Energy identity for anti-self-dual instantons on $\mathbb{C} \times \Sigma$

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April 16, 2008

Abstract

We establish an energy identity for anti-self-dual connections on the product $\mathbb{C} \times \Sigma$ of the complex plane and a Riemann surface. The energy is a multiple of a basic constant that is determined from the values of a corresponding Chern-Simons functional on flat connections and its ambiguity under gauge transformations. For SU(2)-bundles this identity supports the conjecture that the finite energy anti-self-dual instantons correspond to holomorphic bundles over $\mathbb{CP}^1 \times \Sigma$.

Such anti-self-dual instantons on SU(n)- and SO(3)-bundles arise in particular as bubbles in adiabatic limits occurring in the context of mirror symmetry and the Atiyah-Floer conjecture. Our identity proves a quantization of the energy of these bubbles that simplifies and strengthens the involved analysis considerably.

NOTE: The connection need not be anti-self dual. In general, if the energy is finite, then the charge $\frac{1}{8\pi^2}\int\langle F_\Xi \wedge F_\Xi \rangle$ is an integer.

1 Introduction

Let Σ be a Riemann surface and consider the trivial SU(2)-bundle over $\mathbb{C} \times \Sigma$. A connection $\Xi \in \mathcal{A}(\mathbb{C} \times \Sigma)$ on this bundle is a 1-form $\Xi \in \Omega^1(\mathbb{C} \times \Sigma; \mathfrak{su}(2))$ with values in the Lie algebra $\mathfrak{su}(2)$. Gauge transformations $u \in \mathcal{G}(\mathbb{C} \times \Sigma)$ of the bundle are represented by maps $u \in \mathrm{Map}(\mathbb{C} \times \Sigma, \mathrm{SU}(2))$ and act on $\mathcal{A}(\mathbb{C} \times \Sigma)$ by $u^*\Xi = u^{-1}\Xi u + u^{-1}\mathrm{d}u$. We equip $\mathbb{C} \times \Sigma$ with a product metric of the Euclidean metric on $\mathbb{C}$ and a fixed metric on Σ. Then a connection $\Xi \in \mathcal{A}(\mathbb{C} \times \Sigma)$ is called an **ASD instanton** if its curvature is anti-self-dual,

$$F_\Xi + *F_\Xi = 0,$$

where $*$ is the Hodge operator w.r.t. the metric on $\mathbb{C} \times \Sigma$. The curvature 2-form $F_\Xi = \mathrm{d}\Xi + \Xi \wedge \Xi$ transforms under gauge transformations $u \in \mathcal{G}(\mathbb{C} \times \Sigma)$ as $F_{u^*\Xi} = u^{-1}F_\Xi u$, hence the anti-self-duality equation is gauge invariant. Next, we equip $\mathfrak{su}(2)$ with the SU(2)-invariant inner product $\langle \xi, \eta \rangle = -\mathrm{tr}(\xi\eta)$. Then the energy of a connection $\Xi \in \mathcal{A}(\mathbb{C} \times \Sigma)$ is the gauge invariant quantity

$$\mathcal{E}(\Xi) := \tfrac{1}{2} \int_{\mathbb{C}\times\Sigma} |F_\Xi|^2.$$

The main purpose of this note is to establish the following energy identity. Its surprisingly simple proof is given in section 2. For the sake of simplicity we first focus our attention to SU(2)-bundles. Later, we will also indicate how to generalize this result to other structure groups and nontrivial bundles over Σ.

