq 0$)

  $\omega(\cdot, J\cdot)$ metric

[$\boxed{M}$ represents a well posed (elliptic, Fredholm) nonlinear PDE:



$$\bar{\partial}_J u := \partial_s u + J(u)\partial_t u = 0$$

"pseudoholomorphic curve" [Gromov, ...]]

• **morphisms** : $M \xrightarrow{L} N$ Lagrangian correspondence

$L \subset \left(M \times N, (-\omega_M) \times \omega_N\right)$ submanifold, $TL \oplus J \cdot TL = T(M \times N)$

$\underset{(-J_M) \times J_N =: J}{}$ (locally $L \subset M \times N \simeq \mathbb{R}^{m+n} \subset \mathbb{C}^{m+n}$)

Examples:

• **split** : $M_0 \xrightarrow{L_0 \times L_1} M_1$; $L_i \subset M_i$ Lagrangian ; also $M_0 \xrightarrow{L_0} pt$, $pt \xrightarrow{L_1} M_1$

• **graph**: $M \xrightarrow{\mathrm{gr}\,\varphi} N$; $\varphi : (M, \omega_M) \to (N, \omega_N)$ symplectomorphism : $\varphi^* \omega_N = \omega_M$

• **symplectic quotient**: $M \xrightarrow{(\iota \times \pi)\, \mu^{-1}(0)} M /\!\!/ G = \mu^{-1}(0)/G$; $G \to \mathrm{Symp}(M, \omega_M)$, $\mu : M \to \mathrm{Lie}\, G$ moment map

eg. $\mathbb{C}^{n+1} \longrightarrow \mathbb{C}P^n = \{ |z_0|^2 + \ldots + |z_n|^2 - 1 = 0 \} / S^1$

• **morphisms** : $M \xrightarrow{L} N$ Lagrangian correspondence

$L \subset (M\times N, (-\omega_M)\times\omega_N)$ submanifold, $\quad TL \oplus J\cdot TL = TM$

$(-J_M)\times J_N =: J$

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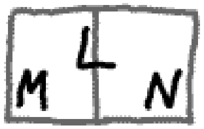
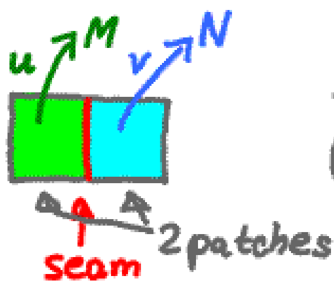


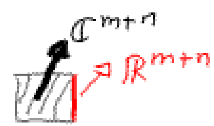
represents a well posed (elliptic, Fredholm) nonlinear PDE:



$$\left\{ \bar\partial_{J_M} u = 0, \; \bar\partial_{J_N} v = 0, \quad u\times v(\text{seam}) \subset L \right\}$$

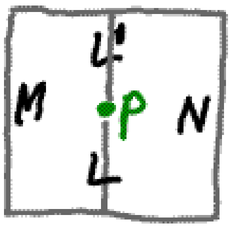
"pseudoholomorphic quilt"

[Perutz, W-W]

(locally )

• **2-morphisms**: Floer homology classes ≈ formal sums of intersection points

$$Mor(L,L') \approx \sum_{p\in L\cap L'} \mathbb{Z}\langle p\rangle \qquad \text{if } L \pitchfork L'$$



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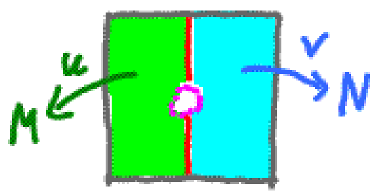


$$\left\{\begin{array}{ll} \bar\partial_{J_M} u = 0 & u\times v\left(\begin{smallmatrix}\text{upper}\\ \text{seam}\end{smallmatrix}\right) \subset L' \\ \bar\partial_{J_N} v = 0 & u\times v\left(\begin{smallmatrix}\text{lower}\\ \text{seam}\end{smallmatrix}\right) \subset L \\ \multicolumn{2}{c}{(u\times v)(\text{puncture}) \longrightarrow p} \end{array}\right\}$$

"pseudoholom. quilt" [W-W]

