FUNCTORIALITY FOR LAGRANGIAN CORRESPONDENCES IN FLOER THEORY

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Abstract. We generalize Lagrangian Floer theory to sequences of Lagrangian correspondences and establish an isomorphism between the Floer homology groups of sequences that are related by the geometric composition of Lagrangian correspondences. On these Floer homologies, we define relative invariants arising from "quilted pseudoholomorphic surfaces": Collections of pseudoholomorphic maps to various target spaces with "seam conditions" in Lagrangian correspondences and boundary conditions in Lagrangian submanifolds. Using these new invariants, we define a composition functor for categories of Lagrangian correspondences in monotone and exact symplectic Floer theory. We show that this functor agrees with geometric composition in the case that the composition is smooth and embedded. As a consequence we obtain "categorification commutes with composition" for Lagrangian correspondences.

Contents

1. Introduction

In this paper we study composition of Lagrangian correspondences in monotone and exact Lagrangian Floer theory. Following Donaldson and Fukaya, one associates to a compact monotone (or noncompact exact) symplectic manifold (M, ω) a category $\mathrm{Don}(M)$ whose objects are certain compact, oriented, relatively spin, monotone (or exact) Lagrangian submanifolds of (M, ω) (which we call *admissible*, see Section 6.1) and whose morphisms are Floer cohomology classes. We use a variation of the usual definition, which we denote $\mathrm{Don}^{\#}(M)$. Given two symplectic manifolds M_0 and M_1 of the same monotonicity type, an admissible Lagrangian correspondence $L_{01} \subset M_0^- \times M_1$ gives rise to a functor

$$\Phi(L_{01}) : \mathrm{Don}^{\#}(M_0) \to \mathrm{Don}^{\#}(M_1).$$

Given a triple M_0, M_1, M_2 of symplectic manifolds and admissible Lagrangian correspondences $L_{01} \subset M_0^- \times M_1$ and $L_{12} \subset M_1^- \times M_2$, the algebraic composition $\Phi(L_{01}) \circ \Phi(L_{12}) : \mathrm{Don}^{\#}(M_0) \to \mathrm{Don}^{\#}(M_2)$ is always defined. On the other hand, one may consider the geometric composition $L_{01} \circ L_{12}$ that was introduced by Weinstein [49, 48]. Under suitable transversality hypotheses, the restriction of the projection $\pi_{02} : M_0^- \times M_1 \times M_1^- \times M_2 \to M_0^- \times M_2$ to

$$L_{01} \times_{M_1} L_{12} := \big(L_{01} \times L_{12}\big) \cap \big(M_0^- \times \Delta_{M_1} \times M_2\big)$$

is an immersion, whose singular Lagrangian image we denote by

$$L_{01} \circ L_{12} \subset M_0^- \times M_2.$$

Our main result is that if $L_{01} \times_{M_1} L_{12}$ is a transverse (hence smooth) intersection and embeds by π_{02} into $M_0^- \times M_2$ then

$$\Phi(L_{01}) \circ \Phi(L_{12}) \cong \Phi(L_{01} \circ L_{12}). \tag{1}$$

In other words, "categorification commutes with composition". If M_1 is not spin, there is also a shift of relative spin structures on the right-hand side. The starting point for this functoriality is an elementary construction of a symplectic category consisting of symplectic manifolds and certain *sequences of Lagrangian correspondences*, explained in Section 2.

There is a slightly stronger version of this result, expressed in the language of 2-categories as follows. Let $\mathrm{Floer}^{\#}$ denote the *Weinstein-Floer* 2-category whose objects are symplectic manifolds, 1-morphisms are sequences of Lagrangian correspondences, and 2-morphisms are Floer cohomology classes; we denote composition of 1-morphisms in this category by $\#$. The maps above extend to a *categorification 2-functor* from $\mathrm{Floer}^{\#}$ to the 2-category of categories Cat. A refinement of the main result says that the concatenation $L_{01} \# L_{12}$ is

2-isomorphic to the geometric composition $L_{01} \circ L_{12}$ as 1-morphisms in Floer$^{\#}$; the formula (1) follows by combining this result with the 2-functor axiom for 1-morphisms.

The functors Φ as well as the composition functor and the natural transformations in this categorification are defined by new Floer type invariants arising from quilted pseudo-holomorphic surfaces. These quilts consist of pseudoholomorphic surfaces (with boundary and strip-like ends in various target spaces) which satisfy seam conditions (mapping certain pairs of boundary components to Lagrangian correspondences) and boundary conditions (mapping other boundary components to simple Lagrangian submanifolds). Similar moduli spaces have been considered by Khovanov and Rozansky [19] under the name of *pseudo-holomorphic foams.*

An isomorphism of Floer cohomologies: A more down-to-earth but weaker version of our result is the following. (For the precise monotonicity and admissibility conditions see Section 5.)

Theorem 1.0.1. *Let M_0, M_1, M_2 be either a triple of exact symplectic manifolds or a triple of compact, monotone symplectic manifolds with the same monotonicity constant, and let*

$$L_0 \subset M_0, \quad L_{01} \subset M_0^- \times M_1, \quad L_{12} \subset M_1^- \times M_2, \quad L_2 \subset M_2^-$$

be compact, monotone, and admissible Lagrangian submanifolds. If $L_{01} \times_{M_1} L_{12}$ is smooth, embeds by π_{02} into $M_0^- \times M_2$, and the Lagrangian image $L_{01} \circ L_{12}$ is monotone and admissible, then there exists a canonical isomorphism

$$HF(L_0 \times L_{12}, L_{01} \times L_2) \xrightarrow{\sim} HF(L_0 \times L_2, L_{01} \circ L_{12}). \tag{2}$$

The injectivity assumption on $\pi_{02}|_{L_{12} \times_{M_1} L_{01}}$ ensures a bijection between the intersections of the Lagrangians $(L_0 \times L_{12}) \cap (L_{01} \times L_2) \cong (L_0 \times L_2) \cap (L_{01} \circ L_{12})$. If these intersections are transverse, then the isomorphism (2) is induced by the identity on the generators of the Floer complex. The Floer differential for $(L_0 \times L_{12}, L_{01} \times L_2)$ counts triples of holomorphic strips in M_0, M_1^-, M_2 (see Figure 1 below). In the standard definition, one would take the width of all three strips to be equal, but in fact one can allow the widths of the strips to differ. (These domains are not conformally equivalent due to the identification between boundary components.) The main difficulty then is to prove that under the stated assumptions and with the width of the middle strip sufficiently close to zero, the triples of holomorphic strips in M_0, M_1^-, M_2 are in one-to-one correspondence with the pairs of holomorphic strips in M_0, M_2 that are counted in the Floer differential for $(L_0 \times L_2, L_{01} \circ L_{12})$.

As in similar situations in Floer theory, the proof is an application of the implicit function theorem, on one hand, and compactness results for certain J-holomorphic strips, on the other. In the limit various kinds of bubbling occur, including a particular “figure eight” bubble that does not appear in the standard theory and must be disallowed by energy quantization and the energy-index relation derived from the monotonicity or exactness assumption. In this paper, only the version with $\mathbb{Z}_2$-coefficients is completely proved; to reduce the length, we banished the discussion of coherent orientations to a separate paper [46]. There should also be versions of this result for Floer cohomology with gradings, coefficients in flat vector bundles, and Novikov rings. We give a detailed proof for the gradings but not for the other versions.

Topological Applications: A consequence of our results is a general prescription for defining topological invariants by decomposing into simple pieces. For example, let Y be a compact manifold and $f : Y \to \mathbb{R}$ a Morse function giving a decomposition $Y =$

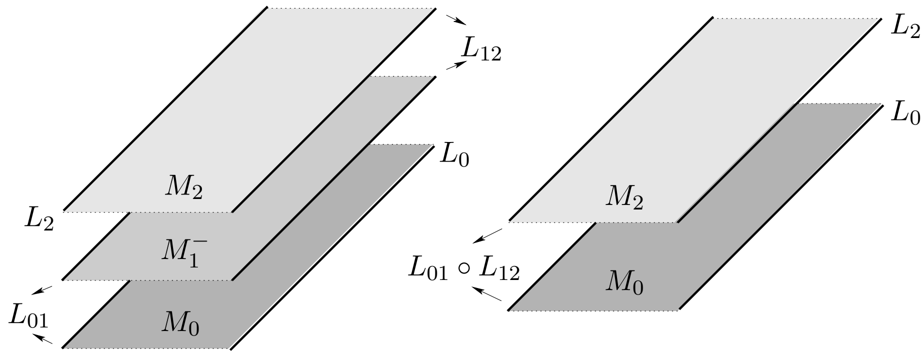

FIGURE 1. Tuples of holomorphic strips that are counted for $HF(L_0 \times L_{12}, L_{01} \times L_2)$ and for $HF(L_0 \times L_2, L_{01} \circ L_{12})$

$Y_{01} \cup \ldots \cup Y_{(l-1)l}$ into simple cobordisms by cutting along non-critical level sets $X_1, \ldots, X_{l-1}$. First one associates to each X_j a monotone symplectic manifold $M(X_j)$, and to each $Y_{(j-1)j}$ with $\partial Y_{(j-1)j} = X_{j-1}^- \sqcup X_j$ a smooth monotone Lagrangian correspondence $L(Y_{(j-1)j}) \subset M(X_{j-1})^- \times M(X_j)$ (taking $M(X_0)$ and $M(X_l)$ to be points.) Second, one checks that the basic moves described by Cerf theory (critical point cancellation, order-of-attaching change, or handle slides) change the sequence of Lagrangian correspondences by replacing adjacent correspondences with an embedded composition, or vice-versa. In other words, the equivalence class of sequences of Lagrangian correspondences by embedded compositions $[L(Y_{01}), \ldots, L(Y_{(l-1)l})]$ does not depend on the choice of the Morse function f. Then the results of this paper provide a group-valued invariant of Y, by taking the Floer homology of the sequence of Lagrangian correspondences.

More categorically speaking, a consequence of our result is that the map assigning to any symplectic or monotone symplectic manifold its Donaldson-Fukaya category extends to a functor from $\mathrm{Symp}^{\#}$, the category of (monotone symplectic manifolds, equivalence classes of sequences of Lagrangian correspondences), to Cat, the category of (categories, isomorphism classes of functors). Consider the category of compact oriented d-dimensional manifolds and equivalence classes of $d+1$-dimensional compact oriented cobordisms. For any other category $\mathcal{C}$ we say following G. Segal that a *(weak) $\mathcal{C}$-valued $d+1$-dimensional topological field theory ($\mathcal{C}$-valued TFT)* is a functor from this cobordism category to $\mathcal{C}$. Composing with our categorification functor shows that *any* $\mathrm{Symp}^{\#}$*-valued TFT gives rise to a* Cat*-valued TFT*. Such symplectic-valued topological field theories should not be confused with the symplectic quantum field theory of Eliashberg et al. [3] which is meant to be a functor from the symplectic cobordism category to the category of vector spaces.

At least formally, there are a number of examples of this construction. In [45] we investigate the theory which uses as symplectic manifolds the moduli spaces of flat bundles with compact structure group on three-dimensional cobordisms containing tangles. In this case, after adding the data of holonomies around the tangles or determinant line bundles and disallowing surfaces that give rise to singular or non-monotone moduli spaces, one obtains a $\mathrm{Symp}^{\#}$-valued TFT and hence a Cat-valued TFT. One can also construct natural transformations for 4-dimensional cobordisms of cobordisms; in other words, a *topological quantum field theory with corners* (roughly speaking; not all the axioms are satisfied) in $2+1+1$ dimensions. These should be thought of as Lagrangian-Floer versions of gauge-theoretic invariants investigated by Donaldson and Floer, in the case without knots, and

Kronheimer-Mrowka and Furata-Steer, in the case with knots, and many other authors. The equivalence of the gauge-theory version, in cases where the invariant has been defined, and the pseudoholomorphic curves version developed here would be a version Atiyah-Floer conjecture. The construction of such theories was suggested by Fukaya in [6], and was one of motivations for the development of Fukaya categories.

Further directions: The original motivation comes from the exact triangle for fibered Dehn twists, which is discussed in the paper [44]. The general theory developed here allow us to formulate the exact triangle as a mirror partner to Horja's exact triangles in [13], so that the third term in the triangle is defined by applying a push-pull functor; this formulation was suggested to us by P. Seidel and I. Smith, see [41]. Applied to the case of flat bundles, this gives exact triangles for the various topological invariants. In particular, for $SU(r)$-bundles on complements of tangles the exact triangle has the same form as the one of Khovanov-Rozansky [18] for $SU(r)$ tangle invariants.

Many of our results have chain-level versions, that is, extensions to Fukaya categories. These are discussed in the paper [25], which is joint work with S. Mau. To each monotone Lagrangian correspondence with minimal Maslov number at least three we define an A_∞ functor, such that the composition of A_∞ functors is homotopic to the A_∞ functor for the geometric composition, if smooth and embedded. Applied to moduli spaces of flat bundles, this gives what might be called a field theory with corners assigning to any surface an A_∞ category. Combining this with Kontsevich's construction of derived A_∞ categories gives a field theory assigning to any surface a triangulated category. These categories are expected to be better behaved; in particular, one might hope they are finitely split-generated. Even more speculatively (in the spirit of Khovanov's work [17]) one might hope for a combinatorial description of the categories which sheds some light on four-dimensional invariants, just as the structure of the tensor categories of representations of quantum groups illuminates the structure of the quantum Chern-Simons invariants. However (coming back to earth) the analytic problems are present already in the homology-level version and best presented in that framework.

Notation: We will frequently refer to the standing assumptions (M1-2), (L1-3), and (G1-2) that can be found on pages 22 - 23.

One notational warning: When dealing with functors we will use functorial notation for compositions, that is $\Phi_0 \circ \Phi_1$ maps an object x to $\Phi_1(\Phi_0(x))$. When dealing with simple maps (like symplectomorphisms, or in the analytic part), we will however stick to the traditional notation $(\phi_1 \circ \phi_0)(x) = \phi_1(\phi_0(x))$.

