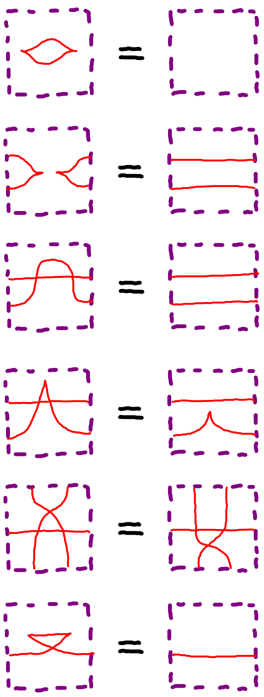
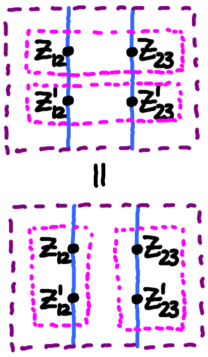


String diagrams in algebra, topology, geometry & analysis



Katrin Wehrheim

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Topological invariants via the symplectic category

Any nice assignment 2-manifold Σ → M_Σ symplectic

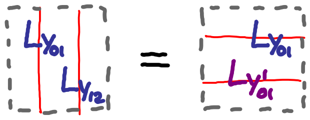
simple 3-cobordism Y → L_Y Lagrangian

(handle attachment)

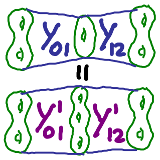
that takes Cerf moves to embedded composition

$$Y_{01} \# Y_{12} = Y'_{01} \# Y'_{12} \longrightarrow L_{Y_{01}} \circ L_{Y_{12}} = L_{Y'_{01}} \circ L_{Y'_{12}}$$

and is natural (



) w.r.t. handle switches



extends to a 2-functor

$$\begin{tikzcd}
Bor_{2+1+1} \arrow[rr] \arrow[dr] & & \mathcal{C}at \\
& Symp \arrow[ur] &
\end{tikzcd}$$

(i.e. "compatible invariants for 2,3,4-manifolds")

2+1 : W-Woodward '07-'15

4 : W '13-...

categories – 0^{th} example

- objects : (topological / vector / ...) spaces X
- morphisms : (continuous / linear / ...) maps $X_1 \xrightarrow{f} X_2$

2-categories – 0th example

- objects : (topological / vector / ...) spaces $\qquad X$
- morphisms : (continuous / linear / ...) maps $\qquad X_1 \xrightarrow{f} X_2$
- 2-morphisms : conjugacy

$$\begin{tikzcd} X_1 \arrow[r, bend left, "f"{name=F}] \arrow[r, bend right, "g"'{name=G}] & X_2 \arrow[Rightarrow, from=F, to=G, "\underline{u}"] \end{tikzcd} \qquad \begin{tikzcd} X_1 \arrow[r, "f"] \arrow[d, "u_1"] & X_2 \arrow[d, "u_2"'] \\ X_1 \arrow[r, "g"'] & X_2 \end{tikzcd}$$

$$g \circ u_1 = u_2 \circ f$$

– these compose:

$$\begin{tikzcd} X_1 \arrow[r, bend left=50, "f"{name=F}] \arrow[r, "g"{name=G, description}] \arrow[r, bend right=50, "h"'{name=H}] & X_2 \arrow[Rightarrow, from=F, to=G, "\underline{u}"] \arrow[Rightarrow, from=G, to=H, "\underline{v}"] \end{tikzcd} = \begin{tikzcd} X_1 \arrow[r, "f"] \arrow[d, "u_1"] \arrow[dd, bend right=60, "v_1 \circ u_1"'] & X_2 \arrow[d, "u_2"'] \arrow[dd, bend left=60, "v_2 \circ u_2"] \\ X_1 \arrow[r, "g"] \arrow[d, "v_1"] & X_2 \arrow[d, "v_2"'] \\ X_1 \arrow[r, "h"] & X_2 \end{tikzcd} = \begin{tikzcd} X_1 \arrow[r, bend left, "f"{name=F}] \arrow[r, bend right, "h"'{name=H}] & X_2 \arrow[Rightarrow, from=F, to=H, "\underline{v} \circ \underline{u}"] \end{tikzcd}$$

