Title: Floer theories in symplectic and low dimensional topology
Katrin Wehrheim (funded 06-09)

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Project Summary

The proposed project belongs into the general realm of interaction between symplectic geometry and low dimensional topology. Important progress in these areas has been made in the last twenty years starting with the work of Donaldson [D1] on smooth four-manifolds, which was based on anti-self-dual instantons, and with the work of Gromov [G] on pseudoholomorphic curves in symplectic manifolds. In both subjects Floer then introduced a new approach to infinite dimensional Morse theory. In the gauge theoretic context the Floer homology groups provide invariants of three-manifolds that fit into a topological quantum field theory with the Donaldson invariants of four-manifolds. In the symplectic context Floer homology is most generally an invariant for pairs of Lagrangian submanifolds, which is closely related to Gromov–Witten invariants.

One primary goal of this project is the proof of the Atiyah–Floer conjecture, which relates the Floer homology of a homology three-sphere to a Floer homology of Lagrangians which arise from moduli spaces of flat bundles associated to a Heegaard splitting. The remaining major difficulty is a version of the large structure limit in a formulation of mirror symmetry for Kähler surfaces. Bubbling effects in this limit can be excluded by purely analytic methods but should be identified geometrically for the context of mirror symmetry. This leads to a conjectural one-to-one correspondence between anti-self-dual connections on $\mathbb{C} \times \Sigma$ and stable holomorphic bundles on $\mathbb{CP}^1 \times \Sigma$.

A second main part of this project, joint with Chris Woodward, is the definition of a class of invariants for three-manifolds and knots using a class of 'monotone' moduli spaces of bundles arising from decompositions of the manifold resp. knot. This is based on the construction of Floer theoretic functors associated to Lagrangian correspondences, which are part of a categorification of symplectic manifolds and Lagrangian correspondences. These invariants should hence fit into a category valued topological quantum field theory in 2+1+1 dimensions resp. for 0-/1-/2-tangles.

Intellectual Merit

The Atiyah-Floer conjecture is a longstanding open question and its solution would be an important step towards understanding the relations between different invariants of homology three-spheres. The large structure limit technique also has an extension to Seiberg-Witten equations, where it would provide one step in a program of Y.-J. Lee towards identifying the Seiberg-Witten and Heegard Floer homology of three-manifolds. Lagrangian correspondences were introduced by Weinstein as additional functors in a symplectic category, and a 2+1+1 topological quantum field theory factoring through a symplectic category has been a vision ever since. The proposed realization moreover yields new tools for computations of Floer homologies and it would geometrically construct knot invariants that are structurally similar to Khovanov's combinatorial invariants.

Broader Impact

More generally, this program aims to further the understanding and exposition of the analytic foundations of gauge theory, pseudoholomorphic curves, and moduli spaces of nonlinear PDE's in general. One such project is to make a revolutionary abstract framework by Hofer-Wysocki-Zehnder accessible to a wider audience through a graduate course and 'users guide' lecture notes. Further expository texts are planned on removable singularity theorems and Lagrangian Floer homology.

Promoting women in mathematics is another important goal of this project. On a large scale she proposes to organize a conference celebrating the achievements of women in science in order to strengthen the visibility of role models. On a smaller scale she plans to introduce a mathematics day at MIT combining a major orientation for female students with an open day for schoolgirls. This is designed to combine networking and mentoring effects and counter the negative effect of stereotypes on career decisions of women.

Introduction and Results From Prior NSF Support

0.1 Floer Theories in Symplectic Geometry and Topology

explain some exciting fundamentals to the non-expert

The Morse complex of a Morse function on a Riemannian manifold, with transverse intersections of stable and unstable manifolds, is generated by the critical points. The boundary operator is defined by counting the connecting orbits of index difference one and the homology of the resulting complex is isomorphic to the homology of the underlying manifold, see e.g. [Mi, Sc]. Floer's idea was to generalize Morse homology to functionals on infinite dimensional spaces, where the critical points have infinite index and coindex, but the relative indices are finite. As in Morse theory, the chain complex is generated by the critical points and the boundary operator is constructed by counting trajectories between critical points. While the gradient flow equations are not well-posed, the trajectories are solutions of certain nonlinear elliptic equations, such that the spaces of connecting orbits are finite dimensional.

The symplectic Floer homology is defined in terms of the symplectic action on the space of paths in a symplectic manifold M which connect two Lagrangian submanifolds L_0 and L_1. In this case the critical points are the intersection points of L_0 and L_1 and the trajectories are pseudo-holomorphic strips with boundary arcs in L_0 and L_1. Under certain monotonicity hypotheses this gives rise to Floer homology groups $\mathrm{HF}^{\mathrm{symp}}(M, L_0, L_1)$ [F2, Oh]. As a special case one obtains invariants $\mathrm{HF}^{\mathrm{symp}}(\varphi)$ for symplectomorphisms φ of a symplectic manifold, where the complex is generated by the fixed points of φ. These were used by Floer [F3] to prove the Arnold conjecture for monotone symplectic manifolds. This proof and the definition of Floer homology for Hamiltonian symplectomorphisms has been extended to all compact symplectic manifolds [HS, FO, LT].

set the stage

In the Lagrangian case Floer homology is not always defined and there is an obstruction theory [FOOO]. It gives rise to the Fukaya category of a symplectic manifold, where essentially the objects are Lagrangian submanifolds, the morphisms are Floer chains, and the Floer differential ∂ and its failure to satisfy $\partial^2 = 0$ are encoded in an A_∞-structure. In certain cases, Seidel has achieved an understanding of the Fukaya category in terms of vanishing cycles of an associated Lefschetz pencil [Se2]. He used this to establish Kontsevich's mirror conjecture for the quartic [Se3], where the Fukaya category appears on the symplectic side.

The second setting in which Floer carried his program through is given by the Chern–Simons functional on the space $\mathcal{A}(Y)$ of SU(2)-connections on a homology 3-sphere Y [F1]. In this case the critical points are the flat connections on Y and the trajectories are anti-self-dual instantons on $\mathbf{R} \times Y$. The resulting instanton Floer homology groups $\mathrm{HF}^{\mathrm{inst}}(Y)$ are smooth invariants of the 3-manifold. They moreover interact naturally with the moduli spaces of anti-self-dual instantons on a 4-manifold X with boundary $\partial X = Y$, giving rise to a topological quantum field theory [D2] that is closely related to the Donaldson invariants of 4-manifolds [D1].

Similarly, the Seiberg–Witten invariants of 4-manifolds are related to a Seiberg–Witten version of Floer homology [Ma, CW, KM2]. Also in the spirit of Seiberg–Witten theory, Ozsváth and Szabó [OS1] introduced a further Floer type 3-manifold invariant: A Heegaard splitting of the 3-manifold gives rise to two totally real tori in the symmetric product $\mathrm{Sym}^g(\Sigma)$. The Heegaard Floer homology is constructed from these tori analogously to Floer's theory for pairs of Lagrangians in a symplectic manifold. Ozsváth and Szabó [OS2] conjecture that the Heegaard Floer homology is isomorphic to an equivariant Seiberg–Witten Floer homology. The structural similarity of these Floer theories is apparent, in particular in comparison to the work of Kronheimer and Mrowka [KM2].