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2. Lagrangian correspondences

Let M be a smooth manifold. A *symplectic form* on M is a closed, non-degenerate two-form ω. A *symplectic manifold* is a smooth manifold equipped with a symplectic form. If (M_1, ω_1) and (M_2, ω_2) are symplectic manifolds, then a diffeomorphism $\varphi : M_1 \to M_2$ is a *symplectomorphism* if $\varphi^*\omega_2 = \omega_1$. Let Symp denote the category whose objects are symplectic manifolds and whose morphisms are symplectomorphisms. The following operations give Symp a structure similar to that of a tensor category.

(a) (Duals) If $M = (M, \omega)$ is a symplectic manifold, then $M^- = (M, -\omega)$ is a symplectic manifold, called the *dual* of M.

(b) (Sums) If $M_j = (M_j, \omega_j), j = 1, 2$ are symplectic manifolds, then $M_1 \cup M_2$ equipped with the symplectic structure ω_1 on M_1 and ω_2 on M_2, is a symplectic manifold.
(c) (Products) Let $M_j = (M_j, \omega_j), j = 1, 2$ be symplectic manifolds, then the Cartesian product $(M_1 \times M_2, \pi_1^*\omega_1 + \pi_2^*\omega_2)$ is a symplectic manifold. (Here $\pi_j : M_1 \times M_2 \to M_j$ denotes the projections.)

Clearly the notion of symplectomorphism is very restrictive; in particular, the symplectic manifolds must be of the same dimension. A more flexible notion of morphism is that of Lagrangian correspondence, defined as follows [49, 48, 11]. Let $M = (M, \omega)$ be a symplectic manifold. A submanifold $L \subset M$ is *isotropic*, resp. *coisotropic*, resp. *Lagrangian* if the ω-orthogonal complement TL^ω satisfies $TL^\omega \subseteq TL$ resp. $TL^\omega \supseteq TL$ resp. $TL^\omega = TL$.

Definition 2.0.2. Let M_1, M_2 be symplectic manifolds. A *Lagrangian correspondence from* M_1 *to* M_2 is a Lagrangian submanifold $L_{12} \subset M_1^- \times M_2$.

Example 2.0.3. The following are examples of Lagrangian correspondences:

(a) (Graphs) If $\varphi_{12} : M_1 \to M_2$ is a symplectomorphism then its graph
$$\mathrm{graph}(\varphi_{12}) = \{(m_1, \varphi_{12}(m_1)) \,|\; m_1 \in M_1\} \subset M_1^- \times M_2$$
is a Lagrangian correspondence.
(b) (Fibered coisotropics) Suppose that $\iota : C \to M$ is a coisotropic submanifold and that the null foliation TC^ω of C is fibrating, that is, there exists a symplectic manifold (B, ω_B) and a fibration $\pi : C \to B$ such that $\iota^*\omega$ is the pull-back $\pi^*\omega_B$. Then
$$(\iota \times \pi) : C \to M^- \times B$$
maps C to a Lagrangian correspondence.
(c) (Level sets of moment maps) Let G be a Lie group with Lie algebra $\mathfrak{g}$. Suppose that G acts on M by Hamiltonian symplectomorphisms generated by a moment map $\mu : M \to \mathfrak{g}^*$. (That is μ is equivariant and the generating vector fields $\mathfrak{g} \to \mathrm{Vect}(M), \xi \mapsto \xi_M$ satisfy $\iota(\xi_M)\omega = -d(\mu, \xi)$.) If G acts freely on $\mu^{-1}(0)$, then $\mu^{-1}(0)$ is a smooth coisotropic fibered over the symplectic quotient $M/\!\!/G = \mu^{-1}(0)/G$, which is a symplectic manifold. Hence we have a Lagrangian correspondence
$$(\iota \times \pi) : \;\mu^{-1}(0) \to M^- \times (M/\!\!/G).$$
The symplectic two-form $\omega_{M/\!\!/G}$ on $M/\!\!/G$ is the unique form on $M/\!\!/G$ satisfying $\pi^*\omega_{M/\!\!/G} = \iota^*\omega$.

Definition 2.0.4. Let M_0, M_1, M_2 be symplectic manifolds and $L_{01} \subset M_0^- \times M_1$, $L_{12} \subset M_1^- \times M_2$ Lagrangian correspondences.

(a) The *dual* Lagrangian correspondence of L_{01} is
$$(L_{01})^t := \{(m_1, m_0) \,|\, (m_0, m_1) \in L_{01}\} \subset M_1^- \times M_0.$$
(b) The *composition* of L_{01} and L_{12} is
$$L_{01} \circ L_{12} := \left\{ (m_0, m_2) \in M_0^- \times M_2 \;\middle|\; \exists m_1 \in M_1 : \begin{array}{c} (m_0, m_1) \in L_{01} \\ (m_1, m_2) \in L_{12} \end{array} \right\} \subset M_0^- \times M_2.$$

Equivalently, $L_{01} \circ L_{12} = \pi_{02}(L_{01} \times_{M_1} L_{12})$ is the image under the projection $\pi_{02} : M_0^- \times M_1 \times M_1^- \times M_2 \to M_0^- \times M_2$ of
$$L_{12} \times_{M_1} L_{01} := (L_{01} \times L_{12}) \cap (M_0^- \times \Delta_1 \times M_2).$$

Here $\Delta_1 \subset M_1^- \times M_1$ denotes the diagonal. $L_{01} \circ L_{12} \subset M_0^- \times M_2$ is an immersed Lagrangian submanifold if $L_{01} \times L_{12}$ intersects $M_0^- \times \Delta_1 \times M_2$ transversally. In general, the composition of smooth Lagrangian submanifolds may not even be immersed. We will be working with the following class of compositions, for which the resulting Lagrangian correspondence is in fact a smooth submanifold.

Definition 2.0.5. We say that the composition $L_{01} \circ L_{12}$ is *embedded* if $L_{12} \times_{M_1} L_{01}$ is cut out transversally (i.e. $(L_{01} \times L_{12}) \pitchfork (M_0^- \times \Delta_1 \times M_2)$) and the projection $\pi_{02} : L_{12} \times_{M_1} L_{01} \to L_{01} \circ L_{12} \subset M_0^- \times M_2$ is an embedding.

Remark 2.0.6. Suppose that the composition $L_{01} \circ L_{12} =: L_{02}$ is embedded.

(a) By the embedding property, for every $(x_0, x_2) \in L_{02}$ there is a unique solution $x_1 \in M_1$ to $(x_0, x_1, x_1, x_2) \in L_{01} \times L_{12}$. Due to the transversality assumption, this solution is given by a smooth map $\ell_1 : L_{02} \to M_1$.
(b) If L_{01} and L_{12} are compact, oriented, and equipped with a relative spin structure, then L_{02} is also compact and inherits an orientation and relative spin structure, see [46].
(c) If $\pi_1(L_{01})$ and $\pi_1(L_{12})$ are torsion, then $\pi_1(L_{02})$ is torsion. If moreover M_0 and M_2 are monotone with the same monotonicity constant, then L_{02} is monotone, see Section 3.1.

More generally, composition of Lagrangian correspondences is defined under clean intersection hypotheses, see [11]. This extension is not needed in the paper, because the version of Floer cohomology used in this paper is invariant under Hamiltonian isotopy, and after such an isotopy transversality may always be achieved.[1]

Composition and duals of Lagrangian correspondences satisfy the following:

(a) (Composition and inversion of graphs) If $\varphi_{01} : M_0 \to M_1$ and $\varphi_{12} : M_1 \to M_2$ are symplectomorphisms, then
$$\operatorname{graph}(\varphi_{01}) \circ \operatorname{graph}(\varphi_{12}) = \operatorname{graph}(\varphi_{12} \circ \varphi_{01}),$$
$$\operatorname{graph}(\varphi_{01})^t = \operatorname{graph}(\varphi_{01}^{-1}).$$
(b) (Identity) If $L_{01} \subset M_0^- \times M_1$ is a Lagrangian correspondence and $\Delta_j \subset M_j^- \times M_j, j = 0, 1$ are the diagonals, then
$$L_{01} = \Delta_0 \circ L_{01} = L_{01} \circ \Delta_1.$$
(c) (Associativity) If $L_{01} \subset M_0^- \times M_1, L_{12} \subset M_1^- \times M_2, L_{23} \subset M_2^- \times M_3$ are Lagrangian correspondences, then
$$(L_{01} \circ L_{12}) \circ L_{23} = L_{01} \circ (L_{12} \circ L_{23}),$$
$$(L_{01} \circ L_{12})^t = (L_{12})^t \circ (L_{01})^t.$$
(d) (Intersections) If $L_{01} \subset M_0^- \times M_1$ is a Lagrangian correspondence, then for any Lagrangian submanifolds $L_0 \subset M_0$, $L_1 \subset M_1$ we have a bijection
$$(L_0 \circ L_{01}) \cap L_1 \xrightarrow{\sim} L_0 \cap (L_1 \circ L_{01}^t).$$

[1]However, one can not necessarily remove all self-intersections of the immersed composition by Hamiltonian isotopy on one correspondence. A basic example is the composition of transverse Lagrangian submanifolds $L, L' \subset M$. Identifying $M \cong M \times \{\mathrm{pt}\} \cong \{\mathrm{pt}\} \times M$ the projection $L \times_M L' \to L \circ L' \subset \{\mathrm{pt}\} \times \{\mathrm{pt}\}$ maps the (finite) intersection $L \cap L'$ to a point.

2.1. **Generalized Lagrangian correspondences.** This subsection describes one resolution of the composition problem, given by passing to sequences of Lagrangian correspondences.

Definition 2.1.1. Let M, M' be symplectic manifolds. A *generalized Lagrangian correspondence* $\underline{L}$ from M to M' consists of

(a) a sequence $N_0, \ldots, N_r$ of any length $r+1 \geq 2$ of symplectic manifolds with $N_0 = M$ and $N_r = M'$,
(b) a sequence $L_{01}, \ldots, L_{(r-1)r}$ of compact Lagrangian correspondences with $L_{(j-1)j} \subset N_{j-1}^- \times N_j$ for $j = 1, \ldots, r$.

Definition 2.1.2. Let $\underline{L}$ from M to M' and $\underline{L}'$ from M' to M'' be two generalized Lagrangian correspondences. Then we define composition

$$(\underline{L}, \underline{L}') := \big(L_{01}, \ldots, L_{(r-1)r}, L'_{01}, \ldots, L'_{(r'-1)r'}\big)$$

as a generalized Lagrangian correspondence from M to M''. Moreover, we define the dual

$$\underline{L}^t := \big(L^t_{(r-1)r}, \ldots, L^t_{01}\big).$$

as a generalized Lagrangian correspondence from M' to M.

Using these notions we define the *symplectic category* $\mathrm{Symp}^\#$ as follows. An extension of this approach, using Floer cohomology spaces to define a 2-category, is given in Section 6.9.

Definition 2.1.3.

(a) The objects of $\mathrm{Symp}^\#$ are smooth symplectic manifolds $M = (M, \omega)$.
(b) The morphisms $\mathrm{Hom}(M, M')$ of $\mathrm{Symp}^\#$ are generalized Lagrangian correspondences from M to M' modulo the equivalence relation $\sim$ generated by
$$\big(\ldots, L_{(j-1)j}, L_{j(j+1)}, \ldots\big) \sim \big(\ldots, L_{(j-1)j} \circ L_{j(j+1)}, \ldots\big)$$
for all sequences and j such that $L_{(j-1)j} \circ L_{j(j+1)}$ is embedded.
(c) The composition of morphisms $[\underline{L}] \in \mathrm{Hom}(M, M')$ and $[\underline{L}'] \in \mathrm{Hom}(M', M'')$ is defined by
$$[\underline{L}] \circ [\underline{L}'] := [(\underline{L}, \underline{L}')] \ \in \mathrm{Hom}(M, M'').$$
(d) The identity in $\mathrm{Hom}(M, M)$ is the equivalence class $[\Delta_M]$ of the diagonal $\Delta_M \subset M^- \times M$.

Note that a sequence of Lagrangian correspondences in $\mathrm{Hom}(M, M')$ can run through any sequence $(N_i)_{i=1,\ldots,r-1}$ of intermediate symplectic manifolds of any length $r - 1 \in \mathbb{N}_0$. Nevertheless, the composition of two such sequences is always well defined. In (c) the new sequence of intermediate symplectic manifolds for $\underline{L} \circ \underline{L}'$ is $(N_1, \ldots, N_{r-1}, N_r = M' = N'_0, N'_1, \ldots, N'_{r'-1})$. This definition descends to the quotient by the equivalence relation $\sim$ since any equivalences within $\underline{L}$ and $\underline{L}'$ combine to an equivalence within $\underline{L} \circ \underline{L}'$. The diagonal defines an identity since $L_{(r-1)r} \circ \Delta_m = L_{(r-1)r}$ is always smooth and embedded. For a discussion of partially defined operations in much more generality see [20].