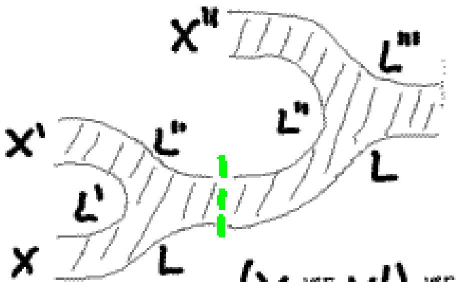
* string diagram defines 2-morphism by viewing boundary □ as puncture

$$\sum_{q\in\cap\,\text{Lagrangians on seams to outside}} \#\left\{\begin{array}{l}\text{pseudoholom. quilts}\\ \text{w. outside puncture} \to q\end{array}\right\}\langle q\rangle$$

* gluing laws for pseudoholomorphic quilts $\Rightarrow$ string diagrams define 2-category structure

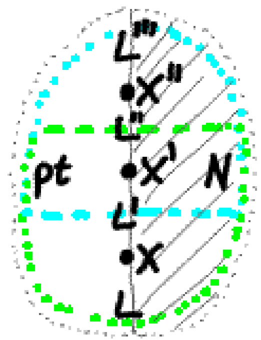
[Donaldson, ..., Mau-W-W]

Ex: $\overset{ver}{\circ}$ associativity for $M=pt$



$(X \overset{ver}{\circ} X') \overset{ver}{\circ} X''$

=



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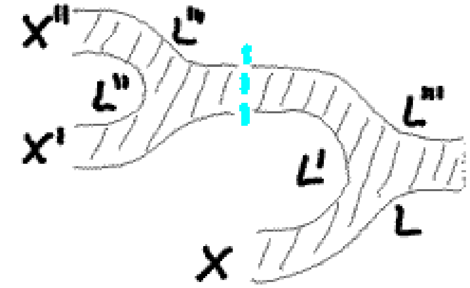

$X \overset{ver}{\circ} (X' \overset{ver}{\circ} X'')$

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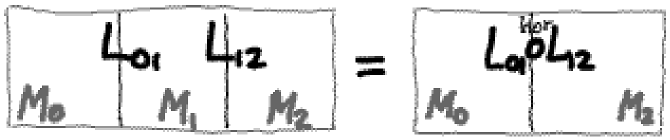
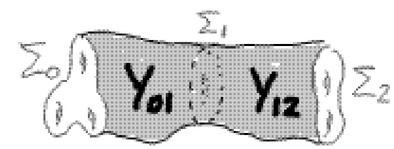
*



is satisfied by allowing sequences $\boxed{L_{01} \overset{hor}{\circ} L_{12} := M_0 \xrightarrow{L_{01}} M_1 \xrightarrow{L_{12}} M_2}$ as morphisms

and interpreting $L \overset{hor}{\circ} L'$ as quilt $L \; L'$

But: In [Ozsvath-Szabo] [Atiyah-Floer] examples, topological composition



corresponds to geometric composition of Lagrangian correspondences:

$$M(\Sigma_j) = \{\tilde{g} : \pi_1(\Sigma_j) \to SU(n)\}/_{SU(n)} \cong \frac{\text{flat } SU(n)\text{-connections}}{\text{gauge}}$$

$$M(\Sigma_0)\times M(\Sigma_1) \supset L(Y_{01}) \cong \left\{\begin{array}{l}\text{flat } SU(n)\text{-connections}\\ \text{on } Y_{01} \text{ mod gauge}\end{array}\right\}\Bigg|_{\partial Y_{01} = \Sigma_0 \cup \Sigma_1}$$

$$\Rightarrow L(Y_{01} \cup Y_{12}) = L(Y_{01}) \circ L(Y_{12})$$

$$\begin{tikzcd} M_0 \arrow[r, "L_{01}"] \arrow[rr, bend right, "L_{01}\circ L_{12}"'] & M_1 \arrow[r, "L_{12}"] & M_2 \end{tikzcd}$$

$$\boxed{L_{01}\circ L_{12} := \pi_{M_0\times M_2}\left(L_{01}\times L_{12} \cap M_0 \times \Delta_{M_1} \times M_2\right)}$$

Generically, $L_{01}\circ L_{12}$ is a Lagrangian immersion.