2-categories – ~~0th~~ NON example

- objects : (topological /vector /...) spaces $\quad X$
- morphisms : (continuous / linear / ...) maps $\quad X_1 \xrightarrow{f} X_2$
- 2-morphisms : conjugacy

$$\begin{tikzcd} X_1 \arrow[r, bend left=50, "f", ""{name=U, below}] \arrow[r, bend right=50, "g"', ""{name=D}] & X_2 \arrow[Rightarrow, from=U, to=D, "\underline{u}"] \end{tikzcd}
\qquad
\begin{tikzcd} X_1 \arrow[r, "f"] \arrow[d, "u_1"] & X_2 \arrow[d, "u_2"'] \\ X_1 \arrow[r, "g"'] & X_2 \end{tikzcd}$$

$$g \circ u_1 = u_2 \circ f$$

– ~~and~~ but are NOT compatible with 1-composition:

$$\begin{tikzcd} X_1 \arrow[r, bend left=50, "f_{12}", ""{name=U1, below}] \arrow[r, bend right=50, "g_{12}"', ""{name=D1}] & X_2 \arrow[r, bend left=50, "f_{23}", ""{name=U2, below}] \arrow[r, bend right=50, "g_{23}"', ""{name=D2}] & X_3 \arrow[Rightarrow, from=U1, to=D1, "\underline{u}"] \arrow[Rightarrow, from=U2, to=D2, "\underline{v}"] \end{tikzcd}
\overset{?}{=}
\begin{tikzcd} X_1 \arrow[r, bend left=40, "f_{23} \circ f_{12}", ""{name=U, below}] \arrow[r, bend right=40, "g_{23} \circ g_{12}"', ""{name=D}] & X_3 \arrow[Rightarrow, from=U, to=D, "\underline{u} \# \underline{v}"] \end{tikzcd}$$

$$\begin{tikzcd} X_1 \arrow[r] \arrow[d, "u_1"] & X_2 \arrow[r] \arrow[d, bend right, "u_2"', ""{name=L}] \arrow[d, bend left, "v_1", ""{name=R}] \arrow[from=L, to=R, phantom, "\overset{?}{\cancel{=}}"] & X_3 \arrow[d, "v_2"] \\ X_1 \arrow[r] & X_2 \arrow[r] & X_3 \end{tikzcd}$$

2-categories in algebra: **Cat** is the 2-category with

- objects: categories X
- morphisms: functors $X_1 \xrightarrow{f} X_2$

with composition $X_1 \xrightarrow{f} X_2 \xrightarrow{g} X_3 = X_1 \xrightarrow{f\#g} X_3$

- 2-morphisms: natural transformations

$$\begin{tikzcd}
X_1 \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow \underline{u}"] & X_2
\end{tikzcd}$$

with two unital, associative compositions

$$\begin{tikzcd}
X_1 \arrow[r, bend left, "f_{12}"] \arrow[r, bend right, "g_{12}"'] \arrow[r, phantom, "\Downarrow \underline{u}"] & X_2 \arrow[r, bend left, "f_{23}"] \arrow[r, bend right, "g_{23}"'] \arrow[r, phantom, "\Downarrow \underline{v}"] & X_3
\end{tikzcd}
=
\begin{tikzcd}
X_1 \arrow[r, bend left, "f_{23} \circ f_{12}"] \arrow[r, bend right, "g_{23} \circ g_{12}"'] \arrow[r, phantom, "\Downarrow \underline{u} \# \underline{v}"] & X_3
\end{tikzcd}$$

$$\begin{tikzcd}[row sep=large]
X_1 \arrow[r, bend left=50, "f"] \arrow[r] \arrow[r, bend right=50, "h"'] \arrow[r, phantom, shift left=2.5ex, "\Downarrow \underline{u}"] \arrow[r, phantom, shift right=2.5ex, "\Downarrow \underline{v}"] & X_2
\end{tikzcd}
=
\begin{tikzcd}
X_1 \arrow[r, bend left, "f"] \arrow[r, bend right, "h"'] \arrow[r, phantom, "\Downarrow \underline{v} \circ \underline{u}"] & X_2
\end{tikzcd}$$

that are compatible:

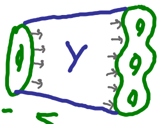
$$(u_{12} \# u_{23}) \circ (v_{12} \# v_{23}) = (u_{12} \circ v_{12}) \# (u_{23} \circ v_{23})$$

$$\begin{tikzcd}
X_1 \arrow[r] \arrow[dr, phantom, "\Downarrow u_{12}"] & X_2 \arrow[r] \arrow[dr, phantom, "\Downarrow u_{23}"] & X_3 \\
X_1 \arrow[r] \arrow[dr, phantom, "\Downarrow v_{12}"] & X_2 \arrow[r] \arrow[dr, phantom, "\Downarrow v_{23}"] & X_3 \\
X_1 \arrow[r] & X_2 \arrow[r] & X_3
\end{tikzcd}$$

2-categories in topology – e.g. dimensions 2+1+1

- **objects** : closed oriented **2-manifolds** Σ

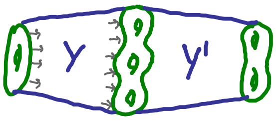
- **morphisms** : closed oriented **3-cobordisms** $\Sigma_1 \xrightarrow{Y} \Sigma_2$
  with collars



$\partial Y = \Sigma_1^- \sqcup \Sigma_2$

- **2-morphisms** :

compositions $\Sigma_1 \xrightarrow{Y} \Sigma_2 \xrightarrow{Y'} \Sigma_3$

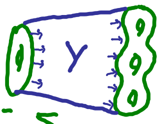


[...& Lurie & ...]

2-categories in topology: **Bor$_{2+1+1}$** is the bicategory with

(associate/unital up to isomorphism)

- **objects** : closed oriented **2-manifolds** Σ
- **morphisms** : closed oriented **3-cobordisms** $\Sigma_1 \xrightarrow{Y} \Sigma_2$
  with collars

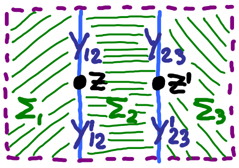


$$\partial Y = \Sigma_1^- \sqcup \Sigma_2$$

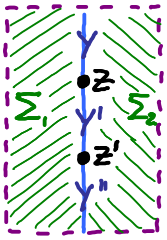
- **2-morphisms** : **4-manifolds** with boundary, corners, modulo diffeomorphisms

All compositions are defined by gluing:

$$\begin{tikzcd} \Sigma_1 \arrow[r, bend left=50, "Y_{12}"{name=U}] \arrow[r, bend right=50, "Y'_{12}"'{name=D}] & \Sigma_2 \arrow[r, bend left=50, "Y_{23}"{name=U2}] \arrow[r, bend right=50, "Y'_{23}"'{name=D2}] & \Sigma_3 \arrow[Rightarrow, from=U, to=D, "Z"] \arrow[Rightarrow, from=U2, to=D2, "Z'"] \end{tikzcd} =$$



$$\begin{tikzcd} \Sigma_1 \arrow[r, bend left=60, "Y"{name=U}] \arrow[r, "Y'"{name=M}] \arrow[r, bend right=60, "Y''"'{name=D}] & \Sigma_2 \arrow[Rightarrow, from=U, to=M, "Z"] \arrow[Rightarrow, from=M, to=D, "Z'"] \end{tikzcd} =$$

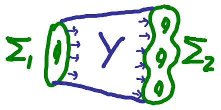


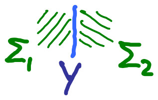
[... & Lurie & ...]

2-categories in topology:

$\underline{\underline{\mathrm{Bor}_{2+1+1}}}$ is the bicategory with

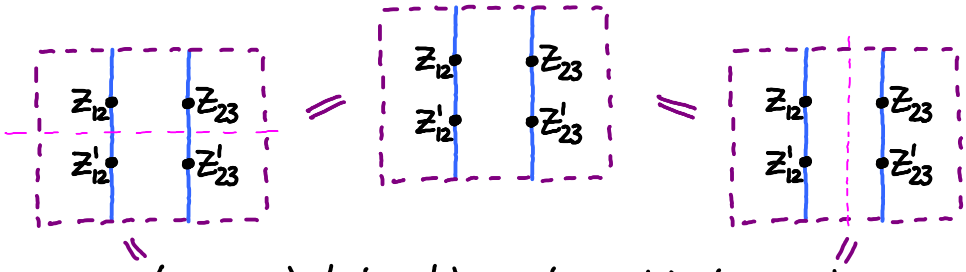
- objects: 2-manifolds Σ
- morphisms: 3-cobordisms
- 2-morphisms: 4-manifolds
  with boundary, corners, modulo diffeomorphisms