Lemma 2.1.4. (a) *If $L_a, L_b \subset M^- \times M'$ are distinct Lagrangian submanifolds, then the corresponding morphisms $[L_a], [L_b] \in \mathrm{Hom}(M, M')$ are distinct.*

(b) *The composition of smooth Lagrangian correspondences* $L \subset M^- \times M'$ *and* $L' \subset M'^- \times M''$ *coincides with the geometric composition,* $[L] \circ [L'] = [L \circ L']$ *if* $L \circ L'$ *is embedded.*

Proof. To see that $L_a \neq L_b \subset M^- \times M'$ define distinct morphisms note that the projection to the (possibly singular) Lagrangian $\pi([\underline{L}]) := L_{01} \circ \ldots \circ L_{(r-1)r} \subset M^- \times M'$ is well defined for all $[\underline{L}] \in \mathrm{Hom}(M, M')$. The rest follows directly from the definitions. □

Remark 2.1.5. The study of Lagrangian correspondences appeared in the study of Fourier integral operators by Hörmander and others. Hörmander's construction associates to any Fourier integral operator $P_{01} \in \mathrm{FIO}(Q_0, Q_1)$ (which in particular induces a smooth map $\mathcal{C}^\infty(Q_0) \to \mathcal{C}^\infty(Q_1)$ between smooth functions on the closed manifolds Q_i) a Lagrangian submanifold $\Lambda_{P_{01}} \in T^*Q_0^- \times T^*Q_1$. Conversely, any homogeneous[2] Lagrangian correspondence $L_{01} \subset T^*Q_0^- \times T^*Q_1$ gives rise to a class of operators $\mathrm{FIO}(L_{01}) \subset \mathrm{FIO}(Q_0, Q_1)$. These constructions satisfy the property [14, Theorem 4.2.2] that if a pair $L_{P_{01}} \subset T^*Q_0^- \times T^*Q_1$, $L_{P_{12}} \subset T^*Q_1^- \times T^*Q_2$ satisfies

(3) $L_{P_{01}} \times L_{P_{12}}$ intersects $T^*Q_0^- \times \Delta_{T^*Q_1} \times T^*Q_2$ transversally and the projection from the intersection to $T^*Q_0^- \times T^*Q_2$ is proper,

then

$$L_{P_{01} \circ P_{12}} = L_{P_{01}} \circ L_{P_{12}}. \tag{4}$$

Define a category $\mathrm{H\ddot{o}rm}^\#$, whose

- objects are compact smooth manifolds,
- morphisms are sequences of Fourier integral operators, modulo the equivalence relation that is generated by $(\ldots, P_{01}, P_{12}, \ldots) \sim (\ldots, P_{01} \circ P_{12}, \ldots)$ for $\Lambda_{P_{01}}, \Lambda_{P_{12}}$ satisfying (3).

The category $\mathrm{H\ddot{o}rm}^\#$ admits a *symbol* functor σ to the symplectic category $\mathrm{Symp}^\#$, given on the level of objects by $Q \mapsto T^*Q$, and on morphisms by assigning to each Fourier integral operator in the sequence the associated Lagrangian correspondence. This remark is continued in Remark 6.9.8 (a).

We conclude this subsection by mentioning special cases of generalized Lagrangian correspondences. The first is the case $M = M'$, which we will want to view separately as a cyclic correspondence, without fixing the "base point" M.

Definition 2.1.6. A *cyclic generalized Lagrangian correspondence* $\underline{L}$ consists of

(a) a cyclic sequence $N_0, N_1, \ldots, N_r, N_{r+1} = N_0$ of symplectic manifolds of any length $r + 1 \geq 1$,
(b) a sequence $L_{01}, \ldots, L_{r(r+1)}$ of compact Lagrangian correspondences with $L_{j(j+1)} \subset N_j^- \times N_{j+1}$ for $j = 0, \ldots, r$.

The second special case is $M = \{pt\}$, which generalizes the concept of Lagrangian submanifolds. Namely, note that any Lagrangian submanifold $L \subset M'$ can be viewed as correspondence $L \subset \{pt\}^- \times M'$.

[2]A Lagrangian correspondence L_{01} is called homogeneous if it lies in the complement of the zero sections, $L_{01} \subset (T^*Q_0^- \setminus 0_{Q_0}) \times (T^*Q_1 \setminus 0_{Q_1})$, and if it is conic, i.e. invariant under positive scalar multiplication in the fibres.

Definition 2.1.7. Let M' be a symplectic manifold. A *generalized Lagrangian submanifold* $\underline{L}$ of M' is a generalized Lagrangian correspondence from a point $M = \{pt\}$ to M'. That is, $\underline{L}$ consists of

(a) a sequence $N_{-r}, \ldots, N_0$ of any length $r \geq 0$ of symplectic manifolds with $N_{-r} = \{\mathrm{pt}\}$ a point and $N_0 = M'$,

(b) a sequence $L_{(-r)(-r+1)}, \ldots, L_{(-1)0}$ of compact Lagrangian correspondences $L_{(i-1)i} \subset N_{i-1}^- \times N_i$.

2.2. **Graded Lagrangians.** Following Kontsevich and Seidel [38] one can define graded Lagrangian subspaces as follows. Let V be a symplectic vector space and let $\mathrm{Lag}(V)$ be the Lagrangian Grassmannian of V. An *N-fold Maslov covering for* V is a $\mathbb{Z}_N$-covering $\mathrm{Lag}^N(V) \to \mathrm{Lag}(V)$ associated to the Maslov class in $\pi_1(\mathrm{Lag}(V))$. A *grading* of a Lagrangian subspace $\Lambda \in \mathrm{Lag}(V)$ is a lift to $\tilde{\Lambda} \in \mathrm{Lag}^N(V)$.

Remark 2.2.1. (a) For any basepoint $\Lambda_0 \in \mathrm{Lag}(V)$ we obtain an N-fold Maslov cover $\mathrm{Lag}^N(V, \Lambda_0)$ given as the homotopy classes of paths $\tilde{\Lambda} : [0,1] \to \mathrm{Lag}(V)$ with base point $\tilde{\Lambda}(0) = \Lambda_0$, modulo loops of Maslov index N. The covering is $\tilde{\Lambda} \mapsto \tilde{\Lambda}(1)$. The base point has a canonical grading given by the constant path $\tilde{\Lambda}_0 \equiv \Lambda_0$. Any path between basepoints Λ_0, Λ_0' induces an identification $\mathrm{Lag}^N(V, \Lambda_0) \to \mathrm{Lag}^N(V, \Lambda_0')$.

(b) For the diagonal $\Delta \subset V^- \times V$ we fix a canonical grading and orientation as follows. We identify the Maslov coverings $\mathrm{Lag}^N(V^- \times V, \Lambda^- \times \Lambda)$ and $\mathrm{Lag}^N(V^- \times V, \Delta)$ by concatenation of the paths

$$(e^{Jt}\Lambda^- \times \Lambda)_{t\in[0,\pi/2]}, \quad (\{(tx + Jy, x + tJy) | x, y \in \Lambda\})_{t\in[0,1]}, \tag{5}$$

where $J \in \mathrm{End}(V)$ is an ω-compatible complex structure on V (i.e. $J^2 = -\mathrm{Id}$ and $\omega(\cdot, J\cdot)$ is symmetric and positive definite). In particular, this induces the *canonical grading* on the diagonal Δ with respect to any Maslov covering $\mathrm{Lag}^N(V^-\times V, \Lambda^-\times\Lambda)$, by continuation. Any identification $\mathrm{Lag}^N(V^-\times V, \Lambda_0^-\times\Lambda_0) \to \mathrm{Lag}^N(V^-\times V, \Lambda_1^-\times\Lambda_1)$ induced by a path in $\mathrm{Lag}^N(V)$ maps the graded diagonal to the graded diagonal, since the product $\gamma^- \times \gamma$ of any loop $\gamma : S^1 \to \mathrm{Lag}(V)$ has Maslov index 0. Similarly, we define a *canonical orientation* on Δ by choosing any orientation on Λ, giving the product $\Lambda^- \times \Lambda$ the product orientation (which is well defined), and extending the orientation over the path (5). This is related to the orientation induced by projection of the diagonal on the second factor by a sign $(-1)^{n(n-1)/2}$, where $\dim(M) = 2n$.

Let M be a symplectic manifold and let $\mathrm{Lag}(M) \to M$ be the fiber bundle whose fiber over $m \in M$ is the space $\mathrm{Lag}(T_mM)$ of Lagrangian subspaces of T_mM. An *N-fold Maslov covering* of M is an N-fold cover $\mathrm{Lag}^N(M) \to \mathrm{Lag}(M)$ whose restriction to each fiber is an N-fold Maslov covering $\mathrm{Lag}^N(T_mM) \to \mathrm{Lag}(T_mM)$. Any choice of Maslov cover for $\mathbb{R}^{2n}$ induces a one-to-one correspondence between N-fold Maslov covers of M and $\mathrm{Sp}^N(2n)$-structures on M. Here $2n = \dim M$ and $\mathrm{Sp}^N(2n)$ is the N-fold covering group of $\mathrm{Sp}(2n)$ associated to the Maslov class in $\pi_1(\mathrm{Sp}(2n))$. (Explicitly, this is realized by using the identity as base point.) An $\mathrm{Sp}^N(2n)$*-structure* on M is an $\mathrm{Sp}^N(2n)$-bundle $\mathrm{Fr}^N(M) \to M$ together with an isomorphism $\mathrm{Fr}^N(M) \times_{\mathrm{Sp}^N(2n)} \mathrm{Sp}(2n) \simeq \mathrm{Fr}(M)$ to the symplectic frame bundle of M. It induces the N-fold Maslov covering

$$\mathrm{Lag}^N(M) = \mathrm{Fr}^N(M) \times_{\mathrm{Sp}^N(2n)} \mathrm{Lag}^N(\mathbb{R}^{2n}).$$

Graded symplectic manifolds (i.e. equipped with Maslov coverings) form a structure similar to that of a tensor category, that is, the notions of duals, disjoint union, and Cartesian product extend naturally to the graded setting. The dual $\mathrm{Lag}^N(M^-)$ of a Maslov covering $\mathrm{Lag}^N(M) \to \mathrm{Lag}(M)$ is the same space with the inverted $\mathbb{Z}_N$-action. We denote this identification by

$$\mathrm{Lag}^N(M) \to \mathrm{Lag}^N(M^-), \qquad \tilde\Lambda \mapsto \tilde\Lambda^-. \tag{6}$$

For Sp^N-structures $\mathrm{Fr}^N(M_0)$ and $\mathrm{Fr}^N(M_1)$ the embedding

$$\mathrm{Sp}^N(2n_0) \times_{\mathbb{Z}_N} \mathrm{Sp}^N(2n_1) \to \mathrm{Sp}^N(2n_0 + 2n_1)$$

induces an $\mathrm{Sp}^N(2n_0+2n_1)$-structure $\mathrm{Fr}^N(M_0 \times M_1)$ on the product and an equivariant map

$$\mathrm{Fr}^N(M_0) \times \mathrm{Fr}^N(M_1) \to \mathrm{Fr}^N(M_0 \times M_1) \tag{7}$$

covering the inclusion $\mathrm{Fr}(M_0) \times \mathrm{Fr}(M_1) \to \mathrm{Fr}(M_0 \times M_1)$. The corresponding product of N-fold Maslov covers on $M_0 \times M_1$ is the N-fold Maslov covering

$$\mathrm{Lag}^N(M_0 \times M_1) := \big(\mathrm{Fr}^N(M_0) \times \mathrm{Fr}^N(M_1)\big) \times_{\mathrm{Sp}^N(2n_0)\times \mathrm{Sp}^N(2n_1)} \mathrm{Lag}^N(\mathbb{R}^{2n_0} \times \mathbb{R}^{2n_1}).$$

Combining this product with the dual yields a Maslov covering for $M_0^- \times M_1$ which we can identify with

$$\mathrm{Lag}^N(M_0^- \times M_1) = \big(\mathrm{Fr}^N(M_0) \times \mathrm{Fr}^N(M_1)\big) \times_{\mathrm{Sp}^N(2n_0)\times \mathrm{Sp}^N(2n_1)} \mathrm{Lag}^N(\mathbb{R}^{2n_0,-} \times \mathbb{R}^{2n_1}).$$

Finally, the inclusion $\mathrm{Lag}(M_0) \times \mathrm{Lag}(M_1) \to \mathrm{Lag}(M_0 \times M_1)$ lifts to a map

$$\mathrm{Lag}^N(M_0) \times \mathrm{Lag}^N(M_1) \to \mathrm{Lag}^N(M_0 \times M_1), \quad (\tilde L_0, \tilde L_1) \mapsto \tilde L_0 \times^N \tilde L_1 \tag{8}$$

with fiber $\mathbb{Z}_N$. It is defined by combining the product (7) with the basic product of the linear Maslov cover $\mathrm{Lag}^N(\mathbb{R}^{2n_0}) \times \mathrm{Lag}^N(\mathbb{R}^{2n_1}) \to \mathrm{Lag}^N(\mathbb{R}^{2n_0} \times \mathbb{R}^{2n_1})$.

Definition 2.2.2. (a) Let M_0, M_1 be two symplectic manifolds equipped with N-fold Maslov covers and let $\phi : M_0 \to M_1$ be a symplectomorphisms. A *grading* of ϕ is a lift of the canonical isomorphism $\mathrm{Lag}(M_0) \to \mathrm{Lag}(M_1)$ to an isomorphism $\phi^N : \mathrm{Lag}^N(M_0) \to \mathrm{Lag}^N(M_1)$, or equivalently, a lift of the canonical isomorphism $\mathrm{Fr}(M_0) \to \mathrm{Fr}(M_1)$ of symplectic frame bundles to an isomorphism $\mathrm{Fr}^N(M_0) \to \mathrm{Fr}^N(M_1)$.

(b) Let $L \subset M$ be a Lagrangian submanifold and M be equipped with an N-fold Maslov cover. A *grading* of L is a lift $\sigma_L^N : L \to \mathrm{Lag}^N(M)$ of the canonical section $\sigma_L : L \to \mathrm{Lag}(M)$.

Remark 2.2.3. (a) The set of graded symplectomorphisms forms a group under composition. In particular, the identity on M has a canonical grading, given by the identity on $\mathrm{Lag}^N(M)$.

(b) Given a one-parameter family ϕ_t of symplectomorphisms with $\phi_0 = \mathrm{Id}_M$, we obtain a grading of ϕ_t by continuity.