0.2 The Atiyah–Floer conjecture

precisely define your main objects and state your goal clearly

A Heegard splitting of a closed 3-manifold M is a decomposition $M = H_0 \cup_\Sigma H_1$ into two handlebodies H_i with common boundary $\partial H_i = \Sigma$. In contrast to Heegard Floer homology one can also use representation spaces to define a pair of Lagrangians $L_{H_0}, L_{H_1} \subset \mathcal{R}_\Sigma$ in a symplectic manifold,

$$L_{H_i} := \mathrm{Hom}\Big(\frac{\pi_1(\Sigma)}{\partial\pi_2(H_i,\Sigma)}, \mathrm{SU}(2)\Big)/\mathrm{SU}(2) \;\subset\; \mathcal{R}_\Sigma := \mathrm{Hom}(\pi_1(\Sigma), \mathrm{SU}(2))/\mathrm{SU}(2),$$

where SU(2) acts by conjugation. These spaces are all singular, but they can be represented as (symplectic) quotients of smooth (Banach-)manifolds in gauge theory,

$$\mathcal{L}_H := \big\{\tilde{A}|_\Sigma \,\big|\, \tilde{A} \in \Omega^1(H;\mathfrak{su}(2)), F_{\tilde{A}} = 0\big\} \;\subset\; \mathcal{A}(\Sigma) := \Omega^1(\Sigma;\mathfrak{su}(2)).$$

More precisely, the symplectic structure on $\mathcal{A}(\Sigma)$ is given by $\omega(\alpha,\beta) = -\int_\Sigma \mathrm{tr}(\alpha\wedge\beta)$. The pull-back action of bundle isomorphisms, represented by the gauge group $\mathcal{G}(\Sigma) := \mathrm{Map}(\Sigma, \mathrm{SU}(2))$, is Hamiltonian with moment map given by the curvature $F_A = \mathrm{d}A + A\wedge A$ of a connection $A \in \mathcal{A}(\Sigma)$.

Assuming that the symplectic Floer homology can be defined in the singular quotient setting, Atiyah and Floer conjectured that it should be naturally isomorphic to the instanton Floer homology of the closed manifold $M = H_0 \cup_\Sigma H_1$, if the latter is an integer homology 3-sphere.

Conjecture 1 (Atiyah, Floer) $\mathrm{HF}^{\mathrm{inst}}(M) \cong \mathrm{HF}^{\mathrm{symp}}(\mathcal{R}_\Sigma, L_{H_0}, L_{H_1})$

The aim of my long term project with Dietmar Salamon is to prove this conjecture via an intermediate Floer homology $\mathrm{HF}^{\mathrm{inst}}([0,1]\times\Sigma, \mathcal{L}_{H_0\sqcup H_1})$ that couples the instanton equation with Lagrangian boundary conditions in the smooth but infinite dimensional setting. (See [Sa, W4] for surveys.) In a first step [SW], based on [W2, W3], we have now defined this intermediate invariant for general 3-manifolds with boundary and a general class of Lagrangian boundary conditions. The following theorem applies in particular when $\mathcal{L} = \mathcal{L}_H$, where H is a disjoint union of handle bodies with $\partial H = \Sigma$, and such that $Y \cup_\Sigma H$ is an integer homology 3-sphere.

state previous results nontechnically

Theorem 2 *Let Y be a compact, oriented 3-manifold with boundary Σ, and let $\mathcal{L} \subset \mathcal{A}(\Sigma)$ be a gauge invariant, monotone, irreducible Lagrangian submanifold, i.e.*

(L1) *$\mathcal{L}$ is a Fréchet submanifold of $\mathcal{A}(\Sigma)$, each tangent space $\mathrm{T}_A\mathcal{L}$ is a Lagrangian subspace of $\Omega^1(\Sigma,\mathfrak{g})$, $\mathcal{L} \subset \mathcal{A}_{\mathrm{flat}}(\Sigma) := \{F_A = 0\}$, and $\mathcal{L}$ is invariant under $\mathcal{G}(\Sigma)$.*

(L2) *The quotient of $\mathcal{L}$ by the based gauge group $\mathcal{G}_z(\Sigma) = \{u : \Sigma \to \mathrm{SU}(2) \,|\, u(z) = \mathbb{1}\}$ is compact, connected, simply connected, and $\pi_2(\mathcal{L}/\mathcal{G}_z(\Sigma)) = 0$.*

(L3) *The zero connection is contained in $\mathcal{L}$ and is nondegenerate. Moreover, every nontrivial flat connection $A \in \mathcal{A}(Y)$ with $A|_\Sigma \in \mathcal{L}$ is irreducible.*

Then the Floer homology $\mathrm{HF}^{\mathrm{inst}}(Y,\mathcal{L})$ is well-defined and independent of the metric and perturbations used to define it.

brag/ describe efforts

For this construction we developed and refined a number of technical tools. In particular, the extension of the compactness results to general Lagrangians is based on new results with Tom Mrowka [MWe] on the topology of the gauge action on $L^2(\Sigma)$, the borderline Sobolev case. These replace an extension result in [W3]: *The $L^2(\Sigma)$-norm on the Lagrangian $\mathcal{L}_H$ controls the*

break project into pieces
display intuitive formulas

corresponding flat connections on H in $L^2(H)$. (Although it is unclear whether the L^2-closure of $\mathcal{L}_H$ is smooth.) The latter however also is a first step towards an isomorphism between the instanton Floer homologies of manifolds with and without boundary,

$$\mathrm{HF}^{\mathrm{inst}}(Y, \mathcal{L}_H) \overset{?}{\cong} \mathrm{HF}^{\mathrm{inst}}(Y \cup H). \tag{1}$$

The monotonicity formulas and coherent orientations for theorem 2 were obtained by identifying the index bundle for the anti-self-duality operator with Lagrangian boundary conditions to one on a corresponding closed manifold. This also provides the linear theory towards (1). As final step towards a proof of conjecture 1 we plan to establish the isomorphism

$$\mathrm{HF}^{\mathrm{inst}}([0,1] \times \Sigma, \mathcal{L}_{H_0 \sqcup H_1}) \overset{?}{\cong} \mathrm{HF}^{\mathrm{symp}}(\mathcal{R}_\Sigma, L_{H_0}, L_{H_1}) \tag{2}$$

by adapting an adiabatic limit argument by Dostoglou–Salamon to the degeneration $\varepsilon \to 0$ of the metric $\mathrm{d}t^2 + \varepsilon^2 g_\Sigma$ on $[0,1] \times \Sigma$.

0.3 Bubbling Analysis

One essential new difficulty in the adiabatic limit for (2) is the occurrence of intermediate bubbles that I can now exclude by the following quantization of energy (which coincides with the charge for anti-self-dual connections). The following is a special cases of the result in [W6] for $\mathbb{C} \times \Sigma$, and can be proven with similar methods in the boundary case.

Theorem 3 *Let Σ be a Riemannian surface and equip $\mathbb{C}$ (or the half space $\mathbb{H} \subset \mathbb{C}$) with the Euclidean metric. Consider a connection $A \in \Omega^1(\mathbb{C} \times \Sigma; \mathfrak{su}(2))$ (or $A \in \Omega^1(\mathbb{H} \times \Sigma; \mathfrak{su}(2))$ with Lagrangian boundary conditions $A|_{\{z\}\times\Sigma} \in \mathcal{L}$ for all $z \in \partial\mathbb{H}$). If the connection has finite energy $\mathcal{E}(A) := \frac{1}{2}\int |F_A|^2 < \infty$, then it has integral charge $\frac{-1}{8\pi^2}\int \mathrm{tr}(F_A \wedge F_A) \in \mathbb{Z}$.*

This integrality is surprising and of independent interest. The bubbles on $\mathbb{C} \times \Sigma$ for Σ a 2-torus also appear in the so-called 'large structure limit', which relates holomorphic bundles to Lagrangian submanifolds in a formulation of Mirror Symmetry for Kähler surfaces. The new energy quantization for these bubbles will significantly strengthen previous results in this area by e.g. Chen and Nishinou. More generally, this charge identity supports a conjecture by Marcos Jardim and myself of a one-to-one correspondence between finite energy anti-self-dual connections on $\mathbb{C} \times \Sigma$ and stable holomorphic bundles over $\mathbb{CP}^1 \times \Sigma$.