Compositions are unital, associative, compatible

because results of gluing are independent of order, e.g



$$(Z_{12} \# Z_{23}) \circ (Z'_{12} \# Z'_{23}) = (Z_{12} \circ Z'_{12}) \# (Z_{23} \circ Z'_{23})$$

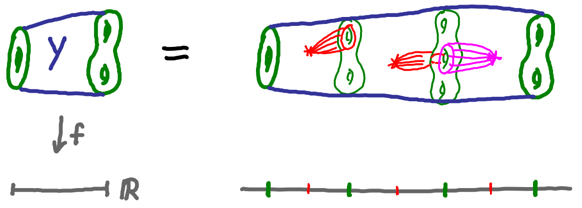
2-categories in topology

Whitney, Morse, Cerf... Gay-Kirby

Morse functions provide decompositions into simple pieces

e.g. in Bor_{2+1}

- morphisms (3-cobordisms) decompose into chains of handle attachments



arising from choices of

- Morse function $f: Y \to \mathbb{R}$
- regular values separating the critical points

$$\Sigma \xrightarrow{Y} \Sigma' = \Sigma = \Sigma_1 \xrightarrow{Y_{12}} \Sigma_2 \xrightarrow{Y_{23}} \Sigma_3 \xrightarrow{Y_{34}} \Sigma_4 = \Sigma'$$

Rmk: Attaching cycles are given by intersecting Morse flow lines in/out of crit. pts with regular level sets.

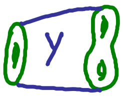
2-categories in topology

Whitney, Morse, Cerf... Gay-Kirby

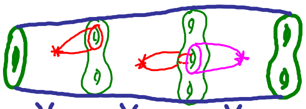
Morse functions provide decompositions into simple pieces
that are unique up to Cerf moves

e.g. in Bor_{2+1}

- morphisms (3-cobordisms) decompose into chains of handle attachments



=

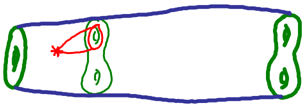


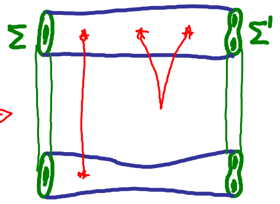
$$\begin{tikzcd} \Sigma \arrow[r, "Y"] & \Sigma' \arrow[r, phantom, "="] & \Sigma=\Sigma_1 \arrow[r, "Y_{12}"] & \Sigma_2 \arrow[r, "Y_{23}"] \arrow[dr, "Y'_{23}"'] & \Sigma_3 \arrow[r, "Y_{34}"] & \Sigma_4=\Sigma' \\ & & & & \Sigma'_3 \arrow[ur, "Y'_{34}"'] & \end{tikzcd}$$

unique up to
- handle switch
- handle cancellation
- cylinder cancellation, diffeomorphism

$Y_{23} \# Y_{34} = Y'_{23} \# Y'_{34}$

=





Rmk: These "Cerf moves" can be understood as generic singularities of a family $(f_s : Y \to \mathbb{R})_{s\in[0,1]}$
and can be graphed on $[0,1]\times Y$

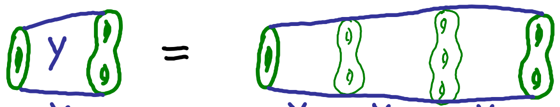
2-categories in topology

Whitney, Morse, Cerf... Gay-Kirby

Morse (2-)functions provide decompositions into simple pieces
that are unique up to (2-)Cerf moves

e.g. in Bor_{2+1+1}

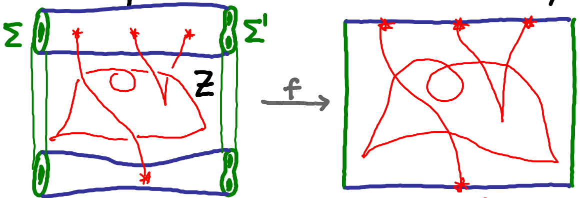
- morphisms (3-cobordisms) decompose into chains of handle attachments



$\Sigma \xrightarrow{Y} \Sigma' \;=\; \Sigma=\Sigma_1 \xrightarrow{Y_{12}} \Sigma_2 \xrightarrow{Y_{23}} \Sigma_3 \xrightarrow{Y_{34}} \Sigma_4=\Sigma'$

unique up to
- handle switch
- handle cancellation
- cylinder cancellation, diffeomorphism