(c) Any choice of grading on the diagonal $\tilde\Delta \in \mathrm{Lag}^N(\mathbb{R}^{2n,-} \times \mathbb{R}^{2n})$ induces a bijection between gradings of a symplectomorphism $\phi : M_0 \to M_1$ and gradings of its graph $\mathrm{graph}(\phi) \subset M_0^- \times M_1$ with respect to the induced Maslov cover $\mathrm{Lag}^N(M_0^- \times M_1)$. Indeed, the graph of the grading, $\mathrm{graph}(\phi^N) \subset (\mathrm{Fr}^N(M_0) \times \mathrm{Fr}^N(M_1))|_{\mathrm{graph}(\phi)}$ is a principal bundle over $\mathrm{graph}(\phi)$ with structure group $\mathrm{Sp}^N(2n)$, $2n = \dim M_0 = \dim M_1$. The graded diagonal descends under the associated fiber bundle construction with $\mathrm{graph}(\phi^N)$ to a section of $\mathrm{Lag}^N(M_0^- \times M_1)|_{\mathrm{graph}(\phi)}$ lifting $\mathrm{graph}(\phi)$. Moreover, this construction is equivariant for the transitive action of $H^0(M_0, \mathbb{Z}_N)$ on both the set of gradings of ϕ and the set of gradings of $\mathrm{graph}(\phi)$.

We will refer to this as the *canonical bijection* when using the canonical grading $\tilde{\Delta} \in \mathrm{Lag}^N(\mathbb{R}^{2n,-} \times \mathbb{R}^{2n})$ in Remark 2.2.1. In particular, the diagonal in $M^- \times M$ has a canonical grading induced by the canonical bijection from the canonical grading of the identity on M.

(d) Any grading σ_L^N of a Lagrangian submanifold $L \subset M$ induces a grading of $L \subset M^-$ via the diffeomorphism $\mathrm{Lag}^N(M^-) \to \mathrm{Lag}^N(M)$.

(e) Given graded Lagrangian submanifolds $L_0 \subset M_0, L_1 \subset M_1$, the product $L_0 \times L_1 \subset M_0 \times M_1$ inherits a grading from (8).

(f) Given a graded symplectomorphism $\phi : M_0 \to M_1$ and a graded Lagrangian submanifold $L \subset M_0$, the image $\phi(L) \subset M_1$ inherits a grading by composition $\sigma^N_{\phi(L)} = \phi^N \circ \sigma_L^N$.

Example 2.2.4. (a) Let $\mathrm{Lag}^2(M)$ be the bundle whose fiber over m is the space of oriented Lagrangian subspaces of T_mM. Then $\mathrm{Lag}^2(M) \to \mathrm{Lag}(M)$ is a 2-fold Maslov covering. A $\mathrm{Lag}^2(M)$-grading of a Lagrangian $L \subset M$ is equivalent to an orientation on L.

(b) By [38, Section 2], any symplectic manifold M with $H^1(M) = 0$ and minimal Chern number N_M admits an N-fold Maslov covering $\mathrm{Lag}^N(M)$ iff N divides $2N_M$. Any Lagrangian with minimal Maslov number N_L admits a $\mathrm{Lag}^N(M)$-grading iff N divides N_L. In particular, if L is simply connected, then $N_L = 2N_M$ and L admits a $\mathrm{Lag}^{2N_M}(M)$ grading.

(c) Suppose that $[\omega]$ is integral, $[\omega] = (1/l)c_1(TM)$, and $\mathcal{L}$ is a line bundle with connection ∇ and curvature $\mathrm{curv}(\nabla) = (2\pi/i)\omega$. This induces a $2l$-fold Maslov cover $\mathrm{Lag}^{2l}(M) \to \mathrm{Lag}(M)$, see [38, Section 2b]. Let $L \subset M$ be a Bohr-Sommerfeld monotone Lagrangian as in Remark 3.1.4. A grading of $\mathcal{L}$ is equivalent to a choice of (not necessarily horizontal) section of $\mathcal{L}|L$ whose l-th tensor power is $\phi_L^{\mathcal{K}}$; that is, a choice of the section $\exp(2\pi i\psi)\phi_L^{\mathcal{L}}$ in (16).

Definition 2.2.5. Let $\Lambda_0, \Lambda_1 \subset V$ be a transverse pair of Lagrangian subspaces in a symplectic vector space V and let $\tilde{\Lambda}_0, \tilde{\Lambda}_1 \in \mathrm{Lag}^N(V)$ be gradings. The *degree* $d(\tilde{\Lambda}_0, \tilde{\Lambda}_1) \in \mathbb{Z}_N$ is defined as follows. Let $\tilde{\gamma}_0, \tilde{\gamma}_1 : [0,1] \to \mathrm{Lag}^N(V)$ be paths with common starting point $\tilde{\gamma}_0(0) = \tilde{\gamma}_1(0)$ and end points $\tilde{\gamma}_j(1) = \tilde{\Lambda}_j$. Let $\gamma_j : [0,1] \to \mathrm{Lag}(V)$ denote their image under the projection $\mathrm{Lag}^N(V) \to \mathrm{Lag}(V)$ and define

$$d(\tilde{\Lambda}_0, \tilde{\Lambda}_1) := \tfrac{1}{2}\dim(\Lambda_0) + I(\gamma_0, \gamma_1) \mod N, \tag{9}$$

where $I(\gamma_0, \gamma_1)$ denotes the Maslov index for the pair of paths as in [43, 33].

Let us recall from [33] that the Maslov index for a pair of paths with regular crossings (in particular with a finite set of crossings $\mathcal{C} := \{s \in [0,1] \,|\, \gamma_0(s) \cap \gamma_1(s) \neq \{0\}\}$) is given by the sum of crossing numbers with the endpoints weighted by $1/2$,

$$I(\gamma_0, \gamma_1) = \frac{1}{2} \sum_{s \in \mathcal{C} \cap \{0,1\}} \mathrm{sign}(\Gamma(\gamma_0, \gamma_1, s)) + \sum_{s \in \mathcal{C} \cap (0,1)} \mathrm{sign}(\Gamma(\gamma_0, \gamma_1, s)).$$

Each crossing operator $\Gamma(\gamma_0, \gamma_1, s)$ is defined on $v \in \gamma_0(s) \cap \gamma_1(s)$ by fixing Lagrangian complements $\gamma_0(s)^c$, $\gamma_1(s)^c$ of $\gamma_0(s), \gamma_1(s)$ and setting

$$\Gamma(\gamma_0, \gamma_1, s)v = \tfrac{d}{dt}\big|_{t=0} \omega(v, w(t) - w'(t)) \tag{10}$$

where $w(t) \in \gamma_0(s)^c$ such that $v+w(t) \in \gamma_0(s+t)$ and $w'(t) \in \gamma_1(s)^c$ such that $v+w'(s+t) \in \gamma_1(s)$.

Remark 2.2.6. The degree can alternatively be defined by fixing $\tilde{\gamma}_0 \equiv \tilde{\Lambda}_0$ and choosing a path $\tilde{\gamma} : [0,1] \to \mathrm{Lag}^N(V)$ from $\tilde{\gamma}(0) = \tilde{\Lambda}_0$ to $\tilde{\gamma}(1) = \tilde{\Lambda}_1$ such that the crossing form $\Gamma(\gamma, \Lambda_0, 0)$ of the underlying path $\gamma : [0,1] \to \mathrm{Lag}(V)$ is positive definite at $s = 0$. Then the degree

$$d(\tilde{\Lambda}_0, \tilde{\Lambda}_1) = \tfrac{\dim \Lambda_0}{2} + I(\Lambda_0, \gamma) = -\sum_{s\in(0,1)} \mathrm{sign}(\Gamma(\gamma, \Lambda_0, s)) = -I'(\gamma, \Lambda_0) \qquad \mathrm{mod}\ N$$

is given by the Maslov index I' of $\gamma|_{(0,1)}$ (not counting the endpoints) relative to Λ_0. Equivalently, we have

$$d(\tilde{\Lambda}_0, \tilde{\Lambda}_1) = I'(\gamma^{-1}, \Lambda_0) \qquad \mathrm{mod}\ N$$

for the reversed path $\gamma^{-1} : [0,1] \to \mathrm{Lag}(V)$ from $\gamma(0) = \Lambda_1$ to $\gamma(1) = \Lambda_0$ such that the crossing form $\Gamma(\gamma^{-1}, \Lambda_0, 1)$ is negative definite at $s = 1$.

Lemma 2.2.7. *(Index theorem for once-punctured disks) Let $\Lambda_0, \Lambda_1 \subset V$ be a transverse pair of Lagrangian subspaces with gradings $\tilde{\Lambda}_0, \tilde{\Lambda}_1 \in \mathrm{Lag}^N(V)$. Then for any smooth path of graded Lagrangian subspaces $\tilde{\Lambda} : [0,1] \to \mathrm{Lag}^N(V)$ with endpoints $\tilde{\Lambda}(j) = \tilde{\Lambda}_j$, $j = 0,1$ we have*

$$d(\tilde{\Lambda}_0, \tilde{\Lambda}_1) = \mathrm{Ind}(D_{V,\Lambda}) \quad \mathrm{mod}\ N.$$

Here $D_{V,\Lambda}$ is any Cauchy-Riemann operator in V on the disk D with one outgoing strip-like end $(0,\infty) \times [0,1] \hookrightarrow D$ and with boundary conditions given by Λ (the projection of $\tilde{\Lambda}$ to $\mathrm{Lag}(V)$) such that $\Lambda(j) = \Lambda_j$ is the boundary condition over the boundary components $(0,\infty) \times \{j\}$, $j = 0,1$ of the end.

Proof. It suffices to prove the index identity for a fixed path $\tilde{\Lambda}$. Indeed, if $\tilde{\Lambda}'$ is any other path with the same endpoints then we have $\mathrm{Ind}(D_{V,\Lambda}) - \mathrm{Ind}(D_{V,\Lambda'}) = \mathrm{Ind}(D_{V,\Lambda}) + \mathrm{Ind}(D_{V,-\Lambda'}) = \mathrm{Ind}(D_{V,\Lambda\#(-\Lambda')})$ by gluing. Here the last Cauchy-Riemann operator is defined on the disk with no punctures and with boundary conditions given by the loop $\Lambda\#(-\Lambda')$. Since the loop lifts to a loop $\tilde{\Lambda}\#(-\tilde{\Lambda}')$ in $\mathrm{Lag}^N(V)$, its Maslov index (and thus index) is 0 modulo N.

By Remark 2.2.6, the degree can be defined by a path $\tilde{\Lambda}$ from $\tilde{\Lambda}_1$ to $\tilde{\Lambda}_0$ whose projection Λ has negative definite crossing form at $s = 1$. The sum of crossing numbers in $d(\tilde{\Lambda}_0, \tilde{\Lambda}_1) = \sum_{s\in(0,1)} \mathrm{sign}(\Gamma(\Lambda, \Lambda_0, s))$ is the Maslov index $I_H(\Lambda)$ in [37, Lemma 11.11] and hence equals to the Fredholm index $\mathrm{Ind}(D_{V,\Lambda})$ over the half space, or the conformally equivalent disk with strip-like end. This conformal isomorphism takes the boundary ends $(-\infty,-1)$ resp. $(1,\infty)$ in the half space $\{\mathrm{Im}\, z \geq 0\}$ (over which Λ equals to Λ_1 resp. Λ_0) to $\{1\} \times (1,\infty)$ resp. $\{0\} \times (1,\infty)$ in the strip-like end. □

Lemma 2.2.8. *The degree map satisfies the following properties.*

(a) *(Additivity) If $V = V' \times V''$ then*

$$d(\tilde{\Lambda}'_0 \times^N \tilde{\Lambda}''_0, \tilde{\Lambda}'_1 \times^N \tilde{\Lambda}''_1) = d(\tilde{\Lambda}'_0, \tilde{\Lambda}'_1) + d(\tilde{\Lambda}''_0, \tilde{\Lambda}''_1)$$

for $\tilde{\Lambda}'_j, \tilde{\Lambda}''_j$ graded Lagrangian subspaces in V', V'' respectively, $j = 0,1$.

(b) *(Multiplicativity) For $\tilde{\Lambda}_0, \tilde{\Lambda}_1$ graded Lagrangian subspaces and any $c \in \mathbb{Z}_N$*

$$d(\tilde{\Lambda}_0, c \cdot \tilde{\Lambda}_1) = c + d(\tilde{\Lambda}_0, \tilde{\Lambda}_1).$$

(c) *(Skewsymmetry) For $\tilde\Lambda_0, \tilde\Lambda_1$ graded Lagrangian subspaces*
$$d(\tilde\Lambda_0,\tilde\Lambda_1) + d(\tilde\Lambda_1,\tilde\Lambda_0) = \dim\Lambda_0 = d(\tilde\Lambda_0,\tilde\Lambda_1) + d(\tilde\Lambda_0^-,\tilde\Lambda_1^-).$$

(d) *(Diagonal) For a transverse pair $\tilde\Lambda_0, \tilde\Lambda_1$ of graded Lagrangian subspaces in V and $\tilde\Delta$ the canonically graded diagonal in $V^- \times V$*
$$d(\tilde\Delta, \tilde\Lambda_0^- \times^N \tilde\Lambda_1) = d(\tilde\Lambda_0,\tilde\Lambda_1).$$

Proof. The first three properties are standard, see [38, Section 2d]. We prove the diagonal property to make sure all our sign conventions match up. For that purpose we fix $\tilde L \in \mathrm{Lag}^N(V)$ and choose the following paths $\tilde\gamma_{..}$ of graded Lagrangian subspaces (with underlying paths $\gamma_{..}$ of Lagrangian subspaces):

- $\tilde\gamma_0 : [-1,1] \to \mathrm{Lag}^N(V)$ from $\tilde\gamma_0(-1) = \tilde L$ to $\tilde\gamma_0(1) = \tilde\Lambda_0$ such that $\tilde\gamma_0|_{[-1,0]} \equiv \tilde L$,
- $\tilde\gamma_1 : [-1,1] \to \mathrm{Lag}^N(V)$ from $\tilde\gamma_1(-1) = \tilde L$ to $\tilde\gamma_1(1) = \tilde\Lambda_1$, such that $\gamma_1|_{[-1/2,0]} \equiv JL \pitchfork L$ and $\gamma_1|_{[-1,-1/2]}$ is a smoothing of $t \mapsto e^{\pi(1+t)J}L$.
- $\tilde\gamma : [-1,1] \to \mathrm{Lag}^N(V^- \times V)$ starting with $\tilde\gamma|_{[-1,-1/2]} = (\tilde\gamma_1^- \times^N \tilde\gamma_0)|_{[-1,-1/2]}$, ending at $\tilde\gamma|_{[0,1]} \equiv \tilde\Delta$, and such that $\gamma|_{[-\frac12,0]}$ is a smoothing of $t \mapsto \{((2t+1)x + Jy, x + (2t+1)Jy) | x,y \in L\}$. (The lift to graded subspaces matches up since $\gamma|_{[-1,0]}$ is exactly the path of (5) which defines $\tilde\Delta$ by connecting it to $\tilde L^- \times \tilde L$.)