Theorem 1.1 *Let $\Xi \in \mathcal{A}(\mathbb{C} \times \Sigma)$ be an ASD instanton. If it has finite energy $\mathcal{E}(\Xi) < \infty$, then actually $\mathcal{E}(\Xi) \in 4\pi^2\mathbb{N}_0$.*

This energy quantization supports a conjectural correspondence between finite energy ASD instantons on $\mathbb{C} \times \Sigma$ and holomorphic bundles over $\mathbb{CP}^1 \times \Sigma$. For $\Sigma = \mathbb{T}^2$ Biquard and Jardim [1] showed that the gauge equivalence classes of ASD instantons with quadratic curvature decay are in one-to-one correspondence to a class of rank 2 stable holomorphic bundles over $\mathbb{CP}^1 \times \mathbb{T}^2$. Here the holomorphic structure induced by an instanton Ξ extends over $\{\infty\} \times \mathbb{T}^2$ to define a bundle E, whose second Chern number is given by the instanton energy, $c_2(E) = \frac{1}{8\pi^2}\int \langle F_\Xi \wedge F_\Xi \rangle$, see [4, §2.3]. By our result this formula continues to give integer (Chern ?) numbers for finite energy instantons and any surface Σ.

Remark 1.2 *Theorem 1.1 extends to ASD instantons on $\mathbb{C} \times P$ for any principal bundle $P \to \Sigma$ with compact structure group* G *as follows:*

Suppose that the Lie algebra $\mathfrak{g}$ is equipped with a G*-invariant metric that satisfies* (H) *below. Then the statement of theorem 1.1 holds with $4\pi^2$ replaced by the constant $\kappa_\mathfrak{g} N_\mathrm{G}^{-1}$ given below.*

On a nontrivial bundle P the gauge transformations are represented by sections in the associated bundle $\mathrm{G}_P = P \times_c \mathrm{G}$ (using the conjugation action on G). We can pick a G-invariant inner product on $\mathfrak{g}$ (and thus on $\mathfrak{g}_P = P \times_{\mathrm{Ad}} \mathfrak{g}$). Then the Maurer-Cartan 3-form on each fibre of G_P induces a closed 3-form $\eta_\mathrm{G} := \frac{1}{12}\langle g^{-1}\mathrm{d}g \wedge [g^{-1}\mathrm{d}g \wedge g^{-1}\mathrm{d}g] \rangle$ on G_P. We need the following assumption.

(H): There exists $\kappa_\mathfrak{g} > 0$ such that $[\kappa_\mathfrak{g}\eta_\mathrm{G}] \in \mathrm{H}^3(\mathrm{G}_P, \mathbb{R})$ is an integral class.

This holds for example with $\kappa_{\mathfrak{so}(3)} = 4\pi^2$ for any SO(3)-bundle when we choose the inner product $-2\mathrm{tr}(\xi\eta)$ for $\xi, \eta \in \mathfrak{so}(3)$. It can also be achieved for any simply connected compact Lie group G,[1] e.g. for the trivial SU(n)-bundles. Finally, N_G is the least common multiple of $\{1, 2, \ldots, n_\mathrm{G}\}$, where n_G denotes the maximal number of connected components that the centralizer of a subgroup in G can have. This is finite since G is compact. For SO(3) we have $N_{\mathrm{SO}(3)} = 1$.

[1] In that case the bundle is automatically trivial and the Lie group G is isomorphic to a product $S_1 \times \ldots \times S_k$ of simply connected, simple, and compact Lie groups S_j with $\pi_3(S_j) \cong \mathbb{Z}$. So we can pick a metric on each factor S_j for which $[\eta_{S_j}] \in \mathrm{H}^3(S_j, \mathbb{R})$ is integral.

One source of interest in the ASD instantons on $\mathbb{C}\times\Sigma$ is the following adiabatic limit. Let $\Sigma \hookrightarrow X \to M$ be a fibre bundle with $\dim X = 4$. Consider ASD instantons Ξ_ε over X with respect to metrics $g_M + \varepsilon^2 g_\Sigma$ for a sequence $\varepsilon \to 0$. If $|F_{\Xi_\varepsilon}|_{\mathrm{fibre}}| + \varepsilon^2 |F_{\Xi_\varepsilon}|_{\mathrm{mix}}|$ converges to a nonzero value, then local rescaling on M (but not in the fibre) yields an ASD instanton on $\mathbb{C}\times\Sigma$ in the limit. This bubbling phenomenon is a central difficulty of the limiting process. Adiabatic limits of this type have fascinating consequences from topology to mathematical physics. They were first considered by Dostoglou-Salamon [3], and recently by Chen [2] and Nishinou [5]. The energy quantization presented here simplifies and strengthens the bubbling analysis and results in all these cases. It can also be used for the Atiyah-Floer conjecture project [6, 9].