More generally, the bubbling analysis in moduli spaces with a monotonicity property (energy–index relation) actually only requires an energy quantization $\mathcal{E} \geq \hbar > 0$. In [W5] I have formulated this well known technique as a general energy quantization principle for sequences of energy density functions. It applies to the adiabatic limit for anti-self-dual instantons as well as to the shrinking of strips in pseudoholomorphic quilts described in the next section.

Theorem 4 *Let D be a Riemannian manifold (possibly with boundary). There exists a constant $\hbar > 0$ depending on $n = \dim D$ and given constants $a, b \geq 0$ such that the following holds:*

DO NOT SQEEZE!

Let $e_i \in \mathcal{C}^2(D, [0,\infty))$ be a sequence of nonnegative functions such that $\boxed{\Delta} e_i \leq A_0 + A_1 e_i + a e_i^{\frac{n+2}{n}}$ and $\boxed{\frac{\partial}{\partial\nu}}\big|_{\partial D} e_i \leq B_0 + B_1 e_i + b e_i^{\frac{n+1}{n}}$ for some constants $A_0, A_1, B_0, B_1 \geq 0$, and with uniformly bounded energy $\int_D e_i \leq E < \infty$. Then there exist finitely many points, $x_1, \ldots, x_N \in D$ and a subsequence such that the e_i are uniformly bounded on every compact subset of $D \setminus \{x_1, \ldots x_N\}$, and there is a concentration of energy $\hbar > 0$ at each x_j (i.e. $\liminf_{i\to\infty} \int_{B_\delta(x_j)} e_i \geq \hbar$ for all $\delta > 0$).

0.4 Functoriality for Lagrangian correspondences

In a joint project with Chris Woodward [WW] we have achieved a categorification for symplectic manifolds and Lagrangian correspondences: To any monotone symplectic manifold M one can associate (following Donaldson and Fukaya) a category $\mathrm{Don}(M)$ whose objects are Lagrangian submanifolds, and whose morphisms are Floer homology classes. To any Lagrangian correspondence, that is a Lagrangian submanifold $L_{01} \subset M_0^- \times M_1$ in the product of two symplectic manifolds M_0, M_1, we define a functor

$$\Phi_{L_{01}} : \mathrm{Don}^{\#}(M_0) \to \mathrm{Don}^{\#}(M_1)$$

between suitable enlargements of these categories. In particular, a simple Lagrangian submanifold $L_0 \subset M_0$ is viewed as correspondence $L_0 \subset \{pt.\}^- \times M_0$ and is mapped to the sequence of Lagrangian correspondences $\Phi_{L_{01}}(L_0) := (L_0, L_{01})$ from $\{pt.\}$ to M_1, which is an object of $\mathrm{Don}^{\#}(M_1)$. In this setting we can define the functor on the morphism level

you can be vague - as long as it's intuitive

$$\Phi_{L_{01}} : \mathrm{HF}(M_0; L_0, L_0') \to \mathrm{HF}(M_1; (L_0, L_{01}), (L_0', L_{01})).$$

Here both the generalized Floer homology for sequences of Lagrangian correspondences and the relative invariant $\Phi_{L_{01}}$ are defined from moduli spaces of *pseudoholomorphic quilts*, whose 'patches' are pseudoholomorphic maps to various symplectic manifolds with 'seams' mapping to Lagrangian correspondences. Our first main result is that the composition with the functor for $L_{12} \subset M_1^- \times M_2$ is isomorphic to the functor associated to the geometric composition of the correspondences, given by $\pi_{02} : (L_{01} \times L_{12}) \pitchfork (M_0 \times \Delta_{M_1} \times M_2) \to L_{01} \circ L_{12} \subset M_0^- \times M_2$, if this is an embedding,

use short, meaningful formulas $\longrightarrow$

$$\Phi_{L_{01}} \circ \Phi_{L_{12}} \cong \Phi_{L_{01} \circ L_{12}}.$$

Moreover, we extend this construction to a 2-functor from a Floer type 2-category of symplectic manifolds and Lagrangian correspondences to the 2-category of categories. One of the basic nontrivial ingredients of this theory is the following, in a more down-to-earth special case.

state results

Theorem 5 *Let M_0, M_1, M_2 be a triple of compact (or exact) symplectic manifolds with the same monotonicity constant, and consider compact, oriented, monotone Lagrangian submanifolds*

$$L_0 \subset M_0, \quad L_{01} \subset M_0^- \times M_1, \quad L_{12} \subset M_1^- \times M_2, \quad L_2 \subset M_2^-.$$

If $L_{01} \times_{M_1} L_{12} := (L_{01} \times L_{12}) \cap (M_0 \times \Delta_{M_1} \times M_2)$ is smooth, π_{02} embeds it into $L_{01} \circ L_{12} \subset M_0^- \times M_2$, and the latter Lagrangian is also monotone, then there exists a canonical isomorphism

$$HF(M_0 \times M_1^- \times M_2; L_0 \times L_{12}, L_{01} \times L_2) \xrightarrow{\sim} HF(M_0 \times M_2^-; L_0 \times L_2, L_{01} \circ L_{12}). \tag{3}$$

sketch proofs and essential difficulties

This isomorphism is induced by the identification $(L_0 \times L_{12}) \cap (L_{01} \times L_2) \cong (L_0 \times L_2) \cap (L_{01} \circ L_{12})$ of the generators of the Floer complex. The Floer differential for $(L_0 \times L_{12}, L_{01} \times L_2)$ counts triples of pseudoholomorphic strips in M_0, M_1, M_2. (Such tuples of strips are the basic examples of pseudoholomorphic quilts, see figure 1.) In the standard definition, one would take the width of all three strips to be equal, but in fact one can allow the widths to differ and prove that, with the width of the middle strip sufficiently close to zero, the triples of pseudoholomorphic strips in M_0, M_1, M_2 are in one-to-one correspondence with the pairs of pseudoholomorphic strips in M_0, M_2 that are counted in the Floer differential for $(L_0 \times L_2, L_{01} \circ L_{12})$. This involves disallowing a "figure eight" bubble (which does not appear in the standard theory) by energy quantization as in theorem 4.

use simple pictures to demonstrate something dramatic

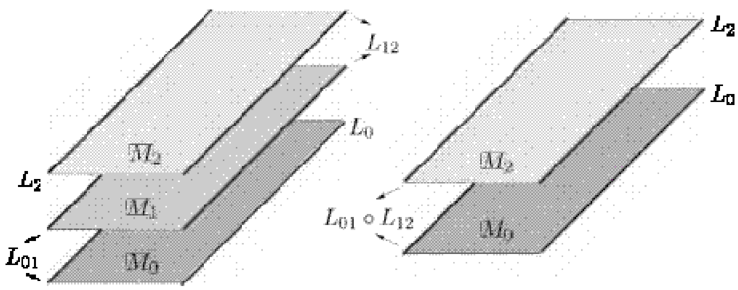

Figure 1: Holomorphic strips that are counted for $HF(L_0 \times L_2, L_{01} \circ L_{12})$ and $HF(L_0 \times L_{12}, L_{01} \times L_2)$

Project Description

The following description of possible projects is organised into three major fields of research and one section summarizing projects of broader impact. Several parts of these are joint work as indicated, some others might become projects for graduate students.

old ongoing project

1 The Atiyah-Floer conjecture and related questions

1.1 Adiabatic limits

The next step in my joint project with Dietmar Salamon is the following.