Note that we have $I(\gamma_0,\gamma_1)|_{[-1,0]} = -\frac12 \dim\Lambda_0$ and $I(\gamma, \gamma_0^- \times \gamma_1)|_{[-1,0]} = I(\gamma_1^-, \gamma_0^-)|_{[-1,0]} + I(\gamma_0,\gamma_1)|_{[-1,0]} = -\dim\Lambda_0$ since $\gamma|_{[-\frac12,0]}$ is transverse to $L^- \times JL$. With these preparations we can calculate
$$\begin{aligned} d(\tilde\Lambda_0,\tilde\Lambda_1) &= \tfrac12 \dim\Lambda_0 + I(\gamma_0,\gamma_1) \;=\; I(\gamma_0,\gamma_1)\big|_{[0,1]} \\ &= I(\Delta, \gamma_0^- \times \gamma_1)\big|_{[0,1]} \\ &= \dim\Lambda_0 + I(\gamma, \gamma_0^- \times \gamma_1) \;=\; d(\tilde\Delta, \tilde\Lambda_0^- \times^N \tilde\Lambda_1). \end{aligned}$$
Here the identity of the Maslov indices over the interval $[0,1]$ follows from identifying the intersections $K(s) := \gamma_0 \cap \gamma_1 \cong \Delta \cap (\gamma_0^- \times \gamma_1)$ and the crossing forms $\Gamma(s), \hat\Gamma(s) : K(s) \to \mathbb{R}$ at regular crossings $s \in [0,1]$ (after a homotopy of the paths to regular crossings). Fix Lagrangian complements $\gamma_0(s)^c$ and $\gamma_1(s)^c$, then for $v \in K(s)$ pick $w_i(t) \in \gamma_i(s)^c$ such that $v + w_i(t) \in \gamma_i(s+t)$. For the corresponding vector $\hat v = (v,v) \in \Delta \cap (\gamma_0^- \times \gamma_1)$ we can pick $\hat w(t) = (0,0) \in \Delta^c$ satisfying $\hat v + \hat w(t) \in \Delta$ and $\hat w'(t) = (w_0, w_1) \in \gamma_0(s)^c \times \gamma_1(s)^c$ satisfying $\hat v + \hat w'(t) \in (\gamma_0 \times \gamma_1)(s+t)$ to identify the crossing forms
$$\begin{aligned} \hat\Gamma(s)\hat v &= \tfrac{d}{dt}\big|_{t=0} (-\omega \oplus \omega)(\hat v, \hat w(t) - \hat w'(t)) \\ &= \tfrac{d}{dt}\big|_{t=0} \big(-\omega(v, -w_0(t)) + \omega(v, -w_1(t))\big) \\ &= \tfrac{d}{dt}\big|_{t=0} \omega(v, w_0(t) - w_1(t)) = \Gamma(s)v. \end{aligned}$$
□

If $L_0, L_1 \subset M$ are $\mathrm{Lag}^N(M)$-graded Lagrangians and intersect transversally then one obtains a *degree map*
$$\mathcal{I}(L_0,L_1) := L_0 \cap L_1 \to \mathbb{Z}_N, \qquad x \mapsto |x| := d(\sigma_{L_0}^N(x), \sigma_{L_1}^N(x)).$$
More generally, if L_0, L_1 do not necessarily intersect transversally, then we can pick a Hamiltonian perturbation $H : [0,1] \times M \to \mathbb{R}$ such that its time 1 flow $\phi_1 : M \to M$ achieves transversality $\phi_1(L_0) \pitchfork L_1$. Then the Hamiltonian isotopy and the grading on L_0

induce a grading on $\phi_1(L_0)$, which is transverse to L_1. The degree map is then defined on the perturbed intersection points, $d : \mathcal{I}(L_0, L_1) := \phi_1(L_0) \cap L_1 \to \mathbb{Z}_N$.

2.3. **Graded Lagrangian correspondences.** In this section we extend the grading and degree constructions to generalized Lagrangian correspondences and discuss their behaviour under geometric composition and insertion of the diagonal.

Definition 2.3.1. Let M and M' be symplectic manifolds equipped with N-fold Maslov coverings. Let $\underline{L} = (L_{01}, \dots, L_{(r-1)r})$ be a generalized Lagrangian correspondence from M to M' (i.e. $L_{(j-1)j} \subset M_{j-1}^- \times M_j$ for a sequence $M = M_1, \dots, M_r = M'$ of symplectic manifolds). A *grading* on $\underline{L}$ consists of a collection of N-fold Maslov covers $\mathrm{Lag}^N(M_j) \to M_j$ and gradings of the Lagrangian correspondences $L_{(j-1)j}$ with respect to $\mathrm{Lag}^N(M_{j-1}^- \times M_j)$, where the Maslov covers on $M_1 = M$ and $M_r = M'$ are the fixed ones.

A pair of graded generalized Lagrangian correspondences $\underline{L_1}$ and $\underline{L_2}$ from M to M' (with fixed Maslov coverings) defines a cyclic Lagrangian correspondence $\underline{L_1}\#(\underline{L_2})^t$, which is graded in the following sense.

Definition 2.3.2. Let $\underline{L} = (L_{01}, \dots, L_{r(r+1)})$ be a cyclic generalized Lagrangian correspondence (i.e. $L_{j(j+1)} \subset M_j^- \times M_{j+1}$ for a cyclic sequence $M_0, M_1, \dots, M_{r+1} = M_0$ of symplectic manifolds). An *N-grading* on $\underline{L}$ consists of a collection of N-fold Maslov covers $\mathrm{Lag}^N(M_j) \to M_j$ and gradings of the Lagrangian correspondences $L_{j(j+1)}$ with respect to $\mathrm{Lag}^N(M_j^- \times M_{j+1})$.

In the following, we will consider a cyclic generalized Lagrangian correspondence $\underline{L}$ and assume that it intersects the generalized diagonal transversally, i.e.

$$\big(L_{01} \times L_{12} \times \ldots \times L_{r(r+1)}\big) \pitchfork \big(\Delta_{M_0}^- \times \Delta_{M_1}^- \times \ldots \times \Delta_{M_r}^-\big)^T, \tag{11}$$

where $\Delta_M^- \subset M \times M^-$ denotes the (dual of the) diagonal and $M_0 \times M_0^- \times M_1 \times \ldots \times M_r^- \to M_0^- \times M_1 \times \ldots \times M_r^- \times M_0$, $Z \mapsto Z^T$ is the transposition of the first to the last factor. In section 3.3 this transversality will be achieved by a suitable Hamiltonian isotopy. It ensures that the above transverse intersection cuts out a finite set, which we identify with the *generalized intersection points*

$$\begin{aligned}\mathcal{I}(\underline{L}) &:= \times_{\Delta_{M_0}}\big(L_{01} \times_{\Delta_{M_1}} L_{12} \ldots \times_{\Delta_{M_r}} L_{r(r+1)}\big)\\ &= \big\{\underline{x} = (x_0, \dots, x_r) \in M_0 \times \ldots \times M_r \,\big|\, (x_0, x_1) \in L_{01}, \dots, (x_r, x_0) \in L_{r(r+1)}\big\}.\end{aligned}$$

Remark 2.3.3. Consider two cyclic generalized Lagrangian correspondences

$$\begin{aligned}\underline{L} &= (L_{01}, \dots, L_{(j-1)j}, L_{j(j+1)}, \dots, L_{r(r+1)}),\\ \underline{L}' &= (L_{01}, \dots, L_{(j-1)j} \circ L_{j(j+1)}, \dots, L_{r(r+1)})\end{aligned}$$

that are equivalent in the category $\mathrm{Symp}^\#$, i.e. the composition $L_{(j-1)j} \circ L_{j(j+1)}$ is embedded in the sense of Definition 2.0.5. Then the generalized intersection points

$$\begin{aligned}\mathcal{I}(\underline{L}) &= \big\{(\dots, x_{j-1}, x_j, x_{j+1}, \dots) \in \ldots \times M_{j-1} \times M_j \times M_{j+1} \ldots \,\big|\\ &\qquad\qquad \dots, (x_{j-1}, x_j) \in L_{(j-1)j}, (x_j, x_{j+1}) \in L_{j(j+1)}, \dots\big\}\\ &= \big\{(\dots, x_{j-1}, x_{j+1}, \dots) \in \ldots \times M_{j-1} \times M_{j+1} \ldots \,\big|\\ &\qquad\qquad \dots, (x_{j-1}, x_{j+1}) \in L_{(j-1)j} \circ L_{j(j+1)}, \dots\big\} = \mathcal{I}(\underline{L}')\end{aligned}$$

are canonically identified, since the intermediate point $x_j \in M_j$ with $(x_{j-1}, x_j) \in L_{(j-1)j}$ and $(x_j, x_{j+1}) \in L_{j(j+1)}$ is uniquely determined by the pair $(x_{j-1}, x_{j+1}) \in L_{(j-1)j} \circ L_{j(j+1)}$.

Now an N-grading on $\underline{L}$ induces an N-fold Maslov covering on $M := M_0^- \times M_1 \times \ldots \times M_r \times M_r^- \times M_0$ and a grading of $L := L_{01} \times L_{12} \times \ldots \times L_{r(r+1)}$. In addition, we have a grading on $\Delta^T := (\Delta^-_{M_0} \times \Delta^-_{M_1} \times \ldots \times \Delta^-_{M_r})^T$ from the canonical grading on each factor. In order to define a degree we then identify generalized intersection points $\underline{x} = (x_0, x_1, \ldots, x_r)$ with the actual intersection points $x = (x_0, x_1, x_1, \ldots, x_r, x_r, x_0) \in L \cap \Delta^T$.

Definition 2.3.4. Let $\underline{L}$ be a graded cyclic generalized Lagrangian correspondence $\underline{L}$ that is transverse to the diagonal (11). Then the *degree* is

$$\mathcal{I}(\underline{L}) \to \mathbb{Z}_N, \qquad \underline{x} \mapsto |\underline{x}| = d(\sigma_L^N(x), \sigma^N_{\Delta^T}(x)).$$

Lemma 2.3.5. *Alternatively, the degree is defined as follows:*

(a) *Pick any tuple of Lagrangian subspaces $\Lambda_i' \in \mathrm{Lag}(T_{x_i}M_i)$, $\Lambda_i'' \in \mathrm{Lag}(T_{x_i}M_i^-)$, $i = 0, \ldots, r$ whose product is transverse to the diagonal, $\Lambda_i' \times \Lambda_i'' \pitchfork \Delta_{T_{x_i}M_i}$. Then there exists a path (unique up to homotopy) $\gamma : [0,1] \to \mathrm{Lag}(T_xM)$ from $\gamma(0) = T_xL$ to $\gamma(1) = \Lambda_0'' \times \Lambda_1' \times \ldots \times \Lambda_r' \times \Lambda_r'' \times \Lambda_0'$ that is transverse to the diagonal at all times, $\gamma(t) \pitchfork T_x\Delta^T$. We lift the grading $\sigma_L^N(x) \in \mathrm{Lag}^N(T_xM)$ along this path and pick preimages under the graded product map (8) to define $\tilde{\Lambda}_i' \in \mathrm{Lag}^N(T_{x_i}M_i)$ and $\tilde{\Lambda}_i'' \in \mathrm{Lag}^N(T_{x_i}M_i^-)$. Then*

$$|\underline{x}| = \sum_{i=0}^r d(\tilde{\Lambda}_i', \tilde{\Lambda}_i''^-).$$

(b) *If $\underline{L}$ has even length $r+1 \in 2\mathbb{N}$ then it defines an N-fold Maslov cover on $\widetilde{M} := M_0^- \times M_1 \times M_2^- \times \ldots \times M_r$ and a pair of graded Lagrangian submanifolds,*

$$L_{(0)} := L_{01} \times L_{23} \times \ldots \times L_{(r-1)r} \subset \widetilde{M},$$
$$L_{(1)} := (L_{12} \times L_{34} \times \ldots \times L_{r(r+1)})^T \subset \widetilde{M}^-,$$

where we denote by $M_1^- \times \ldots \times M_r^- \times M_0 \to M_0 \times M_1^- \times \ldots \times M_r^-$, $Z \mapsto Z^T$ the transposition of the last to the first factor. If $\underline{L}$ has odd length $r+1 \in 2\mathbb{N}+1$ we insert the diagonal $\Delta_{M_0} \subset M_0^- \times M_0 = M_{r+1}^- \times M_0$ (with its canonical grading) before defining a pair of graded Lagrangian submanifolds as above. By (11) the Lagrangians intersect transversally $L_{(0)} \pitchfork L_{(1)}^-$, and this intersection is canonically identified with $\mathcal{I}(\underline{L})$. Then for $\underline{x} \in \mathcal{I}(\underline{L})$ corresponding to $y \in L_{(0)} \cap L_{(1)}^-$ we have

$$|\underline{x}| = |y| = d(\sigma^N_{L_{(0)}}(y), \sigma^N_{L_{(1)}}(y)^-).$$

Proof. In (a) we use the fact that the path γ has zero Maslov index to rewrite

$$d(\sigma_L^N(x), \sigma^N_{\Delta^T}(x)) = d(\tilde{\Lambda}_0' \times^N \tilde{\Lambda}_0'' \times^N \ldots \times^N \tilde{\Lambda}_r' \times^N \tilde{\Lambda}_r'', \tilde{\Delta}^-_{T_{x_0}M_0} \times^N \ldots \times^N \tilde{\Delta}^-_{T_{x_r}M_r}),$$

where we moreover transposed the factors. Now by Lemma 2.2.8 the right hand side can be written as the sum over $d(\tilde{\Lambda}_i' \times^N \tilde{\Lambda}_i'', \tilde{\Delta}^-_{T_{x_i}M_i}) = d(\tilde{\Lambda}_i', \tilde{\Lambda}_i''^-)$.