2 Proof of the energy identity

In the following, $S_r \subset \mathbb{C}$ denotes the circle of radius r centered at 0. We moreover denote by $D_r \subset \mathbb{C}$ the disk of radius r, and we introduce polar coordinates $(r,\phi) \in (0,\infty)\times S^1$ on $\mathbb{C}^* = \mathbb{C}\setminus\{0\}$, with $S^1 = \mathbb{R}/2\pi\mathbb{Z}$. Then on $\mathbb{C}^*\times\Sigma$ we can write a connection $\Xi \in \mathcal{A}(\mathbb{C}\times\Sigma)$ in the splitting

$$\Xi = A(r) + R(r)\mathrm{d}r + \Phi(r)\mathrm{d}\phi$$

with $A(r)\colon S^1 \to \mathcal{A}(\Sigma)$ and $R(r),\Phi(r)\colon S^1 \to \Omega^0(\Sigma,\mathfrak{su}(2))$ for all $r \in (0,\infty)$. The anti-self-duality equation becomes in this splitting

$$\begin{cases} r^{-1}\big(\partial_r\Phi - \partial_\phi R + [\Phi,R]\big) + *F_A = 0, \\ r^{-1}\big(\partial_\phi A - \mathrm{d}_A\Phi\big) - *\big(\partial_r A - \mathrm{d}_A R\big) = 0. \end{cases}$$

By $F_\Xi(r)$ we denote the curvature of $\Xi \in \mathcal{A}(\mathbb{C}\times\Sigma)$ over $S_r\times\Sigma$ (but as a 2-form on $\mathbb{C}\times\Sigma$). Then the curvature of an ASD instanton is

$$\tfrac{1}{2}|F_\Xi(r)|^2 = |F_{A(r)}|^2 + r^{-2}\big|\partial_\phi A(r) - \mathrm{d}_{A(r)}\Phi(r)\big|^2.$$

The energy of an ASD instanton on $D_r\times\Sigma$ can be expressed in terms of the Chern-Simons functional of $B(r) := A(r) + \Phi(r)\mathrm{d}\phi \in \mathcal{A}(S^1\times\Sigma)$,

$$\tfrac{1}{2}\int_{D_r\times\Sigma} |F_\Xi|^2 \;=\; -\tfrac{1}{2}\int_{D_r\times\Sigma} \langle\, F_\Xi \wedge F_\Xi \,\rangle \;=\; -\mathcal{CS}(B(r)).$$

The Chern-Simons functional on connections $B = A + \Phi\mathrm{d}\phi \in \mathcal{A}(S^1\times P)$ is

$$\begin{aligned} \mathcal{CS}(B) &= \tfrac{1}{2}\int_{S^1\times\Sigma} \langle\, B \wedge \big(F_B - \tfrac{1}{6}[B\wedge B]\big)\,\rangle \\ &= \int_{S^1}\int_\Sigma \tfrac{1}{2}\langle\, \partial_\varphi A \wedge A \,\rangle + \langle\, F_A\,,\,\Phi\,\rangle. \end{aligned} \qquad (1)$$

For future reference we note the following identity which shows that the Chern-Simons functional is continuous with respect to the $W^{1,\frac{3}{2}}$-norm. (Note that $W^{1,\frac{3}{2}} \hookrightarrow L^3$ on a 3-manifold.) For all $B, B^0 \in \mathcal{A}(S^1 \times \Sigma)$

$$\begin{aligned}\mathcal{CS}(B) - \mathcal{CS}(B^0) = \int \tfrac{1}{2}\langle\, (F_B + F_{B^0}) \wedge (B - B^0)\,\rangle \\ - \tfrac{1}{12}\int \langle\, [(B - B^0) \wedge (B - B^0)] \wedge (B - B^0)\,\rangle.\end{aligned} \tag{2}$$