Thm: [W-W] $L_{01} \overset{hor}{\circ} L_{12}$ is isomorphic to $L_{01} \circ L_{12}$ if $\underbrace{L_{01} \times L_{12} \pitchfork M_0 \times \Delta_{M_1} \times M_2 \xrightarrow{\pi_{M_0\times M_2}} L_{01} \circ L_{12}}_{\text{"embedded"}}$

and "bubbling is excluded a priori".

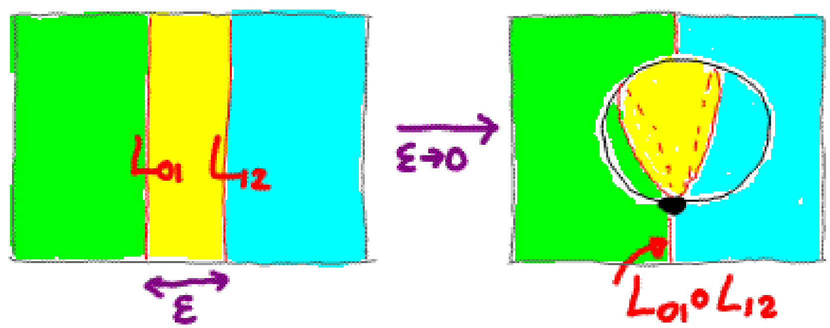
Proof:

=

for counting pseudoholomorphic quilts

by $\overset{\longleftrightarrow}{1}$ $\overset{\longleftrightarrow}{\varepsilon\to 0}$ $\overset{\longleftrightarrow}{1}$ strip shrinking

bubbling problem:
novel singularity
can appear for $\varepsilon \to 0$



[
not a well posed PDE.. yet]

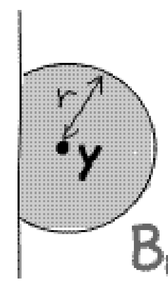
bubble exclusion : by control of energy (e.g. monotone symplectic and Lagrangian manifolds)

and **mean value inequalities** for $e : \mathbb{H}^n \longrightarrow [0,\infty)$

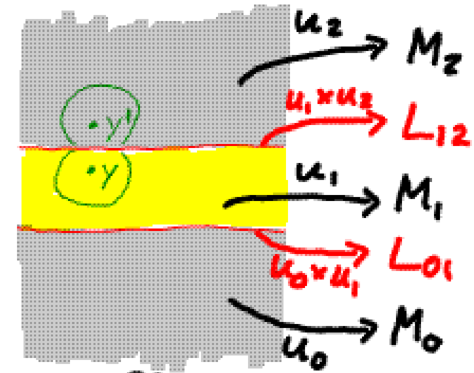
$\mathbb{H}^n = \{ z \in \mathbb{R}^n \mid z_1 \geq 0 \}$

$\forall n \geq 2 \;\exists C \;;\; \forall a,b \geq 0 \;\exists \hbar(a,b) > 0 \;:\; \forall y \in \mathbb{H}^n ,\, r > 0$

$$\left.\begin{array}{l} -\sum_{i=1}^{n} \partial_i^2 e =: \Delta e \leq A_0 + a e^{\frac{n+2}{n}} \\ -\partial_1 e =: \left.\frac{\partial}{\partial \nu}\right|_{\partial \mathbb{H}^n} e \leq B_0 + b e^{\frac{n+1}{n}} \\ \int_{B_r(y) \cap \mathbb{H}^n} e \leq \hbar(a,b) \end{array}\right\} \Rightarrow e(y) \leq C \left(r^{-n} \int_{B_r(y) \cap \mathbb{H}^n} e + A_0 r^2 + B_0 r \right)$$



$B_r(y) \cap \mathbb{H}^n$



Sketch of bubble exclusion:

$n=2$

$e_i = |du_i|^2$

- $\bar\partial_{J_i} u_i = 0 \quad \Rightarrow \quad \Delta e_i \le a\cdot e_i^2$
- $(u_i\times u_{i+1})(\text{seam}) \in L_{i(i+1)} \quad \Rightarrow \quad \frac{\partial}{\partial r}(e_i+e_{i+1})\Big|_{\partial H^2} \le b\,(e_i+e_{i+1})^{3/2}$