In (b) note that a reordering of the factors identifies the pair of graded Lagrangians $(L_{(0)} \times L_{(1)}, \Delta^-_{\widetilde{M}})$ with (L, Δ^T) for r odd. So Lemma 2.2.8 implies

$$d(\sigma_L^N(x), \sigma^N_{\Delta^T}(x)) = d(\sigma^N_{L_{(0)}}(y) \times^N \sigma^N_{L_{(1)}}(y), \tilde{\Delta}^-_{T_{(y,y)}\widetilde{M}}) = d(\sigma^N_{L_{(0)}}(y), \sigma^N_{L_{(1)}}(y)^-).$$

For r even the same argument proves

$$d(\sigma^N_{L_{(0)}}(y), \sigma^N_{L_{(1)}}(y)^-) = d\big(\sigma^N_L(x) \times^N \tilde{\Delta}_{T_{x_0}M_0}, (\tilde{\Delta}^-_{T_{x_0}M_0} \times \ldots \times \tilde{\Delta}^-_{T_{x_r}M_r} \times \tilde{\Delta}^-_{T_{x_0}M_0})^T\big),$$

which equals to $d(\sigma^N_L(x), \sigma^N_{\Delta^T}(x))$ by Lemma 2.3.6 (b) below. □

The following Lemma describes the effect of inserting a diagonal on the grading of generalized Lagrangian correspondences. Part (a) addresses noncyclic correspondences, whereas (b) applies to cyclic correspondences with $\Lambda = T_{(x_0,x_1,\ldots,x_r,x_0)}(L_{01} \times L_{12} \times \ldots \times L_{r(r+1)})$, $K = T_{(x_0,x_0,x_1,\ldots,x_r)}(\Delta^-_{M_0} \times \Delta^-_{M_1} \times \ldots \times \Delta^-_{M_r})$, $V_0 = T_{x_0}M_0$, and $V_1 = T_{(x_1,\ldots,x_r)}(M_1 \times M_1^- \times \ldots \times M_r \times M_r^-)$.

Lemma 2.3.6. *Let V_0, V_1, V_2 be symplectic vector spaces.*

(a) *Let $\tilde{\Lambda}_0 \subset \mathrm{Lag}^N(V_0)$, $\tilde{\Lambda}_{01} \subset \mathrm{Lag}^N(V_0^- \times V_1)$, $\tilde{\Lambda}_{12} \subset \mathrm{Lag}^N(V_1^- \times V_2)$, and $\tilde{\Lambda}_2 \subset \mathrm{Lag}^N(V_2^-)$ be graded Lagrangian subspaces. If the underlying Lagrangian subspaces are transverse then*

$$d(\tilde{\Lambda}_0 \times^N \tilde{\Lambda}_{12}, \tilde{\Lambda}^-_{01} \times^N \tilde{\Lambda}^-_2) = d(\tilde{\Lambda}_0 \times^N \tilde{\Delta}_1 \times^N \tilde{\Lambda}_2, \tilde{\Lambda}^-_{01} \times^N \tilde{\Lambda}^-_{12}).$$

(b) *Let $\tilde{\Lambda} \subset \mathrm{Lag}^N(V_0^- \times V_1 \times V_0)$ and $\tilde{K} \subset \mathrm{Lag}^N(V_0 \times V_0^- \times V_1)$ be graded Lagrangian subspaces. If the underlying Lagrangian subspaces are transverse then*

$$d(\tilde{\Lambda} \times^N \tilde{\Delta}_0, (\tilde{K} \times^N \tilde{\Delta}^-_0)^T) = d(\tilde{\Lambda}, \tilde{K}^T),$$

with the transposition $V_0 \times W \to W \times V_0$, $Z \mapsto Z^T$.

Proof. To prove (a) pick a path $\gamma_{0112} : [0,1] \to \mathrm{Lag}(V_0 \times V_1^- \times V_1 \times V_2^-)$ from $\gamma_{0112}(0) = \Lambda^-_{01} \times \Lambda^-_{12}$ to a split Lagrangian subspace $\gamma_{0112}(1) = \Lambda'_0 \times \Lambda'_1 \times \Lambda''_1 \times \Lambda'_2$ that is transverse to $\Lambda_0 \times \Delta_1 \times \Lambda_2$ at all times and hence has Maslov index $I(\gamma_{0112}, \Lambda_0 \times \Delta_1 \times \Lambda_2) = 0$. We can homotope this path with fixed endpoints to $\gamma_{0112} = \gamma_{01} \times \gamma_{12} : [0,1] \to \mathrm{Lag}(V_0 \times V_1^-) \times \mathrm{Lag}(V_1 \times V_2^-)$ that may intersect $\Lambda_0 \times \Delta_1 \times \Lambda_2$ but still has vanishing Maslov index. We lift the grading along the paths γ_{01} and γ_{12} and pick preimages under the graded product map (8) to obtain gradings $\tilde{\Lambda}'_0 \in \mathrm{Lag}^N(V_0)$, $\tilde{\Lambda}'_1 \in \mathrm{Lag}^N(V_1^-)$, $\tilde{\Lambda}''_1 \in \mathrm{Lag}^N(V_1)$, $\tilde{\Lambda}'_2 \in \mathrm{Lag}^N(V_2^-)$. With these we calculate, using Lemma 2.2.8

$$\begin{aligned} d(\tilde{\Lambda}_0 \times^N \tilde{\Lambda}_{12}, \tilde{\Lambda}^-_{01} \times^N \tilde{\Lambda}^-_2) &= d(\tilde{\Lambda}_0 \times^N \tilde{\Lambda}''^-_1 \times^N \tilde{\Lambda}'^-_2, \tilde{\Lambda}'_0 \times^N \tilde{\Lambda}'_1 \times^N \tilde{\Lambda}^-_2) \\ &= d(\tilde{\Lambda}_0, \tilde{\Lambda}'_0) + d(\tilde{\Lambda}''^-_1, \tilde{\Lambda}'_1) + d(\tilde{\Lambda}'^-_2, \tilde{\Lambda}^-_2) \\ &= d(\tilde{\Lambda}_0, \tilde{\Lambda}'_0) + d(\tilde{\Delta}_1, \tilde{\Lambda}'_1 \times^N \tilde{\Lambda}''_1) + d(\tilde{\Lambda}_2, \tilde{\Lambda}'_2) \\ &= d(\tilde{\Lambda}_0 \times^N \tilde{\Delta}_1 \times^N \tilde{\Lambda}_2, \tilde{\Lambda}'_0 \times^N \tilde{\Lambda}'_1 \times^N \tilde{\Lambda}''_1 \times^N \tilde{\Lambda}'_2) \\ &= d(\tilde{\Lambda}_0 \times^N \tilde{\Delta}_1 \times^N \tilde{\Lambda}_2, \tilde{\Lambda}^-_{01} \times^N \tilde{\Lambda}^-_{12}). \end{aligned}$$

The first and last degree identity are due to the vanishing of the Maslov index

$$0 \;=\; I(\Lambda_0 \times \Delta_1 \times \Lambda_2, \gamma_{01} \times \gamma_{12}) \;=\; I(\Lambda_0 \times \gamma^-_{12}, \gamma_{01} \times \Lambda^-_2) = 0.$$

The identity of these Maslov indices follows from identifying the intersections $K(s) := (\Lambda_0 \times \gamma^-_{12}(s)) \cap (\gamma_{01}(s) \times \Lambda^-_2) \cong (\Lambda_0 \times \Delta_1 \times \Lambda_2) \cap (\gamma_{01} \times \gamma_{12})$ and the crossing form $\Gamma(s), \hat{\Gamma}(s) : K(s) \to \mathbb{R}$ given by (10) at regular crossings $s \in [0,1]$. Fix Lagrangian complements $\gamma_{01}(s)^c \subset V_0 \times V_1^-$ and $\gamma_{12}(s)^c \subset V_1 \times V_2^-$, then for $v = (v_0, v_1, v_2) \in K(s)$ we can pick $(w_1, w_2)(t) \in \gamma_{12}(s)^c$ such that $v + (0, w_1, w_2)(t) \in \Lambda_0 \times \gamma_{12}(s+t)$ and $(w'_0, w'_1)(t) \in \gamma_{01}(s)^c$ such that $v + (w'_0, w'_1, 0)(t) \in \gamma_{01}(s+t) \times \Lambda_2$. For the corresponding vector $\hat{v} = (v_0, v_1, v_1, v_2) \in$

$(\Lambda_0\times\Delta_1\times\Lambda_2^-)\cap(\gamma_{01}^-\times\gamma_{12}^-)$ we have $\hat v+(0,0,0,0)\in(\Lambda_0\times\Delta_1\times\Lambda_2)$ and $\hat v+(w_0',w_1',w_1,w_2)(t)\in(\gamma_{01}\times\gamma_{12})(s+t)$. With this we identify the crossing forms

$$\begin{aligned}\hat\Gamma(s)\hat v &= \tfrac{d}{dt}\big|_{t=0}(\omega_0\oplus-\omega_1\oplus\omega_1\oplus-\omega_2)\big(\hat v,(0,0,0,0)-(w_0',w_1',w_1,w_2)(t)\big)\\ &= \tfrac{d}{dt}\big|_{t=0}\big(-\omega_0(v_0,w_0')-\omega_1(v_1,w_1-w_1')+\omega_2(v_2,w_2)\big)\\ &= \tfrac{d}{dt}\big|_{t=0}(\omega_0\oplus-\omega_1\oplus\omega_2)\big(v,(0,w_1,w_2)(t)-(w_0',w_1',0)(t)\big)=\Gamma(s)v.\end{aligned}$$

This proves (a). To prove (b) we pick a path $\gamma:[0,1]\to\mathrm{Lag}(V_0^-\times V_1\times V_0)$ from $\gamma(0)=\Lambda$ to a split Lagrangian subspace $\gamma(1)=\Lambda_0^-\times\Lambda_1\times\Lambda_0'\in\mathrm{Lag}(V_0^-)\times\mathrm{Lag}(V_1)\times\mathrm{Lag}(V_0)$ that is transverse to K^T at all times and hence has Maslov index

$$0=I(\gamma,K^T)=I(\gamma\times\Delta_0,(K\times\Delta_0^-)^T).$$

Here the equality of Maslov follows directly from the identification of the trivial intersections $(\gamma\times\Delta_0)\cap(K\times\Delta_0^-)^T\cong\gamma\cap K^T=\{0\}$. Now we can lift the grading along γ to obtain gradings $\tilde\Lambda_0\in\mathrm{Lag}^N(V_0)$, $\tilde\Lambda_1\in\mathrm{Lag}^N(V_1)$, $\tilde\Lambda_0'\in\mathrm{Lag}^N(V_0)$. With these we calculate, using part (a) and the fact that gradings are invariant under simultaneous transposition of both factors

$$\begin{aligned}d(\tilde\Lambda\times^N\tilde\Delta_0,(\tilde K\times^N\tilde\Delta_0^-)^T) &= d(\tilde\Lambda_0^-\times^N\tilde\Lambda_1\times^N\tilde\Lambda_0'\times^N\tilde\Delta_0,(\tilde K\times^N\tilde\Delta_0^-)^T)\\ &= d(\tilde\Lambda_0'\times^N\tilde\Delta_0\times^N\tilde\Lambda_0^-\times^N\tilde\Lambda_1,\tilde\Delta_0^-\times^N\tilde K)\\ &= d(\tilde\Lambda_0'\times^N\tilde K^-,\tilde\Delta_0^-\times^N(\tilde\Lambda_0^-\times^N\tilde\Lambda_1)^-)\\ &= d(\tilde\Delta_0\times^N(\tilde\Lambda_0^-\times^N\tilde\Lambda_1),\tilde\Lambda_0'^-\times^N\tilde K)\\ &= d(\tilde K^-,\tilde\Lambda_0'^-\times^N(\tilde\Lambda_0^-\times^N\tilde\Lambda_1)^-)\\ &= d(\tilde\Lambda_0'\times^N\tilde\Lambda_0^-\times^N\tilde\Lambda_1,\tilde K)\\ &= d(\tilde\Lambda_0^-\times^N\tilde\Lambda_1\times^N\tilde\Lambda_0',\tilde K^T)\ =\ d(\tilde\Lambda,\tilde K^T)\end{aligned}$$

□

In the rest of this section we investigate the effect of Weinstein composition on the grading of Lagrangian correspondences. This requires a generalization of Viterbo's index calculations [43].