state inter-mediate goals

Conjecture 6 *Under the assumptions of theorem 2 there is a natural isomorphism*

$$\mathrm{HF}^{\mathrm{inst}}(Y, \mathcal{L}_H) \cong \mathrm{HF}^{\mathrm{inst}}(Y \cup_\Sigma H).$$

sketch approach

Our idea of proof is to choose a family of embeddings $\psi_\varepsilon : Y \hookrightarrow Y \cup_\Sigma H =: \tilde{Y}$ starting from $\psi_1 = \mathrm{Id}_Y$ and shrinking $\tilde{Y} \setminus \psi_\varepsilon(Y)$ to the 1-dimensional core of H for $\varepsilon \to 0$. Then the anti-self-dual instantons on $\mathbb{R} \times \tilde{Y}$ (for a fixed metric g on $\tilde{Y}$) pull back to anti-self-dual instantons on $\mathbb{R} \times Y$ with respect to the metrics $\psi_\varepsilon^* g$, which degenerate on ∂Y for $\varepsilon \to 0$. On the other hand, an anti-self-dual instanton on $\mathbb{R} \times Y$ with boundary values in $\mathcal{L}_H$ can be pushed forward by ψ_ε and extended as in section 0.2 to an almost anti-self-dual instanton on $\mathbb{R} \times \tilde{Y}$. As $\varepsilon \to 0$, one should be able to pass from the genuine boundary value problem to solutions on the closed manifold $\tilde{Y}$. The linear theory (identification of indices and orientations) has largely been covered in [SW].

If conjecture 6 is true, then the Atiyah–Floer conjecture reduces to the following analogon of the Atiyah–Floer conjecture for mapping tori (by Dostoglou and Salamon).

Conjecture 7 *Any Heegaard splitting $H_0 \cup_\Sigma H_1$ of a homology 3-sphere induces a natural isomorphism*

$$\mathrm{HF}^{\mathrm{inst}}([0,1] \times \Sigma, \mathcal{L}_{H_0} \times \mathcal{L}_{H_1}) \cong \mathrm{HF}^{\mathrm{symp}}(\mathcal{R}_\Sigma, L_{H_0}, L_{H_1}).$$

The proof of the required correspondence between pseudoholomorphic curves in the moduli space of flat connections and anti-self-dual instantons – both with Lagrangian boundary conditions

– should be an adaptation of the adiabatic limit argument in [DS]. More precisely, if we choose a split metric $dt^2 + \varepsilon^2 g_\Sigma$ on $[0,1]\times\Sigma$ for a fixed metric g_Σ on Σ and any $\varepsilon > 0$, then the anti-self-duality equation with Lagrangian boundary conditions becomes

$$\begin{cases} \partial_s A - \mathrm{d}_A\Phi + *(\partial_t A - *\mathrm{d}_A\Psi) = 0, \\ \partial_s\Psi - \partial_t\Phi + [\Phi,\Psi] + \varepsilon^{-2} * F_A = 0, \\ A(s,i) \in \mathcal{L}_{H_i} \qquad \text{for } s\in\mathbb{R}, i\in\{0,1\}, \end{cases} \tag{4}$$

for $A : \mathbb{R}\times[0,1] \to \mathcal{A}(\Sigma)$ and $\Phi,\Psi : \mathbb{R}\times[0,1] \to \Omega^0(\Sigma,\mathfrak{su}(2))$. As $\varepsilon \to 0$ one should be able to pass from these ASD instantons to pseudoholomorphic curves in the sense of (6). Here one crucial point is to study three different bubbling effects corresponding to the ratio between the divergence of the curvature and the convergence $\varepsilon \to 0$. One extreme case corresponds to the bubbling for holomorphic curves that also has to be studied in part 1.2 below. The other extreme case corresponds to bubbling for anti-self-dual instantons as in theorem 2. The intermediate case can be understood more geometrically as the convergence of semi-locally rescaled solutions to a nontrivial anti-self-dual instanton on $\mathbb{R}^2\times\Sigma$ with finite energy. These are of independent interest, see part 1.5 below. In the present monotone setting, all these bubbling effects can be dealt with by an energy quantization argument as in theorem 4.

give details – but make sure the overall argument is clear without reading/understanding all of them

refer back/forward

1.2 Floer homology for singular symplectic quotients

A trajectory of the symplectic Floer homology $HF^{symp}(\mathcal{R}_\Sigma, L_{Y_0}, L_{Y_1})$ should be a pseudoholomorphic map $u : \mathbb{R}\times[0,1] \to \mathcal{R}_\Sigma$ with boundary values in L_{H_0} and L_{H_1},

$$\partial_s u + J\partial_t u = 0, \qquad u(s,i) \in L_{H_i} \quad \text{for } s\in\mathbb{R}, i = 0,1. \tag{5}$$

The almost complex structure J is induced by the Hodge operator of a $[0,1]$-family of metrics on Σ. If u takes values in the irreducible representations (where $\mathcal{R}_\Sigma$ is smooth), then (5) is meaningful and u lifts to a map $A : \mathbb{R}\times[0,1] \to \mathcal{A}(\Sigma)$ satisfying

$$\begin{cases} \partial_s A - \mathrm{d}_A\Phi + *(\partial_t A - \mathrm{d}_A\Psi) = 0, \\ *F_A = 0, \\ A(s,i) \in \mathcal{L}_{H_i} \quad \text{for } s\in\mathbb{R}, i = 0,1. \end{cases} \tag{6}$$

Here $\mathrm{d}_A\Phi$ and $\mathrm{d}_A\Psi$ are viewed as infinitesimal gauge transformations and A uniquely determines $\Phi,\Psi : \mathbb{R}\times[0,1] \to \Omega^0(\Sigma,\mathfrak{su}(2))$ via $\Delta_A\Phi = \mathrm{d}_A\partial_s A$, $\Delta_A\Psi = \mathrm{d}_A\partial_t A$. This is since $\Delta_A = \mathrm{d}_A^*\mathrm{d}_A$ is invertible for irreducible $A\in\mathcal{A}(\Sigma)$. If A is allowed to become reducible, then Φ and Ψ have some extra freedom that makes the moduli space of solutions (A,Φ,Ψ) infinite dimensional. We expect however that one can use perturbations of (6) to obtain finite dimensional smooth moduli spaces of trajectories in the cases that are relevant for $HF^{symp}(\mathcal{R}_\Sigma, L_{H_0}, L_{H_1})$, i.e. when at least one critical point is irreducible.

short sections

This setup should generalize to more general symplectic quotients. In particular, quotients of finite dimensional group actions could be used as test case for this definition.

clear titles

1.3 Products in the Atiyah-Floer conjecture

If H_0, H_1, H_2 are three handle bodies with boundary Σ such that the manifold $H_i \cup_\Sigma \bar{H}_j$ is a homology 3-sphere for $i \neq j$, then there are product morphisms on all three Floer homologies in conjectures 6 and 7 that should be intertwined by our isomorphisms. For the symplectic Floer homology these are the Donaldson products

$$\mathrm{HF}^{\mathrm{symp}}(\mathcal{R}_\Sigma, L_{H_0}, L_{H_1}) \otimes \mathrm{HF}^{\mathrm{symp}}(\mathcal{R}_\Sigma, L_{H_1}, L_{H_2}) \to \mathrm{HF}^{\mathrm{symp}}(\mathcal{R}_\Sigma, L_{H_0}, L_{H_2}).$$

These should be defined by counting holomorphic triangles $D \to \mathcal{R}_\Sigma$ in the sense of part 1.2 with boundary conditions on the L_{H_i} for the three boundary arcs of D. The product on the instanton Floer homology takes the form

$$\mathrm{HF}^{\mathrm{inst}}(H_0 \cup \bar{H}_1) \otimes \mathrm{HF}^{\mathrm{inst}}(H_1 \cup \bar{H}_2) \to \mathrm{HF}^{\mathrm{inst}}(H_0 \cup \bar{H}_2).$$