$\Rightarrow$ If singularity forms $\left(\text{for a sequence } (u_0,u_1,u_2)_{k\in\mathbb{N}}\right)$ then energy concentrates

$$e_1(y_k)+e_2(y'_k) \xrightarrow[k\to\infty]{} \infty$$

$$\int_{B_{r_k}(y,y')} e_1+e_2 \ge \hbar(a,b) > 0$$

$r_k \to 0$

Thm [WW]: $\left(\begin{smallmatrix}\text{monotone}\\ \text{or}\\ \text{exact}\end{smallmatrix}\right)$ Symplectic manifolds and Lagrangian correspondences with geometric composition (when embedded) can be extended to a **symplectic 2-category** **Symp**.

Cor.: There exists a 2-functor $\boxed{\mathrm{Symp} \to \mathrm{Cat}}$ associating

$$M \mapsto H_*\mathcal{F}^\#(M) \ \left(\begin{smallmatrix}\text{homology of}\\ \text{extended Fukaya category}\end{smallmatrix}\right), \quad \left(M_0 \xrightarrow{L_{01}} M_1\right) \mapsto \left(\underbrace{\Phi_{L_{01}} : H\mathcal{F}^\#(M_0) \to H\mathcal{F}^\#(M_1)}_{\text{functor}}\right)$$

Thm [Ma'u-W-W]: There exists a "chain level" A_∞-functor $\mathcal{F}^\#(M_0^- \times M_1) \to \mathcal{F}un(\mathcal{F}^\#(M_0), \mathcal{F}^\#(M_1))$

$$L_{01} \mapsto C\Phi_{L_{01}}$$

$$L_{01} \cap L_{01}' \ni p \mapsto A_\infty\text{- natural transf.}$$

Cor.: Any (partial)$^{\circledast}$ functor $\mathcal{Bor}_{n+1} \to \mathcal{Symp}$ gives rise to a "topological quantum field theory" $\mathcal{Bor}_{n+1} \to \mathcal{Cat}$.

$\circledast$

all objects: n-manifolds  $\ldots \Sigma_g$ $\longrightarrow$ M_{Σ_g} symplectic

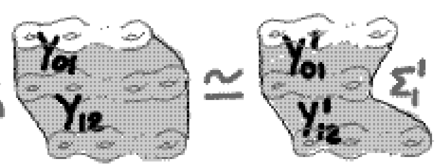
"prime" morphisms: handle attachment / trivial cobordism $\longrightarrow$ $L_Y \subset M_{\Sigma_g}^- \times M_{\Sigma_{g+1}}$ Lagrangian

$L_{\Sigma\times[0,1]} = \text{diagonal} \subset M_\Sigma^- \times M_\Sigma$

"moves" between prime decompositions: $Y_{01} \cup_{\Sigma_1} Y_{12} \simeq Y'_{01} \cup_{\Sigma'_1} Y'_{12}$ $\longrightarrow$ $L_{Y_{01}} \circ L_{Y_{12}} = L_{Y'_{01}} \circ L_{Y'_{12}}$

embedded geometric composition

- handle cancellation
- $\sim$ switch
- ($\cdot$ $\sim$ slide)
- diffeomorphism
- trivial cancellation $Y \cup (\Sigma\times[0,1]) \simeq Y$



[Cerf theory] $\longleftrightarrow$ [WW construction of $\mathcal{Symp}$]

Cor.: Any (partial) functor $\mathcal{T}op_{n+1} \to \mathcal{S}ymp$ gives rise to a "topological quantum field theory" $\mathcal{T}op_{n+1} \to \mathcal{C}at$.