- 2-morphisms (4-manifolds) decompose into "topological string diagrams"



critical set is a smooth 1-submanifold

critical values can have cusps > and crossings ×

arising from choice of

Morse 2-function $f: Z \to [0,1]^2$

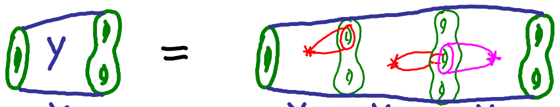
2-categories in topology

Whitney, Morse, Cerf... Gay-Kirby

Morse (2-)functions provide decompositions into simple pieces
that are unique up to (2-)Cerf moves

e.g. in Bor_{2+1+1}

- morphisms (3-cobordisms) decompose into chains of handle attachments

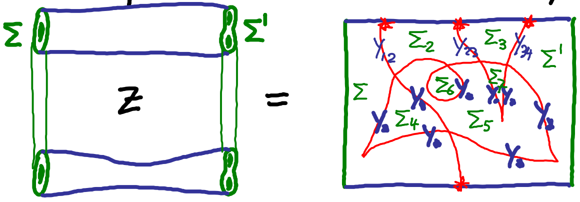


$$\begin{tikzcd} \Sigma \arrow[r, "Y"] & \Sigma' & = & \Sigma=\Sigma_1 \arrow[r, "Y_{12}"] & \Sigma_2 \arrow[r, "Y_{23}"] \arrow[dr, "Y'_{23}"'] & \Sigma_3 \arrow[r, "Y_{34}"] & \Sigma_4=\Sigma' \\ & & & & & \Sigma'_3 \arrow[ur, "Y'_{34}"'] & \end{tikzcd}$$

unique up to
- handle switch
- handle cancellation
- cylinder cancellation, diffeomorphism

$$Y_{23} \# Y_{34} = Y'_{23} \# Y'_{34}$$

- 2-morphisms (4-manifolds) decompose into "topological string diagrams"



- patches labeled with 2-manifolds
- seams labeled with 3-cobordisms
- crossings ✕, cusps >

categories in symplectic geometry

Weinstein 1980s

- **objects :** symplectic manifolds $X = (X, \omega)$

ω closed 2-form
$\omega \wedge \ldots \wedge \omega = \text{volume}$

- **morphisms :** symplectomorphisms $X_1 \xrightarrow{f} X_2$

f diffeomorphism
$f^* \omega_2 = \omega_1$

and more general ($L = \text{graph}\, f$)

Lagrangian relations $X_1 \xrightarrow{L} X_2$

$L \subset X_1^- \times X_2$
$\frac{1}{2}$dim. submanifold
$(-\omega_1) \times \omega_2 |_L = 0$

geometric composition $X_1 \xrightarrow{f} X_2 \xrightarrow{g} X_3 = X_1 \xrightarrow{g \circ f} X_3$

generalizes to $X_1 \xrightarrow{L_{12}} X_2 \xrightarrow{L_{23}} X_3 = X_1 \longrightarrow X_3$

$$L_{12} \circ L_{23} = \pi \left(L_{12} \times L_{23} \cap X_1 \times \Delta_{X_2} \times X_3 \right)$$

$$= \left\{ (x_1, x_3) \,\middle|\, \exists x_2 : \begin{array}{l} (x_1, x_2) \in L_{12} \\ (x_2, x_3) \in L_{23} \end{array} \right\}$$

↯

but is problematic for
- nontransverse intersections $\underbrace{L_{12} \times L_{23} \cap X_1 \times \Delta_{X_2} \times X_3}$
- nonimmersed projection $\pi : \quad \to X_1 \times X_3$

categories in linear symplectic geometry

Weinstein 1980s

- objects : symplectic manifolds $X = (X, \omega)$

ω closed 2-form
$\omega \wedge \ldots \wedge \omega = \text{volume}$

- morphisms : symplectomorphisms $X_1 \xrightarrow{f} X_2$

f diffeomorphism
$f^* \omega_2 = \omega_1$

and more general ($L = \text{graph}\, f$)

Lagrangian relations $X_1 \xrightarrow{L} X_2$

$L \subset X_1^- \times X_2$
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generalizes to $X_1 \xrightarrow{L_{12}} X_2 \xrightarrow{L_{23}} X_3 = X_1 \longrightarrow X_3$

$$L_{12} \circ L_{23} = \pi \left(L_{12} \times L_{23} \cap X_1 \times \Delta_{X_2} \times X_3 \right)$$
$$= \left\{ (x_1, x_3) \mid \exists x_2 : \begin{array}{l} (x_1, x_2) \in L_{12} \\ (x_2, x_3) \in L_{23} \end{array} \right\}$$

This yields a category of symplectic vector spaces & linear Lagrangian relations.