It is defined by counting anti-self-dual instantons on a cobordism X that is obtained by gluing $\mathbb{R}\times H_i$ to the three boundary components of $D \times \Sigma$. Finally, let $K := [0,1] \times \Sigma$, then our analytic setup for the anti-self-duality equation with Lagrangian boundary conditions also includes the domain $D \times \Sigma$, which should yield an intermediate product

$$\mathrm{HF}^{\mathrm{inst}}(K, \mathcal{L}_{H_0} \times \mathcal{L}_{H_1}) \times \mathrm{HF}^{\mathrm{inst}}(K, \mathcal{L}_{H_1} \times \mathcal{L}_{H_2}) \to \mathrm{HF}^{\mathrm{inst}}(K, \mathcal{L}_{H_0} \times \mathcal{L}_{H_2}).$$

1.4 Instanton Floer homology for 3-manifolds

In a first step we extended the instanton Floer homology with Lagrangian boundary conditions to general gauge invariant Lagrangian submanifolds $\mathcal{L} \subset \mathcal{A}(\Sigma)$. One might be able to go one step further, also allowing for different bundles. For the nontrivial SO(3)-bundle one should then find an isomorphism between the Floer homology as defined in theorem 2 and Fukaya's Floer homology for 3-manifolds with boundary [Fu]. Moreover, Fukaya suggested that one might try to define an invariant for more general closed 3-manifolds Y starting from a Heegard splitting $Y = H_0 \cup_\Sigma H_1$ and using the symplectic Floer homology for perturbations of the Lagrangians $L_{H_0}, L_{H_1} \subset \mathcal{R}_\Sigma$ (whose intersection should be transverse to the singular locus). It might be one step easier to use the new instanton Floer homology $\mathrm{HF}^{\mathrm{inst}}([0,1] \times \Sigma, \mathcal{L}_0, \mathcal{L}_1)$ for perturbations $\mathcal{L}_i$ of the $\mathcal{L}_{H_i}$ such that $\mathcal{L}_0 \cap \mathcal{L}_1$ contains no nontrivial reducible connection. The main issue would be to understand the wall crossing effects for different choices of perturbations.

1.5 Anti-self-dual instantons on $\mathbb{C} \times \Sigma$ and holomorphic bundles on $\mathbb{CP}^1 \times \Sigma$

give connection to other problems/ fields

The adiabatic limit in (4) without boundary conditions is considered as the 'large structure limit' in a formulation of Mirror Symmetry for Kähler surfaces. It would thus be interesting to obtain a full description of the limit by identifying the bubbles as geometric objects. The energy identity in theorem 3 for the intermediate bubbles $A \in \mathcal{A}_{\mathrm{ASD}}(\mathbb{C} \times \Sigma)$, anti-self-dual instantons on $\mathbb{C} \times \Sigma$, seems to indicate an underlying topological object whose invariants would fix the energy. Indeed, if $A \in \mathcal{A}_{\mathrm{ASD}}(\mathbb{C} \times \Sigma)$ extends to a connection on a closed manifold, then the corresponding bundle would have second Chern number $-\frac{1}{4\pi^2}\int \mathrm{tr}(F_A \wedge F_A) = \frac{1}{4\pi^2}\mathcal{E}(A)$. In general, there will be holonomy obstructions to the extension of A, but the holomorphic structure associated to A might still extend to a holomorphic bundle over $\mathbb{CP}^1 \times \Sigma$. A result in this direction was proven by Biquard and Jardim [BJ]: Anti-self-dual instantons on $\mathbb{C} \times \mathbb{T}^2$ with quadratic curvature decay are in one-to-one correspondence to stable holomorphic bundles over $\mathbb{CP}^1 \times \mathbf{T}^2$.

describe non-central facts unprecise

With this background, Marcos Jardim and myself conjecture a one-to-one correspondence between finite energy anti-self-dual instantons on $\mathbb{C} \times \Sigma$ and stable holomorphic bundles over $\mathbb{CP}^1 \times \Sigma$. A major step towards this would be to prove that finite energy implies quadratic curvature decay.

1.6 Analogues for Seiberg-Witten and Heegard Floer homology

I have convinced myself that the elliptic theory for anti-self-dual instantons with Lagrangian boundary conditions as in theorem 2 should work analogously for the Seiberg-Witten equation with Lagrangian boundary conditions, and the bubbling effects are excluded as usual in Seiberg-Witten theory. Following ideas of Tom Mrowka, one might use this to define a Seiberg-Witten Floer theory for 3-manifolds with boundary. This would have to deal with reducible critical points analogously to the theory by Kronheimer–Mrowka, where the reducible points are blown up and one then deals with a generalized Morse theory on a manifold with boundary.

Yi-Jen Lee has set up a program for proving an isomorphism between the Heegaard Floer homology defined by Ozsváth and Szabó and a Seiberg-Witten Floer homology. It is based on techniques of Taubes but will also make use of a Floer theory as above, or at least of the associated moduli spaces. An adiabatic limit analysis analogous to section 1.1 should thus be helpful for the program towards this isomorphism.

2 The algebraic and analytic structure of Floer theories

A general Floer theory for Lagrangian intersections has to overcome several problems. On the technical level, holomorphic disks need to be somewhere injective in order for standard transversality methods (choice of generic almost complex structure or Hamiltonian perturbation) to apply. More conceptionally, multiply covered disks form orbifold singularities in the moduli space of holomorphic disks. Finally, even transverse, simple disk bubbles can occur as codimension 1 phenomenon in moduli spaces of holomorphic curves, thus obstructing $\partial^2 = 0$ in the Floer theory. These bubbling phenomena cancel in special monotone or symmetric cases, but in other cases the algebraic structure arising from the moduli spaces of Floer trajectories must be more general than a homology theory.

However, even in the easiest cases there are very few explicit calculations of Floer homologies beyond Floer's proof of the Arnold conjecture [F2], identifying $\mathrm{HF}(L, L)$ with the homology $\mathrm{H}_*(L)$ in the absence of disk bubbles. (Notable exceptions are Heegard Floer homology for knots, which now even has a combinatorial description [MOS], some calculations from complex geometry [C, Oh], and predictions via Mirror symmetry in the current work of Fukaya-Seidel-Smith.)

2.1 Abstract transversality theory

There are now several approaches to the mentioned transversality problems: First, Fukaya-Oh-Ono-Ohta [FO, FOOO] developed a technique of constructing a 'virtual fundamental cycle' from the 'obstruction bundle' formed by the (finite dimensional) cokernels over the solution set. A more geometric approach, using intersections with complex hypersurfaces, is being developed by Cieliebak-Mohnke. Finally, Hofer-Wysocki-Zehnder are currently finalizing a new abstract nonlinear Fredholm theory [H, HWZ]. They construct a smooth structure on compactified moduli spaces from a general implicit function theorem for transverse 'scale smooth Fredholm sections' in a 'strong bundle' over a 'polyfold'. Moreover, they provide a general transversality construction in the presence of finite symmetry groups as well as interactions of moduli spaces (i.e. when 'bubbling' in the boundary of one moduli space is described in terms of 'gluing' of other moduli spaces). This setup was developed for the construction of Symplectic Field Theory, but it is in a general formulation that should allow for the application to other moduli spaces arising from nonlinear PDE's.

state goal

My aim is to use the latter setup to streamline the construction of some Floer theories, e.g. construct an equivariant instanton Floer homology for general 3-manifolds along the lines of [AB], at the same time providing test cases for the abstract Fredholm theory. The equivariant instanton case, for example, requires an extension of the transversality theory to finite dimensional symmetry groups. A more tangible setting, to which the abstract theory should apply directly, is pseudodoholomorphic curves with Lagrangian boundary conditions. I have already done some basic constructions for this case, see section 2.2, and am also hoping to extend this work towards establishing the basic polyfold charts and Fredholm properties for a relative symplectic field theory, where Lagrangian boundary conditions arise from cobordisms of Legendrians.

claim experience

bold goals...