Ex.:

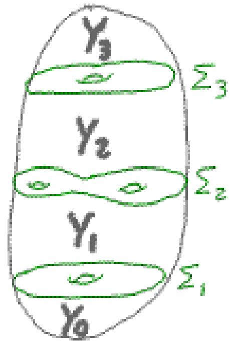
Perutz-Lekili	: 2+1	TQFT containing	Ozsvath-Szabo ("Heegård Floer") 3-mfd invariant
Perutz + work	: 2+1+1	—— " ——	conj. Seiberg-Witten 4-mfd —"—
W.-W.	: 2+1	—— " ——	Atiyah-Floer type 3-mfd —"—
W.-W.-work	: 2+1+1	—— " ——	conj Donaldson type 4-mfd —"—

...

2010 3-dim topological invariants via "functoriality"

Y 3-manifold $\xrightarrow[\text{decompose into simple pieces}]{[\text{Cerf}, \ldots]}$ $Y = \bigcup_{i} Y_i$ $\xrightarrow{\text{go symplectic}}$ $\underline{L}(Y, \text{choice})$ $\xrightarrow[\text{count holomorphic curves}]{[WW]}$ $HF(\underline{L}) = Mor(\underline{L}, 1)$

! choice !



go symplectic:

- ...
- symmetric products & attaching cycles
- SU(n) representations

$\underline{L}(Y, \text{choice})$:

$$pt \leftarrow L(Y_3) - M(\Sigma_3) \leftarrow L(Y_2) - M(\Sigma_2) \leftarrow L(Y_1) - M(\Sigma_1) \leftarrow L(Y_0) - pt$$

$HF(\underline{L}) = Mor(\underline{L}, 1)$ $\downarrow$

topological invariant of Y

! independent of choice !

another sample invariant for $\begin{pmatrix} Y \text{ closed 3-manifold} \\ [f: Y \to S^1] \text{ homotopy class of } S^1\text{-valued function} \end{pmatrix}$

using

- cyclic connected Cerf theory [Gay-Kirby + W-W]: Decompositions of Y along regular, connected level sets of $f: Y \to S^1$ exist and are unique up to Cerf moves: cancellation / switch of critical points

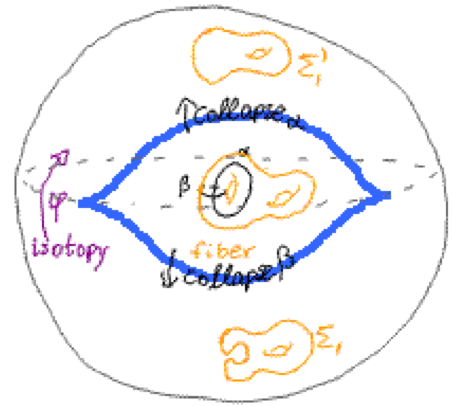
- $\{\rho : \pi_1(\Sigma \setminus pt) \to SU(n) \text{ hom.} \mid \rho(\text{pt}) = -\mathbb{1}\} / SU(n)$ smooth, monotone symplectic
  $\{ \text{— " —}\ Y \setminus \text{line} \text{ — " — } \text{line} \text{ — " —}\} / SU(n)$ — " — Lagrangian
  compression body: all crit pts of same index (1 or 2)

next:

[Perutz, Lekili, Gay-Kirby, Williams]:

$$X^4 \xrightarrow{f} S^2$$ broken Lefschetz fibration

exist (with connected fibers) and are unique up to moves

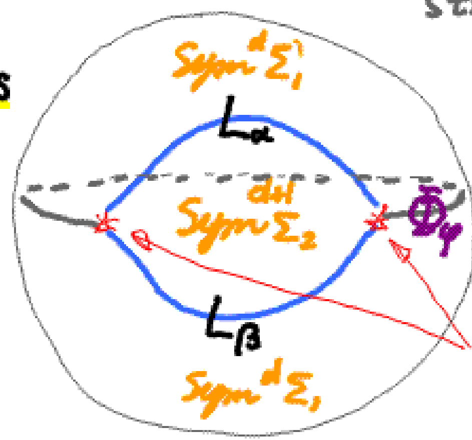


... and very close to a 2-categorical string diagram on S^2 ...

⇝ define 4-mfd invariants

by

- symplectic labeling
- checking moves



induced symplectomorphism

$1_\varphi \in HF(\Phi_\eta, \Phi_\varphi) \cong HF(L_\beta, L_\alpha, gr\,\Phi_\varphi)$