First, we lift the composition map to Maslov covers. Let M_0,M_1,M_2 be symplectic manifolds equipped with N-fold Maslov coverings $\mathrm{Lag}^N(M_j)$, $j=0,1,2$. We equip the products $M_i^-\times M_j$ and $M_0^-\times M_1\times M_1^-\times M_2$ with the induced Maslov coverings $\mathrm{Lag}^N(M_i^-\times M_j)$ resp. $\mathrm{Lag}^N(M_0^-\times M_1\times M_1^-\times M_2)$. We denote by

$$\mathcal T(M_1)\subset\mathrm{Lag}(M_0^-\times M_1\times M_1^-\times M_2)\big|_{M_0\times\Delta_{M_1}\times M_2}$$

the subbundle whose fibre over (m_0,m_1,m_1,m_2) consists of the Lagrangian subspaces $\Lambda_{0112}\subset T_{(m_0,m_1,m_1,m_2)}(M_0^-\times M_1\times M_1^-\times M_2)$ that are transverse to the diagonal $\Delta_{0112}:=T_{m_0}M_0\times\Delta_{T_{m_1}M_1}\times T_{m_2}M_2$. The linear composition of Lagrangian subspaces extends a smooth map

$$\circ:\mathcal T(M_1)\to\mathrm{Lag}(M_0^-\times M_2),\qquad \Lambda_{0112}\mapsto\pi_{M_0\times M_2}\big(\Lambda_{0112}\cap\Delta_{0112}\big).$$

The preimage of $\mathcal T(M_1)$ in the Maslov cover will be denoted by

$$\mathcal T^N(M_1)\subset\mathrm{Lag}^N(M_0^-\times M_1\times M_1^-\times M_2)\big|_{M_0\times\Delta_{M_1}\times M_2}.$$

Finally, recall that we have a canonical grading of the diagonal $\tilde{\Delta}_{M_1} \in \mathrm{Lag}^N(M_1^- \times M_1)$ and its dual $\tilde{\Delta}_{M_1}^- \in \mathrm{Lag}^N(M_1 \times M_1^-)$, and let us denote another exchange of factors by $\mathrm{Lag}^N(M_0^- \times M_2 \times M_1 \times M_1^-) \to \mathrm{Lag}^N(M_0^- \times M_1 \times M_1^- \times M_2)$, $\tilde{\Lambda} \mapsto \tilde{\Lambda}^T$.

Lemma 2.3.7. *The linear composition $\circ : \mathcal{T}(M_1) \to \mathrm{Lag}(M_0^- \times M_2)$ lifts to a unique smooth map $\circ^N : \mathcal{T}^N(M_1) \to \mathrm{Lag}^N(M_0^- \times M_2)$ with the property that*

$$\circ^N\big((\tilde{\Lambda}_{02} \times^N \tilde{\Lambda}_{11})^T\big) = d(\tilde{\Lambda}_{11}, \tilde{\Delta}_{M_1}^-) \cdot \tilde{\Lambda}_{02}. \tag{12}$$

for all graded Lagrangians $\tilde{\Lambda}_{02} \in \mathrm{Lag}^N(M_0^- \times M_2)$ and $\tilde{\Lambda}_{11} \in \mathrm{Lag}^N(M_1 \times M_1^-)$, such that the underlying Lagrangian $\Lambda_{11} \in \mathrm{Lag}(M_1 \times M_1^-)$ is transverse to the diagonal.

Proof. We denote by $\mathrm{Lag}(\mathbb{R}^{2n})$ the Lagrangian Grassmannian in $\mathbb{R}^{2n}$, write $\dim M_i = 2n_i$, and abbreviate $\mathbb{R}_{0112} := \mathbb{R}^{2n_0,-} \times \mathbb{R}^{2n_1} \times \mathbb{R}^{2n_1,-} \times \mathbb{R}^{2n_2}$. Let $\mathcal{T} \subset \mathrm{Lag}(\mathbb{R}_{0112})$ be the subset of Lagrangian subspaces meeting the diagonal $\mathbb{R}^{2n_0} \times \Delta_{\mathbb{R}^{2n_1}} \times \mathbb{R}^{2n_2}$ transversally. The linear composition map

$$\mathrm{Lag}(\mathbb{R}_{0112}) \supset \mathcal{T} \to \mathrm{Lag}(\mathbb{R}^{2n_0,-} \times \mathbb{R}^{2n_2}), \qquad \Lambda \mapsto \pi_{\mathbb{R}^{2n_0} \times \mathbb{R}^{2n_2}}\big(\Lambda \cap (\mathbb{R}^{2n_0} \times \Delta_{\mathbb{R}^{2n_1}} \times \mathbb{R}^{2n_2})\big)$$

is $\mathrm{Sp}(2n_0) \times \mathrm{Sp}(2n_1) \times \mathrm{Sp}(2n_2)$-equivariant, and lifts to a unique $\mathrm{Sp}^N(2n_0) \times \mathrm{Sp}^N(2n_1) \times \mathrm{Sp}^N(2n_2)$-equivariant map

$$\mathrm{Lag}^N(\mathbb{R}_{0112}) \supset \mathcal{T}^N \to \mathrm{Lag}^N(\mathbb{R}^{2n_0,-} \times \mathbb{R}^{2n_2}) \tag{13}$$

with the property (12). On the other hand, the restriction of $\mathrm{Fr}(M_0) \times \mathrm{Fr}(M_1) \times \mathrm{Fr}(M_1) \times \mathrm{Fr}(M_2)$ to $M_0 \times \Delta_{M_1} \times M_2$ admits a reduction of the structure group to $\mathrm{Sp}(2n_0) \times \mathrm{Sp}(2n_1) \times \mathrm{Sp}(2n_2)$, and similarly the restriction

$$\mathrm{Fr}^N_{0112} := \big(\mathrm{Fr}^N(M_0) \times \mathrm{Fr}^N(M_1) \times \mathrm{Fr}^N(M_1) \times \mathrm{Fr}^N(M_2)\big)\big|_{M_0 \times \Delta_{M_1} \times M_2}$$

admits a reduction of the structure group to $\mathrm{Sp}^N(2n_0) \times \mathrm{Sp}^N(2n_1) \times \mathrm{Sp}^N(2n_2)$. This group acts on $\mathrm{Lag}^N(\mathbb{R}_{0112})$ by the diagonal action of $\mathrm{Sp}^N(2n_1)$ on $\mathbb{R}^{2n_1} \times \mathbb{R}^{2n_1,-}$. Finally, we use the associated fiber bundle construction to identify

$$\begin{aligned}
&\mathrm{Lag}^N(M_0^- \times M_1 \times M_1^- \times M_2)\big|_{M_0 \times \Delta_{M_1} \times M_2} \\
&\cong \mathrm{Fr}^N_{0112} \times_{\mathrm{Sp}^N(2n_0) \times \mathrm{Sp}^N(2n_1) \times \mathrm{Sp}^N(2n_1) \times \mathrm{Sp}^N(2n_2)} \mathrm{Lag}^N(\mathbb{R}_{0112}) \\
&\cong \big(\mathrm{Fr}^N(M_0) \times \mathrm{Fr}^N(M_1) \times \mathrm{Fr}^N(M_2)\big) \times_{\mathrm{Sp}^N(2n_0) \times \mathrm{Sp}^N(2n_1) \times \mathrm{Sp}^N(2n_2)} \mathrm{Lag}^N(\mathbb{R}_{0112})
\end{aligned}$$

and

$$\mathrm{Lag}^N(M_0^- \times M_2) = \big(\mathrm{Fr}^N(M_0^-) \times \mathrm{Fr}^N(M_2)\big) \times_{\mathrm{Sp}^N(2n_0) \times \mathrm{Sp}^N(2n_2)} \mathrm{Lag}^N(\mathbb{R}^{2n_0,-} \times \mathbb{R}^{2n_2}).$$

Then the forgetful map on the first factor and the equivariant map (13) on the second factor define the unique lift $\circ^N$. □

Now consider two graded Lagrangian correspondences $L_{01} \subset M_0^- \times M_1$ and $L_{12} \subset M_1^- \times M_2$ and suppose that the composition $L_{01} \circ L_{12} =: L_{02} \subset M_0^- \times M_2$ is smooth and embedded. The canonical section $\sigma_{L_{02}} : L_{02} \to \mathrm{Lag}(M_0^- \times M_2)$ is given by the linear composition $\circ$ applied to $(\sigma_{L_{01}} \times \sigma_{L_{12}})|_{L_{01} \times_{\Delta_{M_1}} L_{12}}$. The gradings $\sigma^N_{L_{01}}$, $\sigma^N_{L_{12}}$ induce a grading on L_{02},

$$\sigma^N_{L_{02}} := \circ^N\big(\sigma^N_{L_{01}} \times^N \sigma^N_{L_{12}}\big)\big|_{L_{01} \times_{\Delta_{M_1}} L_{12}}, \tag{14}$$

where the map $\times^N$ is defined in (8) and we identify $L_{02} \cong L_{01} \times_{\Delta_{M_1}} L_{12}$.

Proposition 2.3.8. *Let $L_0 \subset M_0$, $L_{01} \subset M_0^- \times M_1$, $L_{12} \subset M_1^- \times M_2$, and $L_2 \subset M_2^-$ be graded Lagrangians such that the composition $L_{01} \circ L_{12} =: L_{02}$ is embedded. Then, with respect to the induced grading on L_{02}, the degree map $\mathcal{I}(L_0 \times L_2, L_{02}) \to \mathbb{Z}_N$ is the pull-back of the degree map $\mathcal{I}(L_0 \times L_{12}, L_{01} \times L_2) \to \mathbb{Z}_N$ under the canonical identification [3] of intersection points.*

Proof. Suppose for simplicity that Hamiltonian perturbations have been applied to the Lagrangians L_0, L_2 such that $\mathcal{I}(L_0 \times L_2, L_{02})$ (and hence also $\mathcal{I}(L_0 \times L_{12}, L_{01} \times L_2)$) is the intersection of transverse Lagrangians. Then we need to consider $(m_0, m_1, m_2) \in (L_0 \times L_{12}) \cap (L_{01} \times L_2)$, which corresponds to $(m_0, m_2) \in (L_0 \times L_2) \cap L_{02}$. We abbreviate the tangent spaces of the Lagrangians by $\Lambda_j = T_{m_j} L_j$, $\Lambda_{ij} = T_{(m_i,m_j)} L_{ij}$, and $\Delta_1 = \Delta_{T_{m_1} M_1}$ and their graded lifts by $\tilde{\Lambda}_j = \sigma^N_{L_j}(m_j)$, $\tilde{\Lambda}_{ij} = \sigma^N_{L_{ij}}(m_i, m_j)$, and $\tilde{\Delta}_1 = \tilde{\Delta}_{T_{m_1} M_1}$. Then we claim that

$$
\begin{aligned}
d(\tilde{\Lambda}_0 \times^N \tilde{\Lambda}_{12}, \tilde{\Lambda}_{01}^- \times^N \tilde{\Lambda}_2^-) &= d(\tilde{\Lambda}_0 \times^N \tilde{\Delta}_1 \times^N \tilde{\Lambda}_2, \tilde{\Lambda}_{01}^- \times^N \tilde{\Lambda}_{12}^-) \\
&= d(\tilde{\Lambda}_0 \times^N \tilde{\Lambda}_2, \tilde{\Lambda}_{01}^- \circ^N \tilde{\Lambda}_{12}^-).
\end{aligned} \tag{15}
$$

The first identity is Lemma 2.3.6. To prove (15) we begin by noting the transverse intersection $\Lambda_{02} \pitchfork \Lambda_0 \times \Lambda_2$. We denote $\tilde{\Lambda}_{02} := \tilde{\Lambda}_{01} \circ^N \tilde{\Lambda}_{12}$ (hence $\tilde{\Lambda}_{02}^- = \tilde{\Lambda}_{01}^- \circ^N \tilde{\Lambda}_{12}^-$) and pick a path $\tilde{\gamma}_{02} : [0,1] \to \mathrm{Lag}^N(T_{m_0} M_0^- \times T_{m_2} M_2)$ from $\tilde{\gamma}_{02}(0) = \tilde{\Lambda}_0 \times^N \tilde{\Lambda}_2$ to $\tilde{\gamma}_{02}(1) = \tilde{\Lambda}_{02}^-$ whose crossing form with $\Lambda_0 \times \Lambda_2$ at $s = 0$ is positive definite and hence by Remark 2.2.6

$$
d(\tilde{\Lambda}_0 \times^N \tilde{\Lambda}_2, \tilde{\Lambda}_{02}^-) = -I'(\gamma_{02}, \Lambda_0 \times \Lambda_2).
$$

Here I' denotes the Maslov index of a pair of paths (the second one is constant), not counting crossings at the endpoints. Next, fix a complement $L_{11} \in \mathrm{Lag}(T_{(m_1,m_1)} M_1 \times M_1^-)$ of the diagonal. Then both $(\Lambda_{02} \times^N L_{11})^T$ and $\Lambda_{01} \times \Lambda_{12}$ are transverse to $T_{m_0} M_0 \times \Delta_1 \times T_{m_2} M_2$ and their composition is Λ_{02}. By Lemma 2.3.9 below we then find a path γ_{0112} and lift it to $\tilde{\gamma}_{0112} : [0,1] \to \mathrm{Lag}^N(T_{(m_0,m_1,m_1,m_2)} M_0^- \times M_1 \times M_1^- \times M_2)$ from $\tilde{\gamma}_{0112}(0) = [\tilde{\Lambda}_{02} \times^N \tilde{L}_{11}]^T$ to $\tilde{\gamma}_{0112}(1) = \tilde{\Lambda}_{01} \times^N \tilde{\Lambda}_{12}$ whose composition $\circ(\gamma_{0112}) = \Lambda_{02}$ is constant and that has no crossings with $\Lambda_0 \times \Delta_1 \times \Lambda_2$ (by the transversality $\gamma_{0112} \cap (\Lambda_0 \times \Delta_1 \times \Lambda_2) = \Lambda_{02} \cap (\Lambda_0 \times \Lambda_2) = \{0\}$). Here the grading of $\tilde{L}_{11}$ is determined by continuation along this path. Since the composition $\circ(\gamma_{0112})$ is constant this continuation yields

$$
\tilde{\Lambda}_{02} = \circ^N(\tilde{\gamma}_{0112}) = \circ^N((\tilde{\Lambda}_{02} \times^N \tilde{L}_{11})^T) = d(\tilde{L}_{11}, \tilde{\Delta}_1^-) \cdot \tilde{\Lambda}_{02}.
$$

Here we also used (12), and we deduce that $d(\tilde{L}_{11}, \tilde{\Delta}_1^-) = 0 \mod N$. Furthermore, we fix a path $\tilde{\gamma}_{11} : [0,1] \to \mathrm{Lag}^N(T_{(m_1,m_1)} M_1^- \times M_1)$ from $\tilde{\gamma}_{11}(0) = \tilde{\Delta}_1$ to $\tilde{\gamma}_{11}(1) = \tilde{L}_{11}^-$ whose crossing form with Δ_1 at $s = 0$ is positive definite, and thus

$$
-I'(\gamma_{11}, \Delta_1) = d(\tilde{\Delta}_1, \tilde{L}_{11}^-) = d(\tilde{L}_{11}, \tilde{\Delta}_1^-) = 0 \mod N.
$$

[3] Here it suffices to allow for Hamiltonian perturbation on M_0 and M_2, i.e. replacing L_0, L_2 with $L_0' := \phi_1^{H_0}(L_0)$, $L_2' := (\phi_1^{H_2})^{-1}(L_2)$. Then for every $(m_0, m_2) \in (L_0' \times L_2') \cap L_{02}$ there is a unique $m_1 \in M_1$ such that $(m_0, m_1) \in L_{01}$, $(m_1, m_2) \in L_{12}$, and hence $(m_0, m_1, m_2) \in (L_0' \times L_{12}) \cap (L_{01} \times L_2')$. The same identification will be used in (35).

