A Polyfold Cheat Sheet by Katrin Wehrheim

Polyfold theory was developed in [HWZ] in order to *regularize* moduli spaces of pseudoholomorphic curves. In the simplest cases, this is achieved by describing a Gromov-compactified moduli space $\overline{\mathcal{M}} = \sigma^{-1}(0)$ as the zero set of a section and associating to it a cobordism class $[(\sigma+p)^{-1}(0)]$ of perturbed zero sets obtained from the following core theorem.

M-Polyfold Regularization Theorem: Let $\mathcal{E} \to \mathcal{B}$ be a strong M-polyfold bundle modeled on scale Hilbert spaces, and let $\sigma : \mathcal{B} \to \mathcal{E}$ be a scale smooth Fredholm section such that $\sigma^{-1}(0) \subset \mathcal{B}$ is compact. Then there exists a class of sc$^+$ perturbation sections $\nu : \mathcal{B} \to \mathcal{E}$ supported near $\sigma^{-1}(0)$ such that $\sigma+\nu$ is transverse to the zero section. As a consequence, $(\sigma+\nu)^{-1}(0)$ carries the structure of a smooth compact manifold. Moreover, for any other such perturbation $\nu' : \mathcal{B} \to \mathcal{E}$ there exists a smooth compact cobordism between $(\sigma + \nu')^{-1}(0)$ and $(\sigma + \nu)^{-1}(0)$.

Similar polyfold regularization theorems exist for moduli spaces with isotropy as well as boundary and corners; in general yielding transverse multivalued perturbations, whose zero sets are compact weighted branched orbifolds with boundary and corners, unique up to appropriate cobordism. The following is a glossary for the new language used to formulate these theorems; introductions to the underlying ideas can be found in e.g. [FFGW, 2.1] and various lecture videos [WWW].

Scale Banach / Hilbert space: A *Banach / Hilbert space* such as $L^2(S^1)$ with additional scale structure such as $(H^k(S^1))_{k\in\mathbb{N}_0}$. *(In finite dimensions: vector space with norm / inner product.)*

Scale differentiability / smoothness: A notion of *differentiability / smoothness* for maps between scale Banach spaces such that (classically nonwhere differentiable) reparametrization actions such as $S^1 \times L^2(S^1) \to L^2(S^1), (t,u) \mapsto u(t+\cdot)$ are scale smooth, and the chain rule holds.
(In finite dimensions: classical differentiability / smoothness.)

M-polyfold: Alternative notion to *Banach manifold*, designed to allow a notion of smooth structure on a space $\mathcal{B}$ of (not necessarily pseudoholomorphic) maps modulo reparametrization whose (nodal and non-nodal) domains vary over a Deligne-Mumford type space. *(In finite dimensions: manifold.)*

Strong M-polyfold bundle: Analogue to the notion of *Banach bundle* – a bundle $\mathcal{E} \to \mathcal{B}$ over an M-polyfold $\mathcal{B}$ whose fibers are scale Banach spaces. *(In finite dimensions: vector bundle.)*

Polyfold (with boundary/corners): Alternative notion to *Banach orbifold (with boundary/corners)*, generalizing the notion of M-polyfold to allow for maps with nontrivial isotropy and Deligne-Mumford spaces with boundary/corners. *(In finite dimensions: orbifold (with boundary/corners).)*

Strong polyfold bundle: Analogue to the notion of *Banach orbi-bundle* – a bundle $\mathcal{E} \to \mathcal{B}$ over a polyfold $\mathcal{B}$ whose fibers are scale Banach spaces. *(In finite dimensions: orbi-bundle.)*

Scale-smooth Fredholm section of strong (M-)polyfold bundle: Analogue to the notion of *Fredholm section in a Banach (orbi-)bundle*, which applies to the Cauchy-Riemann operator as section $\sigma = \bar{\partial}_J : \mathcal{B} \to \mathcal{E}$ of appropriate (M-)polyfold bundles. *(In finite dimensions: smooth section.)*

sc$^+$ section of strong (M-)polyfold bundle: Analogue to the notion of *compact perturbation of a Fredholm section*. Examples are reparametrization-invariant 0-th order perturbations of the Cauchy-Riemann operator. *(In finite dimensions: smooth section.)*

Regularization with boundary and corners: The polyfold regularization theorem generalizes directly to Fredholm sections $\sigma : \mathcal{B} \to \mathcal{E}$ over (M-)polyfolds with boundary and corners in various versions corresponding to the notion of transversality to the boundary strata and admissible perturbations (arising from compatibility requirements between perturbations and gluing/breaking).

General sc$^+$ perturbations can always be chosen such that $\sigma + \nu$ is "neatly transverse" and hence $(\sigma+\nu)^{-1}(0)$ is a compact manifold (resp. weighted branched orbifold) with boundary and corners, whose corner strata are given by its intersection with the corresponding boundary strata of $\mathcal{B}$.

References

[FFGW] O. Fabert, J. Fish, R. Golovko, K. Wehrheim, *Polyfolds: A first and second look*, arxiv:1210.6670.

[HWZ] H. Hofer, K. Wysocki, E. Zehnder, *A General Fredholm Theory I–III* and further publications, 2007–present.

[WWW] www.math.ias.edu/∼joelfish/polyfold_outline.html and youtube playlist of IHES summer school 2015.