The Chern-Simons functional is not gauge invariant, but its ambiguity on gauge orbits is determined by the degree of the gauge transformations (as maps to $\mathrm{SU}(2) \cong S^3$): For all $B \in \mathcal{A}(S^1 \times \Sigma)$ and $u \in \mathcal{G}(S^1 \times \Sigma)$

$$\mathcal{CS}(B) - \mathcal{CS}(u^*B) = 4\pi^2 \deg(u) \in 4\pi^2\,\mathbb{Z} \tag{3}$$

For a general (possibly nontrivial) bundle $P \to \Sigma$ one has to fix a flat reference connection. Then connections are given by 1-forms with values in $\mathfrak{g}_P$ and the Chern-Simons functional depends on the choice of this reference connection only up to an additive constant. (The proof of theorem 1.1 will show that a flat connection exists.) The right hand side of (2) is then given by $\int u^*\eta_{\mathrm{G}}$. So under the assumption (H) we have $\mathcal{CS}(B) - \mathcal{CS}(u^*B) \in \kappa_{\mathfrak{g}}\,\mathbb{Z}$.

The second point that affects the constant in the energy identity is the possible values of the Chern-Simons functional on flat connections. The following result holds for SU(2)- and SO(3)-bundles, and we will give the argument for a general bundle $P \to \Sigma$, indicating how to proceed for other structure groups.

Lemma 2.1 *For every flat connection $B \in \mathcal{A}_{\mathrm{flat}}(S^1 \times \Sigma)$ there is a gauge transformation $u \in \mathcal{G}(S^1 \times \Sigma)$ such that $\mathcal{CS}(u^*B) = 0$, and consequentially $\mathcal{CS}(B) = 4\pi^2 \deg(u) \in 4\pi^2\,\mathbb{Z}$.*

Proof: Any flat connection B on $S^1 \times \Sigma$ corresponds to a holonomy representation $\rho : \pi_1(\Sigma) \to \mathrm{G}$ and an element $g \in S_{\mathrm{im}\,\rho} \subset \mathrm{G}$; the holonomy around S^1 which lies in the centralizer of $\mathrm{im}\,\rho$. We will use parallel transport along S^1 and a homotopy $g^n \sim \mathbb{1} \subset S_{\mathrm{im}\,\rho}$ (for some $n \leq n_{\mathrm{G}}$) to bring B into the form $A + \Phi \mathrm{d}\phi$ with S^1-independent A, for which the Chern–Simons functional trivially vanishes. Then $n\mathcal{CS}(B) \in \kappa_{\mathfrak{g}}\mathbb{Z}$ by (3).

More precisely, we periodically extend B to to a connection in $\mathcal{A}_{\mathrm{flat}}(\mathbb{R} \times P)$. Then there is a gauge transformation $u\colon \mathbb{R} \to \mathcal{G}(P)$ such that $u(0) \equiv \mathbb{1}$ and $u^*B \in \mathcal{A}_{\mathrm{flat}}(\mathbb{R} \times P)$ has no $\mathrm{d}\phi$-component. Thus the curvature component $\partial_\phi(u^*B)$ vanishes, and hence $u^*B \equiv A^0 \in \mathcal{A}_{\mathrm{flat}}(P)$. The gauge transformation is found by parallel transport, i.e. solving $\partial_\phi u = -\Phi u$. So due to the periodicity of Φ we obtain the twisted periodicity $u(\phi + 2\pi) = u(\phi)u(2\pi)$ for the gauge transformation. Unless $u(2\pi) \equiv \mathbb{1}$ this does not define a gauge transformation on $S^1 \times P$. However, we know that $u(2\pi)$ lies in the isotropy subgroup $\mathcal{G}_{A^0}$, since $u(2\pi)^*A^0 = u(2\pi)^*B(2\pi, \cdot)|_\Sigma = u(0)^*B(0, \cdot)|_\Sigma = A^0$. If $\mathcal{G}_{A^0}$ is connected, then we can multiply u with a path within $\mathcal{G}_{A^0}$ from $\mathbb{1}$ to $u(2\pi)^{-1}$ to obtain the required gauge transformation $w \in \mathcal{G}(S^1 \times P)$. It satisfies $w^*B = A^0 + \Phi^0 \mathrm{d}\phi$ with