As part of this project I will be teaching a 2-semester graduate course (starting spring 2007) and I am planning to write a lecture script, highlighting the essential abstract theory and describing its application to pseudoholomorphic disks.

...need not all be accomplished

2.2 Generalizations of Floer homology for Lagrangian intersections

have fun!

The algebraic problems due to disk bubbling have been resolved in an obstruction theory by Fukaya-Oh-Ono-Ohta [FOOO], which encodes the moduli spaces of holomorphic disks in an A_∞-structure. However, to the dismay of some geometers, this algebra is generated by a countable collection of currents on each Lagrangian. An alternative, finitely generated structure was proposed by Cornea-Lalonde [CL] using so-called clusters, which are trees of holomorphic disks connected by Morse flow lines. The feature of cluster moduli spaces compared to [FOOO] is that a disk bubbling off in a 1-parameter family is no longer considered a boundary point, i.e. contribution to ∂^2, but the family is continued by allowing a flow line of a Morse function between the attaching points of the disk. However, [CL] disallows ghost bubbles and chooses a more symmetric algebra and orientations in order to obtain a homology theory. Nevertheless, I conjectured an equality of the two theories in the unobstructed case: The vanishing of [FOOO] obstruction classes should correspond to vanishing of the [CL] free terms, in which case a linearization of the [CL] fine Floer complex should coincide with the [FOOO] generalized Floer homology obtained from obstruction corrections to the differential. In the general (obstructed) case it has been observed (inter alia by Fukaya and Seidel) that the cluster moduli spaces correspond to stable trees that would occur in the A_∞ perturbation lemma [Se4, Proposition 1.12] when lifting the [FOOO] A_∞-structure through the [HL] equivalence between de Rham theory for currents and Morse theory.

utterly nontechnical project description

In a joint project with Peter Albers we are using this perturbation lemma as guideline to define an actual A_∞-algebra from cluster moduli spaces. This should be a finitely generated equivalent of the [FOOO] A_∞-algebra for general Lagrangian submanifolds. In order to capture the classical cup product as in [FOOO], we have to allow for ghost bubbles in the cluster trees of disks. Eventually, we hope to achieve a construction of the Fukaya A_∞-category whose morphism spaces are finitely generated, in particular by critical points of a Morse function for the isomorphism spaces. In the process of this project, I am moreover planning to formulate the compactification and transversality theory for the cluster moduli spaces in the new abstract [HWZ] Fredholm theory described in section 2.1. In this new framework, a disk bubble can indeed be described as interior point of a 'polyfold'. The compactified space of clusters (stable trees of holomorphic discs) should then be the zero set of a 'scale smooth Fredholm section' on a 'bundle' over the 'polyfold'. This point of view also seems to simplify orientation issues at ghost bubbles.

2.3 Relations on monotone Floer homology for Lagrangian intersections

you can sell as project
what you already did ...
.. just haven't published

Theorem 5 provides a new relation between the Floer homologies of monotone Lagrangian submanifolds. This may allow for calculations of the Floer homology in special cases. For example, if a Hamiltonian G-action on a symplectic manifold M gives rise to a smooth symplectic quotient $M/\!\!/G = \mu^{-1}(0)/G$, then it defines a smooth Lagrangian correspondence $\mu^{-1}(0) \hookrightarrow M^- \times M/\!\!/G$ and its self-composition, $\Lambda_\mu := \{(x,y) \in \mu^{-1}(0)^2 \,|\, [x] = [y] \in M/\!\!/G\} \subset M^- \times M$. Under suitable monotonicity and transversality assumptions one should then obtain the following identity between Floer homologies on $M/\!\!/G$ and $M \times M$ as corollary of theorem 5.

Conjecture 8 *Let $\mathcal{L}_0, \mathcal{L}_1 \subset M$ be monotone Lagrangian submanifolds transverse to $\mu^{-1}(0)$ such that the intersections $\mathcal{L}_i \cap \mu^{-1}(0)$ embed to monotone Lagrangians $L_i \subset M/\!\!/G$. Then there is a natural isomorphism* $\mathrm{HF}(M/\!\!/G; L_0, L_1) \cong \mathrm{HF}(M \times M; \mathcal{L}_0 \times \mathcal{L}_1, \Lambda_\mu)$.

Previously, the only known relation on Floer homology for pairs of (not Hamiltonian isotopic) Lagrangian submanifolds was Seidel's exact triangle for Dehn twists around Lagrangian spheres. Under suitable monotonicity assumptions it can be stated as follows.

Theorem 9 (Seidel) *Let $\tau_C : M \to M$ denote the Dehn twist around a Lagrangian sphere $C \subset M$. Then for any monotone pair of Lagrangians $L_0, L_1 \subset M$ there exists an exact triangle*

$$\begin{tikzcd}
\mathrm{HF}(M; \tau_C(L_0), L_1) \arrow[rr] & & \mathrm{HF}(M; L_0, L_1) \arrow[dl, dashed] \\
& \mathrm{HF}(M \times M; L_0 \times C, C \times L_1). \arrow[ul, dashed] &
\end{tikzcd} \tag{7}$$

Following a setup of Seidel–Smith, it is part of my joint project with Chris Woodward to extend this result to Dehn twists around coisotropics $C \subset M$ that are spherically fibred over a symplectic base B. (These occur as vanishing cycles of symplectic Lefschetz fibrations of Morse–Bott type. They give rise to Lagrangian correspondences $C \hookrightarrow M^- \times B$ and their transpose $C^t \hookrightarrow B^- \times M$.) In that case, the bottom of the exact triangle is replaced by $\mathrm{HF}(M \times B^- \times M; L_0 \times C^t, C \times L_1)$ and the corresponding maps can be defined in terms of holomorphic sections of Lefschetz fibrations over quilted surfaces similar to the ones described in section 0.4.

3 Floer Field theory

new project in progress

3.1 A_∞ functors associated to Lagrangian correspondences

In a joint project with Sikimeti Mau and Chris Woodward we are working on extending the categorification for symplectic manifolds and Lagrangian correspondences in section 0.4 to an A_∞-categorification. For that purpose we realize Stasheff's multiplihedron as a moduli space of quilted disks. Under suitable monotonicity assumptions we can use it to construct A_∞-functors $\Phi_{L_{ij}}$ associated to Lagrangian correspondences $L_{ij} \subset M_i^- \times M_j$. These functors will be defined on the level of Floer homologies analogous to Seidel's construction of the Fukaya category [Se4] in the exact (or monotone) case. With similar analytical tools as for theorem 5 we can prove that the composition $\Phi_{L_{01}} \circ \Phi_{L_{12}}$ of such A_∞-functors is isomorphic to the functor $\Phi_{L_{01} \circ L_{12}}$ for the geometric composition of the correspondences, in the case that the latter is smooth and embedded.

This shrinking of strips in pseudoholomorphic quilts gives rise to "figure eight bubbles", for which we conjecture a removal of singularity to a quilted pseudoholomorphic sphere with seams that are tangential at a marked point. In monotone cases this bubbling can be disallowed by energy considerations, but in general it will constitute an obstruction to the isomorphism of Floer homologies in theorem 5 as well as the isomorphism $\Phi_{L_{01}} \circ \Phi_{L_{12}} \cong \Phi_{L_{01} \circ L_{12}}$. In analogy to the obstruction theory [FOOO] for $\partial^2 = 0$ we expect that the isomorphisms and functors in the monotone case can be replaced by morphisms on the chain level, which then satisfy A_∞-type relations, encoding the bubbling of both disks and "figure eight bubbles". This would be the natural extension of the categorification in section 0.4 to general symplectic manifolds and Lagrangian correspondences.