But nonlinear composition (at best) yields immersed Lagrangians.

The symplectic category Symp is given by

W-Woodward
"cheap trick"

- objects : symplectic manifolds X
- morphisms : composable chains of Lagrangian relations

$$X \xrightarrow{\underline{L}} X' = \left(X = X_0 \xrightarrow{L_{01}} X_1 \xrightarrow{L_{12}} \dots X_{n-1} \xrightarrow{L_{(n-1)n}} X_n = X' \right) \qquad L_{ij} \subset X_i^- \times X_j$$

modulo geometric composition "when embedded"

$$\left(X_1 \xrightarrow{L_{12}} X_2 \xrightarrow{L_{23}} X_3 \right) \sim X_1 \xrightarrow{L_{12} \circ L_{23}} X_3 \qquad \underbrace{L_{12} \times L_{23} \pitchfork X_1 \times \Delta_{X_2} \times X_3}_{\pi : \quad \to X_1 \times X_3 \ \text{injective}}$$

- composition # by concatenation

$$[X_1 \xrightarrow{\underline{L}_{12}} \dots \to X_2] \# [X_2 \xrightarrow{\underline{L}_{23}} \dots \to X_3] := [X_1 \underbrace{\to \overset{\underline{L}_{12}}{\dots} \to X_2 \to \overset{\underline{L}_{23}}{\dots} \to}_{\underline{L}_{12} \# \underline{L}_{23}} X_3]$$

2-categories in symplectic geometry – via analysis

W-Woodward

Thm: There is a 2-category Symp with

*"monotonicity"

- objects : nice* symplectic manifolds
- morphisms : composable chains of nice* Lagrangian relations

$$X \xrightarrow{\underline{L}} X' = \left(X = X_0 \xrightarrow{L_{01}} X_1 \xrightarrow{L_{12}} \dots X_{n-1} \xrightarrow{L_{(n-1)n}} X_n = X' \right) \qquad L_{ij} \subset X_i^- \times X_j$$

- composition by concatenation $X_1 \xrightarrow{\underline{L}_{12}} X_2 \xrightarrow{\underline{L}_{23}} X_3 := X_1 \xrightarrow{\underline{L}_{12} \# \underline{L}_{23}} X_3$
- 2-morphisms : quilted Floer homology classes ("qFH")
- 2-compositions defined by realizing symplectic string diagrams as moduli spaces of pseudoholomorphic quilts

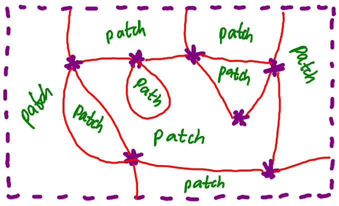
* isomorphisms

$$\begin{tikzcd} X_1 \arrow[r, "L_{12}"] \arrow[rr, bend right, "L_{12}\circ L_{23}"'] \arrow[r, phantom, "\Downarrow\alpha"', shift right=4] & X_2 \arrow[r, "L_{23}"] \arrow[r, phantom, "\Uparrow\beta"', shift right=4] & X_3 \end{tikzcd}$$

$$\alpha \circ \beta = 1_{L_{12} \# L_{23}}$$
$$\beta \circ \alpha = 1_{L_{12} \circ L_{23}}$$

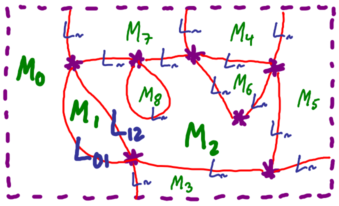
when $L_{12} \circ L_{23}$ "embedded"

Symplectic string diagrams have



- patches ▨
- seams —
- punctures *

Symplectic string diagrams have



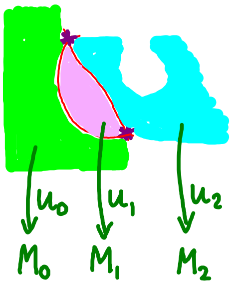
- patches labeled with symplectic manifolds
- seams labeled with Lagrangian relations
- punctures labeled with qFH classes