$\partial_\phi A^0 = 0$ but possibly nonzero Φ^0. Now compare (1) to see that $\mathcal{CS}(w^*B) = 0$, and so $\mathcal{CS}(B) = 4\pi^2 \deg(w)$ by (3).

For SO(3)-bundles, any isotropy subgroup is connected since any centralizer (of the holonomy subgroup) in SO(3) is connected. Thus the proof is finished. For a general Lie group whose centralizers have up to $n_{\rm G}$ components, one finds that $u(2\pi)^n$ is homotopic to the identity for some integer $n \leq n_{\rm G}$. Then an "n-fold cover" $B^{(n)}$ of B can be put into a gauge whose Chern-Simons functional vanishes, and thus $\mathcal{CS}(B) = n^{-1}\mathcal{CS}(B^{(n)}) \in \kappa_{\mathfrak{g}} n^{-1}\,\mathbb{Z} \subset \kappa_{\mathfrak{g}} N_{\rm G}^{-1}\,\mathbb{Z}$ if (H) holds.

For SU(2) we would have $n_{\rm G} = 2$ due to the centralizer $\{\mathbb{1}, -\mathbb{1}\}$. However, since the isotropy element $u(2\pi) = -\mathbb{1}$ is a constant, we do not need to go to a cover. More generally suppose that $u(2\pi) = \exp(2\pi\xi)$ for some constant $\xi \in \mathfrak{g}$. Let $v(\phi) := \exp(-\phi\xi)$, then $w := uv \in \mathcal{G}(S^1 \times \Sigma)$ and $w^*B = v^{-1}A^0 v - \xi \mathrm{d}\phi$ (and both are of class $W^{1,\infty}$). Then using $F_{A^0} = 0$ and $\mathrm{d}\xi = 0$ we obtain

$$\begin{aligned}\mathcal{CS}(w^*B) &= \int_{S^1}\int_\Sigma \tfrac{1}{2}\langle\, v^{-1}[\xi, A^0]v \wedge v^{-1}A^0 v\,\rangle \\ &= \int_{S^1}\int_\Sigma \langle\, \xi, A^0 \wedge A^0\,\rangle \;=\; -\int_{S^1}\int_\Sigma \langle\, \xi, \mathrm{d}A^0\,\rangle \;=\; 0.\end{aligned}$$

□

In the subsequent proof of the energy identity we work with a general bundle $P \to \Sigma$ and only for the final conclusion use the knowledge from lemma 2.1 on the possible values of the Chern-Simons functional on flat connections.

Proof of theorem 1.1: Let $B(r) \in \mathcal{A}(S^1 \times \Sigma)$ be given by Ξ on $S_r \times \Sigma$, then $|F_{B(r)}|^2 = |F_{A(r)}|^2 + |\partial_\varphi A(r) - \mathrm{d}_{A(r)}\Phi(r)|^2 \leq \frac{1}{2}r^2|F_\Xi(r)|^2$ for $r \geq 1$, and hence

$$\int_1^\infty r^{-1}\|F_{B(r)}\|^2_{L^2(S^1\times\Sigma)}\mathrm{d}r \;\leq\; \mathcal{E}(\Xi) < \infty.$$