3.2 A category valued TQFT in $2+1+1$ dimensions

In the project with Chris Woodward we have constructed a categorification 2-functor from a symplectic 2-category of (symplectic manifolds, generalized Lagrangian correspondences, Floer homology classes) to the 2-category of (categories, functors, natural transformations), see section 0.4. Using moduli spaces of bundles we are now working on defining a 2-functor from the 2-category of (2-manifolds, 3-cobordisms, 4-cobordisms) to the symplectic 2-category, thus defining a category valued topological quantum field theory. In particular, this will yield new invariants for 3- and 4-manifolds. By an extension of the Atiyah-Floer conjecture our constructions should be equivalent to a gauge-theoretic approach that was described by Fukaya [Fu]. The rough idea of our constructions is the following:

connect it all if possible

To a compact oriented surface Σ one can associate the moduli space $M(\Sigma)$ of semistable vector bundles of rank r and degree d with fixed determinant. (Equivalently, $M(\Sigma)$ is a space of $U(r)$-representations of the fundemental group of the punctured surface, with central holonomy around the puncture.) For r, d coprime $M(\Sigma)$ is a smooth, compact, monotone symplectic manifolds, to which we have associated a Donaldson-Fukaya category $\mathrm{Don}^\#(M(\Sigma))$.

To a compact oriented cobordism Y between surfaces $\partial Y = \Sigma_-^- \sqcup \Sigma_+$ one would ideally associate a Lagrangian correspondence $L(Y) \subset M(\Sigma_-)^- \times M(\Sigma_+)$ given by the restrictions of bundles on Y to the boundary. (These would correspond to $U(r)$-representations of $\pi_1(Y \setminus I)$ with central holonomy around a line I connecting the punctures in $\Sigma_\pm$. In general this does not define a smooth submanifold, but when Y is a simple cobordism (e.g. if it supports a Morse function with one critical point such that the boundary are level sets) then $L(Y)$ is indeed a smooth, monotone Lagrangian correspondence. We can thus use a decomposition $Y = Y_{01} \cup_{\Sigma_1} \ldots \cup_{\Sigma_{k-1}} Y_{(k-1)k}$ of the cobordism along level sets Σ_i of a Morse function that separate the critical points to define a sequence $\underline{L}(Y) = (L(Y_{01}), \ldots, L(Y_{(k-1)k}))$ of Lagrangian correspondences $L(Y_{i(i+1)}) \subset M(\Sigma_i)^- \times M(\Sigma_{i+1})$ from $M(\Sigma_0) = M(\Sigma_-)$ to $M(\Sigma_k) = M(\Sigma_+)$. This exactly is a morphism in our symplectic category, so we have associated to it the functor

$$\Phi(Y) := \Phi_{L(Y_{01})} \circ \ldots \circ \Phi_{L(Y_{(k-1)k})} : \mathrm{Don}^\#(M(\Sigma_-)) \to \mathrm{Don}^\#(M(\Sigma_+)).$$

To see that this is well defined we have to compare two decompositions of Y that are related by a sequence of Cerf moves (handle slide, cancellation, and change of order in handle attaching). Each of these can be described in terms of cobordism splitting of the form $\boxed{Y_{01} \cup Y_{12} = Y_{02}}$, where Y_{02} contains at most 2 critical points and its associated Lagrangian correspondence $\boxed{L(Y_{02}) = L(Y_{01}) \circ L(Y_{12})}$ is the embedded geometric composition of the two correspondences associated to the pieces. The functoriality of the symplectic categorification then implies $\boxed{\Phi_{L(Y_{01})} \circ \Phi_{L(Y_{12})} = \Phi_{L(Y_{02})}}$.

display crucial identities

Finally, to a 4-cobordism Z between 3-cobordisms $Y_\pm$ we associate a natural transformation $\Pi(Z) : \Phi(Y_-) \to \Phi(Y_+)$ by a similar splitting into simple cobordisms.

3.3 Floer field theory for tangles

In another part of my project with Chris Woodward we are varying the constructions in the previous section to define a categorification 2-functor from the 2-category of (oriented finite subsets of S^2, tangles in $S^2 \times [0,1]$, cobordisms of tangles in $S^2 \times [0,1]^2$). Again, we can use our general 2-functor from the symplectic 2-category to the 2-category of categories. In this case, the basic smooth, monotone symplectic manifolds are given by the moduli spaces of parabolic bundles, that is flat bundles over the punctured S^2 with fixed conjugacy classes of holonomies around the punctures for certain choices of conjugacy classes as in [MWo]. Tangles $T \subset S^2 \times [0,1]$ connect these punctures, and the corresponding moduli spaces of flat bundles over $S^2 \times [0,1] \setminus T$ will depend on T, in contrast to the line $I \subset Y$ in the previous section. Again, we use a decomposition of T into simple tangles to define a sequence $\underline{L}(T)$ of Lagrangian correspondences. To check the independence of the corresponding functor on Donaldson-Fukaya categories one then has to check that the Reidemeister moves on tangles correspond to embedded compositions of Lagrangian correspondences.

To obtain invariants for a knot (or link) K we can add trivial strands such that the moduli spaces on S^2 are points. Then one can construct the Floer homology $\mathrm{HF}(\underline{L}(K))$ of the corresponding sequence of Lagrangian correspondences and show that it is independent of the decomposition. The latter is based on the variant $\boxed{HF(\ldots, L_{01}, L_{12}, \ldots) \cong HF(\ldots, L_{01} \circ L_{12}, \ldots)}$ of theorem 5 for embedded compositions $L_{01} \circ L_{12}$. This construction is similar to one (based on a braid representation) suggested by Seidel, Smith, and Manolescu, and is conjecturally related to the invariants constructed by Seidel-Smith [SS], Khovanov [K2, K1], and Khovanov-Rozansky [KR]. In particular, we expect to obtain the analogue of Khovanov's exact triangles for resolutions of crossings from the exact triangle (7) for fibered Dehn twists on the moduli spaces of parabolic bunbles, which Seidel associated to braid moves on the punctures in S^2. Similar gauge theoretic constructions of knot invariants have been investigated by Collin-Steer [CS] and Kronheimer.

4 Synergistic Projects $\to$ broader impact

4.1 Exposition of Analytic Foundations

One of my wider goals is to further the general understanding and provide accessible expositions of the analytic foundations of gauge theory, pseudoholomorphic curves, and moduli spaces of nonlinear PDE's in general.

brag

In a first step this lead to a textbook [W1] on Uhlenbeck compactness. An extension of this exposition to bubbling effects is in progress. As a first step I gave an elementary exposition and proof of a general energy quantization principle, theorem 4. For the next step, removal of singularities, I keep collecting material. From [W3] I have an alternative, simplified proof of the removal of codimension 2 singularities [Sl] in a special case. An approach of Tom Mrowka to the gauge action in the borderline Sobolev topology, which we use in [MWe], should lead to an even more general result. In addition, the current 'J-holomorpic curves' graduate student seminar at MIT has lead to the realization that an approach of Ivashkovich and Shevchishin [IS] may provide a direct removable singularity proof for pseudoholomorphic curves in general almost complex manifolds, with only totally real boundary conditions. A compilation of these results with easily accessible proofs should be a helpful reference to researchers in the field and a guidance to those entering.

show true scholar-ship

On a larger scale, the analytic techniques used to define invariants from moduli spaces of solutions of nonlinear PDE's have become very sophisticated. Some crucial techniques can only be found in highly technical papers, adapted to special settings. This seems to pose a problem not only for graduate students entering the field and for researchers who mainly work on applying the invariants, but it also leads to the effect that many analytic papers have to redo essentially known constructions in a somewhat new setting for lack of general references. It is my hope that the work of Hofer-Wysocki-Zehnder [H, HWZ] described in section 2.1 will offer a solution for some of these problems. It provides an abstract framework that should make the constructions of compactifications, transversality, and gluing with coherent perturbations/orientations much more transparent. Its new language should also be suitable to phrase analytic results within the construction of a special moduli space in a modular manner that will make them directly applicable in other settings of only roughly similar type. Finally, [HWZ] provides several abstract constructions, e.g. of smooth structures on compactified solution spaces, transverse perturbations of sections, solution counting, and even of continuation maps in invariance proofs. Instead of adapting these constructions to a special case one should be able to just verify the assumptions of these abstract theorems. For that purpose, however, one needs to learn this radically new language, which essentially replaces large parts of differential geometry and functional analysis by concepts more suitable for the description of moduli spaces. Due to its scale and sheer volume the best way to make this theory accessible to a larger audience seems to be through lectures and workshops.