They represent quilts of (illustrated only for $i=0,1,2\subset\{0,\dots,8\}$)

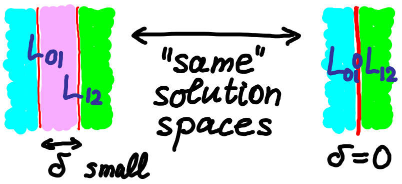
pseudoholomorphic maps $\partial_s u_i = -J_i(u_i)\,\partial_t u_i$

with Lagrangian seam conditions $(u_0,u_1)(\llcorner)\subset L_{01}$

$(u_1,u_2)(\urcorner)\subset L_{12}$



Proof of Thm uses
strip shrinking:



Topological invariants via the symplectic category

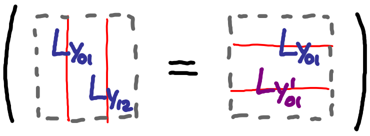
Any nice assignment 2-manifold $\Sigma \to M_\Sigma$ symplectic

simple 3-cobordism $Y \to L_Y$ Lagrangian
(handle attachment)

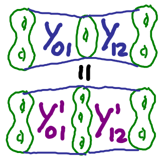
that takes Cerf moves to embedded composition

$$Y_{01} \# Y_{12} = Y'_{01} \# Y'_{12} \longrightarrow L_{Y_{01}} \circ L_{Y_{12}} = L_{Y'_{01}} \circ L_{Y'_{12}}$$

and is natural



w.r.t. handle switches



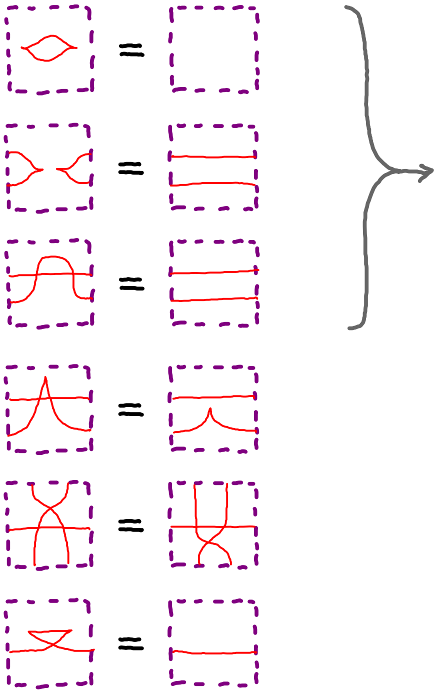
extends to a 2-functor

$$\begin{tikzcd} \mathrm{Bor}_{2+1+1} \arrow[rr] \arrow[dr] & & \mathcal{C}at \\ & \mathrm{Symp} \arrow[ur] & \end{tikzcd}$$

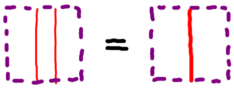
(i.e. "compatible invariants for 2,3,4-manifolds")

2+1 : W-Woodward '07-'15
4 : W '13-...

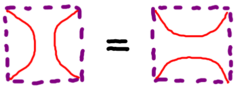
Proof: 2-Cerf moves in Bor_{2+1+1} reduce in Symp to

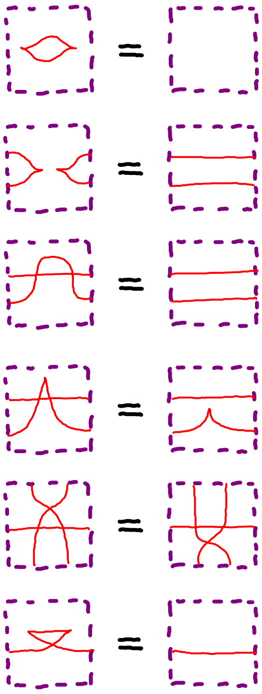


- strip shrinking

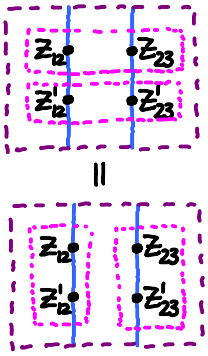


- naturality





Thank you!



String diagrams
in algebra
topology
geometry
analysis