Thus we find a sequence $r_i \to \infty$ with $\|F_{B(r_i)}\|_{L^2(S^1\times\Sigma)} \to 0$. By Uhlenbeck's weak compactness [7] we then find a further subsequence, gauge transformations $u_i \in \mathcal{G}(S^1 \times P)$, and a flat limit connection $B_\infty \in \mathcal{A}_{\rm flat}(S^1 \times P)$ such that

$$\|u_i^*B(r_i) - B_\infty\|_{W^{1,2}(S^1\times\Sigma)} \to 0. \tag{4}$$

More precisely, the Uhlenbeck compactness theorem (also see [8, Theorem A] with $p = 2 \geq \frac{1}{2}\dim(S^1 \times \Sigma)$) provides a subsequence $B_i := B(r_i)$ that converges in the weak $W^{1,2}$-topology to a flat (and hence smooth) connection $B_\infty \in \mathcal{A}_{\rm flat}(S^1 \times P)$. Since the Sobolev embedding $W^{1,2} \hookrightarrow L^4$ is compact (in dimension 3) we can moreover assume that $B_i \to B_\infty$ in the L^4-norm. So by the local slice theorem (e.g. [8, Theorem 8.1]) one can – for a further subsequence and sequence of gauge transformations – achieve the additional relative Coulomb gauge condition

$$\mathrm{d}^*_{B_\infty}(u_i^*B_i - B_\infty) = 0.$$

Moreover, we have

$$\mathrm{d}_{B_\infty}(u_i^*B_i - B_\infty) = u_i^{-1}F_{B_i}u_i - \tfrac{1}{2}[(u_i^*B_i - B_\infty)\wedge(u_i^*B_i - B_\infty)].$$

So from the regularity of the Hodge decomposition of 1-forms (e.g. [8, Theorem 5.1]) one obtains the convergence (4) in the $W^{1,2}$-norm.

Now we have $\mathcal{CS}(u_i^* B(r_i)) \to \mathcal{CS}(B^\infty)$ due to the convergence of $u_i^* B(r_i)$ and (2). On the other hand the energy is finite, so

$$\mathcal{E}(\Xi) \;=\; -\lim_{i\to\infty} \tfrac{1}{2} \int_{D_{r_i}\times\Sigma} \langle\, F_\Xi \wedge F_\Xi \,\rangle \;=\; -\lim_{i\to\infty} \mathcal{CS}(B(r_i)).$$

This shows that $\mathcal{CS}(B(r_i))$ also converges. Now for an SU(2)-bundle we have $\mathcal{CS}(B^\infty) \in 4\pi^2\,\mathbb{Z}$ from lemma 2.1. Thus $\mathcal{CS}(B(r_i)) = \mathcal{CS}(u_i^* B(r_i)) + 4\pi^2 \deg(u_i)$ must converge to some value in $4\pi^2\,\mathbb{Z}$. This proves the claim since that limit is also the energy $\mathcal{E}(\Xi)$.

For a general bundle under the assumption (H) we know that $\mathcal{CS}(u_i^* B(r_i))$ converges to a value in $\kappa_{\mathfrak{g}} N_{\mathrm{G}}^{-1}\,\mathbb{Z}$. Since $\mathcal{CS}(B(r_i)) - \mathcal{CS}(u_i^* B(r_i)) \in \kappa_{\mathfrak{g}}\,\mathbb{Z}$ we must have $\mathcal{E}(\Xi) = \lim \mathcal{CS}(B(r_i)) \in \kappa_{\mathfrak{g}} N_{\mathrm{G}}^{-1}\,\mathbb{Z} + \kappa_{\mathfrak{g}}\,\mathbb{Z} = \kappa_{\mathfrak{g}} N_{\mathrm{G}}^{-1}\,\mathbb{Z}$, which proves remark 1.2. □

Acknowledgments

I would like to thank Benoit Charbonneau, Kenji Fukaya, Marcos Jardim, Tom Mrowka, and Dietmar Salamon for valuable discussions. Support by the Swiss National Science Foundation and the US National Science Foundation is gratefully acknowledged.

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