After learning the theory through a lecture course myself I have thus been giving expository lectures, organized a 'polyfold scale calculus and the cluster complex' workshop at IAS Princeton, and will now be giving a lecture course 'Polyfold-Fredholm theory and generalized Lagrangian Floer theory' starting in spring 2007 at MIT. The goal of this 2 semester course will be to give a full presentation of the abstract theory, though focussing on definitions and concepts rather than technical proofs, and to illustrate its usage on the cluster moduli spaces of holomorphic disks described in section 2.2. In the same spirit I am hoping to make the lecture notes into a brief 'users guide' for researchers who wish to apply the abstract theory. A blown-up version of the notes might grow into an exposition of Lagrangian Floer theory.

4.2 Effects of Stereotypes and Role Models on Women in Mathematics

don't just quote your-own status as under-represented – show that you care about others and get involved

While it seems apparent that the contact with role models increases the likelihood of talented women actually pursueing mathematics, recent results on the effect of stereotypes [DH] suggest an even stronger reason to make women in mathematics more visible: It was shown that the confrontation with negative stereotypes actually has a direct negative influence on the mathematical performance. From my own every day experience I can attest that the existence of female mathematicians still comes as a surprise to most people, with no exception for scientists and even women. Most of us theoretically know all about gender equality but our unconscious judgements still follow the stereotypes of not expecting women in mathematics and undervalueing their scientific achievements while focussing more on social skills.

In the last few years I have mostly concentrated my efforts on furthering the networking of women in mathematics. In my experience this seems to come more natural in private rather than institutionalized settings, and with the increasing number of (at least junior) women it only seems to need a little activation energy. The visibility of female role models and overcoming the negative stereotypes, however, seems to still require more decisive action. I hope that I am now in a position where I can make contributions in that direction. A first large scale idea is to organize a conference, celebrating the achievements of women in science. The main content should be a series of public lectures by highly distinguished scientists, not specifically announced as gender equality events, but with speakers who happen to be women. Gigliola Staffilani and myself are currently soliciting advice and searching for possible co-organizers from other departments of MIT.

On a smaller scale, I am planning to adapt the 'mathematics open day for schoolgirls' which I organized at ETH Zürich for several years. This is mainly an orientation event, giving an idea of the mathematics beyond calculus, providing information on the structure of a major, and showcasing research in pure and applied mathematics and career opportunities for mathematics majors. In the educational setting of the U.S. it seems more appropriate to target this program at undergraduates around the time of their choice of major. After a test run, however, I am hoping to also invite girls from schools in the Boston area, adding a mentoring effect by them meeting students from MIT. The organizational meetings can also serve as natural networking opportunity for female graduate students, postdocs, and faculty. The speakers should include some distinguished male mathematicians, but only women would be explicitly invited to participate – with the pitch that they get a special invitation since we scientifically know that many of them are highly talented for mathematics although they may not believe in it. So maybe one day our numbers of majors will prove the stereotypes wrong!

sooo much easier to remember/guess while reding

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<u>all</u> your papers / preprints

Biographical Sketch: Katrin Wehrheim

(a) Professional Preparation:

Universität Hamburg (Germany)	Mathematics	Vordiplom 1995
Universität Hamburg (Germany)	Physics	Vordiplom 1995
Imperial College (UK)	Physics	Diploma 1996
ETH Zürich (Switzerland)	Mathematics	Diploma with distinction 1998
ETH Zürich (Switzerland)	Mathematics	Doctor of Mathematics 2002
ETH Zürich (Switzerland)	Global Analysis	Postdoc 2002–2003

(b) Appointments:

Massachusetts Institute of Technology	Assistant Professor	2005–present
Institute for Advanced Study	Member	2004–2006
Princeton University	Visiting Fellow	2005–2006
Princeton University	Instructor	2003–2004
ETH Zürich (Switzerland)	Postdoctoral Fellow	2002–2003
ETH Zürich (Switzerland)	Research Assistant	2001-2002
ETH Zürich (Switzerland)	Teaching Assistant	1996-2001
Universität Hamburg (Germany)	MATLAB-Programmer	1995
German Electron Synchrotron (DESY)	Laboratory Assistant	1993/1994

(c) Publications:

1. T.S.Mrowka, K.Wehrheim, *L^2-topology and Lagrangians in the space of connections over a Riemann surface*, in preparation.
2. K.Wehrheim, C.T.Woodward, *Functoriality for Lagrangian correspondences in Floer theory*, http://www-math.mit.edu/~katrin, 1–107.
3. D.A.Salamon, K.Wehrheim, *Instanton Floer homology with Lagrangian boundary conditions*, math.SG/0607318, 1–141.
4. K.Wehrheim, *Energy identity for anti-self-dual instantons on $\mathbb{C} \times \Sigma$*, Math. Res. Lett. 13 (2006), no. 1, 161–166.
5. K.Wehrheim, *Lagrangian boundary conditions for anti-self-dual instantons and the Atiyah-Floer conjecture*, J. Symplectic Geom. 3 (2005), no. 4, 703–747.
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9. K.Wehrheim, *Banach space valued Cauchy-Riemann equations with totally real boundary conditions*, Comm. Contemp. Math. 6 (2004), no. 4, 601–635.

10. K.Wehrheim, *Uhlenbeck Compactness*, EMS Series of Lectures in Mathematics, 2004.

(d) Synergistic Activities:

1. Organizational involvement in 'Program for Women in Mathematics' at IAS Princeton and 'Emmy Noether Ring' at Princeton university, departmental counsellor for female students and organiser of annual Mathematics Open Day for schoolgirls at ETH Zürich (1998 – 2003).
2. Member (and at times vice chair) of departmental teaching committee (working on curriculum and degree requirements) at ETH Zürich (1998 – 2003).
3. Organiser of 'Polyfold Scale Calculus and the Cluster Complex' Workshop at IAS Princeton (October 2005), bringing together experts in two new fields to start a project of combining the two approaches, including two minicourses for approx. 20 graduate students and postdocs.
4. Expository textbook on Uhlenbeck compactness.
5. Referee for Mathematische Zeitschrift, Journal of Symplectic Geometry, Geometrical and Functional Analysis, Topology, Advances in Mathematics, Communications in Mathematical Physics, Journal of Geometry and Physics, Inventiones, Journal of the AMS, Annals of Mathematics, International Mathematics Research Notices.

(e) Collaborators and Other Affiliations:

1. Collaborators and Co-Editors: Peter Albers (NYU), Sikimeti Mau (Rutgers), Tom Mrowka (MIT), Dietmar Salamon (ETH Zürich), Chris Woodward (Rutgers)
2. Graduate and Postdoctoral Advisors: Dietmar Salamon (ETH Zürich), Eduard Zehnder (ETH Zürich)
3. Thesis Advisor and Postgraduate-Scholar Sponsor to: none as of 10/2006