Now the concatenated path $(\tilde\gamma_{02}\times\tilde\gamma_{11})^T\#\tilde\gamma_{0112}^-$ connects $\tilde\Lambda_0\times^N\tilde\Delta_1\times^N\tilde\Lambda_2$ to $\tilde\Lambda_{01}^-\times^N\tilde\Lambda_{12}^-$ with positive definite crossing form at $s=0$, and (15) can be verified,

$$\begin{aligned}
&d(\tilde\Lambda_0\times^N\tilde\Delta_1\times^N\tilde\Lambda_2,\tilde\Lambda_{01}^-\times^N\tilde\Lambda_{12}^-)\\
&=-I'((\gamma_{02}\times\gamma_{11})^T\#\gamma_{0112}^-,\Lambda_0\times\Delta_1\times\Lambda_2)\\
&=-I'(\gamma_{02},\Lambda_0\times\Lambda_2)-I'(\gamma_{11},\Delta_1)-I'(\gamma_{0112}^-,\Lambda_0\times\Delta_1\times\Lambda_2)\\
&=-I'(\gamma_{02},\Lambda_0\times\Lambda_2)=d(\tilde\Lambda_0\times^N\tilde\Lambda_2,\tilde\Lambda_{02}^-).
\end{aligned}$$

□

Lemma 2.3.9. *Let V_0,V_1,V_2 be symplectic vector spaces, $\Lambda_{02}\subset V_0^-\times V_2$ a Lagrangian subspace, and denote by*

$$\mathcal{T}_{\Lambda_{02}}\subset\mathrm{Lag}(V_0^-\times V_1\times V_1^-\times V_2)$$

the subset of Lagrangian subspaces $\Lambda\subset V_0^-\times V_1\times V_1^-\times V_2$ with $\Lambda\pitchfork(V_0\times\Delta_{V_1}\times V_2)=:\hat\Lambda_{02}$ and $\pi_{02}(\hat\Lambda_{02})=\Lambda_{02}$. Then $\mathcal{T}_{\Lambda_{02}}$ is contractible.

Proof. We fix metrics on V_0, V_1, and V_2. Then we will construct a contraction $(\rho_t)_{t\in[0,1]}$, $\rho_t:\mathcal{T}_{\Lambda_{02}}\to\mathcal{T}_{\Lambda_{02}}$ with $\rho_0=\mathrm{Id}$ and $\rho_1\equiv\Psi(\Lambda_{02}\times(\Delta_1)^\perp)$, where $\Psi:V_0^-\times V_2\times V_1\times V_1^-\to V_0^-\times V_1\times V_1^-\times V_2$ exchanges the factors. To define $\rho_t(\Lambda)$ we write $\Lambda=\hat\Lambda_{02}\oplus\hat\Lambda_{11}$, where $\hat\Lambda_{11}$ is the orthogonal complement of $\hat\Lambda_{02}$ in Λ. Now $\hat\Lambda_{02}$ is the image of $(\mathrm{Id}_{V_0},i_1,i_1,\mathrm{Id}_{V_2}):\Lambda_{02}\to V_0^-\times V_1\times V_1^-\times V_2$ for a linear map $i_1:\Lambda_{02}\to V_1$ and $\hat\Lambda_{11}$ is the image of $(j_0,\mathrm{Id}_{V_1}+j_1,-\mathrm{Id}_{V_1}+j_1,j_2):V_1\to V_0^-\times V_1\times V_1^-\times V_2$ for linear maps $j_i:V_1\to V_i$. One can check that

$$\rho_t(\Lambda):=\mathrm{im}\big(\mathrm{Id}_{V_0},t\cdot i_1,t\cdot i_1,\mathrm{Id}_{V_2}\big)\oplus\mathrm{im}\big(t\cdot j_0,\mathrm{Id}_{V_1}+t^2\cdot j_1,-\mathrm{Id}_{V_1}+t^2\cdot j_1,t\cdot j_2\big)$$

is an element of $\mathcal{T}_{\Lambda_{02}}$ for all $t\in[0,1]$ and defines a smooth contraction. □

Finally, we identify the index on the two complexes in Theorem 1.0.1, using a result of Viterbo.

Lemma 2.3.10. *Let $L_0\subset M_0$, $L_{01}\subset M_0^-\times M_1$, $L_{12}\subset M_1^-\times M_2$, and $L_2\subset M_2^-$ be graded Lagrangians such that the composition $L_{01}\circ L_{12}=:L_{02}$ is embedded. Consider a map $u_{02}=(u_0,u_2):\mathbb{R}\times[0,1]\to M_0\times M_2$ taking boundary values in $(L_0\times L_2,L_{01}\circ L_{12})$, and limiting to constants $\underline{x}^\pm$ as $s\to\pm\infty$. Let $u_{012}=(u_0,\bar u_1,u_2):\mathbb{R}\times[0,1]\to M_0\times M_1\times M_2$ be the corresponding map, which takes boundary values in $(L_0\times L_{12},L_{01}\times L_2)$ and satisfies $\partial_t\bar u_1=0$. Then the Maslov-Viterbo indices are equal, $I(u_{02})=I(u_{012})$.*

Proof. We can homotope (u_0,u_2) and simultaneously $\bar u_1$ to maps that are constant outside of the compact subset $[0,1]\times[0,1]\subset\mathbb{R}\times[0,1]$. Let $u_{0112}=(u_0,\overline{u}_1,\overline{u}_1,u_2^T)$, where $u_2^T(s,t)=u_2(s,\frac12-t)$, with boundary in $(L_0\times\Delta_1\times L_2,L_{01}\times L_{12})$. Then we have

$$I(u_{0112})=I(u_{02})$$

by a special case of [43, Proposition 3], applied to the coisotropic submanifold $Q=M_0\times\Delta_1\times M_2$. To prove the Lemma it remains to identify the Maslov indices

$$I(u_{0112})=I(u_{012}).$$

For that purpose we need to consider the paths of Lagrangian subspaces given by $\gamma_0(s)=T_{u_0(s,0)}L_0$, $\gamma_{01}(s)=T_{u_0(s,1),\overline{u}(s)}L_{01}$, $\gamma_{12}(s)=T_{\overline{u}(s),u_2(s,0)}L_{12}$, $\gamma_2(s)=T_{u_2(s,1)}L_2$ for $s\in[0,1]$. Then the identity of Maslov indices $I(\gamma_0\times\gamma_{12},\gamma_{01}^-\times\gamma_2^-)=I(\gamma_0\times\Delta_1\times\gamma_2,\gamma_{01}^-\times\gamma_{12}^-)$ follows as in Lemma 2.3.6. □

3. Floer cohomology for Lagrangian correspondences

The main content of this section is a review of the construction of graded Floer cohomology in monotone and exact cases by Floer, Oh, and Seidel. In 3.3 we then formalize the extension of Floer cohomology to sequences of Lagrangian correspondences.

3.1. **Monotonicity.** Let (M,ω) be a symplectic manifold. Let $\mathcal{J}(M,\omega)$ denote the space of compatible almost complex structures on (M,ω). Any $J \in \mathcal{J}(M,\omega)$ gives rise to a complex structure on the tangent bundle TM; the first Chern class $c_1(TM) \in H^2(M,\mathbb{Z})$ is independent of the choice of J. Throughout, we will use the following standing assumptions on all symplectic manifolds:

(M1): (M,ω) is *monotone*, that is for some $\tau \geq 0$

$$[\omega] = \tau c_1(TM).$$

(M2): If $\tau > 0$ then M is compact. If $\tau = 0$ then M is (necessarily) noncompact but satisfies "bounded geometry" assumptions as in [37].

Note here that we treat the exact case $[\omega] = 0$ as special case of monotonicity (with $\tau = 0$). Next, we denote the index map by

$$c_1 : \pi_2(M) \to \mathbb{Z}, \quad u \mapsto (c_1, u_*[S^2]).$$

The *minimal Chern number* $N_M \in \mathbb{N}$ is the positive generator of its image.

Associated to a Lagrangian submanifold $L \subset M$ are the Maslov index and action (i.e. symplectic area) maps

$$I : \pi_2(M,L) \to \mathbb{Z}, \qquad A : \ \pi_2(M,L) \to \mathbb{R}.$$

Our standing assumptions on all Lagrangian submanifolds are the following:

(L1): L is *monotone*, that is

$$2A(u) = \tau I(u) \quad \forall u \in \pi_2(M,L)$$

where the $\tau \geq 0$ is (necessarily) that from (M1).

(L2): L is compact and oriented.

Any homotopy class $[u] \in \pi_2(M,L)$ that is represented by a nontrivial J-holomorphic curve $u : (D,\partial D) \to (M,L)$ (for D the unit disk) has positive action $A([u]) = \int u^*\omega > 0$. Monotonicity with $\tau > 0$ then implies that the index is also positive. So, for practical purposes, we define the (effective) *minimal Maslov number* $N_L \in \mathbb{N}$ as the generator of $I(\{[u] \in \pi_2(M,L) | A([u]) > 0\}) \subset \mathbb{N}$. If M and L are exact ($\tau = 0$), then no nonconstant holomorphic disks can exist, so we have $N_L = \infty$.

If the Lagrangian submanifold L is oriented then the index $I(u)$ is always even. So the orientation and monotonicity assumption on L imply $N_L \geq 2$. For most purposes[4] we could drop the orientation assumption and replace (L2) by

(L2'): L is compact and has minimal Maslov number $N_L \geq 2$.

[4]For nonoriented Lagrangians or Lagrangian correspondences all constructions and results in this paper extend directly to the ungraded Floer cohomologies with $\mathbb{Z}_2$-coefficients. We do not discuss orientations of moduli spaces in this case. The grading also directly extends if we drop the assumption (G2) of compatibility with orientations.

Now (L1) and (L2) or (L2') imply that any nontrivial holomorphic disk must have $I(u) \geq 2$, which excludes disk bubbling in transverse moduli spaces of index 0 and 1. In order for the Floer cohomology groups to be well defined we will also have to make the following additional assumption.

(L3): L has minimal Maslov number $N_L \geq 3$.

Moreover, we will restrict our considerations to Maslov coverings and gradings that are compatible with orientations, that is we make the following additional assumptions on the grading of the symplectic manifolds M and Lagrangian submanifolds $L \subset M$. (In the case $N = 2$ these assumptions reduce to (L2).)

(G1): M is equipped with a Maslov covering $\mathrm{Lag}^N(M)$ for N even, and the induced 2-fold Maslov covering $\mathrm{Lag}^2(M)$ is the one described in Example 2.2.4 (i).

(G2): L is equipped with a grading $\sigma_L^N : L \to \mathrm{Lag}^N(M)$, and the induced 2-grading $L \to \mathrm{Lag}^2(M)$ is the one given by the orientation of L.

In the following we discuss topological situations which ensure monotonicity.

Lemma 3.1.1. *Suppose that M is monotone and $L \subset M$ is a Lagrangian such that $\pi_1(L)$ is torsion (that is, every element has finite order). Then L is monotone and the minimal Maslov number is at least $2N_M/k$ where k is the maximum of orders of elements of $\pi_1(L)$.*

Proof. Let $u : (D, \partial D) \to (M, L)$ and let $k(u)$ be the order of the restriction of u to the boundary in $\pi_1(L)$. After passing to a $k(u)$-fold cover $\tilde{u}$, we may assume that the restriction of $\tilde{u}$ to ∂D is homotopically trivial in L. By adding the homotopy we obtain a sphere $v : S^2 \to M$ with

$$k(u)I(u) = I(\tilde{u}) = 2c_1(v)$$

divisible by $2N_M$. For the relation between the first Chern class and the Maslov index see e.g. [26, Appendix]. The similar identity for the actions (due to $\omega|_L = 0$) completes the proof. □

Definition 3.1.2. We say that a tuple $\underline{L} = (L_e)_{e\in\mathcal{E}}$ is *monotone* with monotonicity constant $\tau \geq 0$ if the following holds: Let Σ be any connected compact surface with nonempty boundary $\partial\Sigma = \sqcup_{e\in\mathcal{E}} C_e$ (with C_e possibly empty or disconnected). Then for every map $u : \Sigma \to M$ satisfying $u(C_e) \subset L_e$ we have the action-index relation

$$2\int u^*\omega = \tau I(u^*TM, (u^*TL_e)_{e\in\mathcal{E}}),$$

where I is the sum of the Maslov indices of the totally real subbundles $(u|_{C_e})^*TL_e$ in some fixed trivialization of u^*TM.

In practice, we will need the action-index relation only for finitely many surfaces. For example, the action-index relation for disks and annuli suffices to define Floer cohomology for a pair of Lagrangians. The following is a minor generalization of [29, Proposition 2.7].

Lemma 3.1.3. *If M is monotone, each $L_e \subset M$ is monotone, and the image of each $\pi_1(L_e)$ in $\pi_1(M)$ is torsion, then the tuple $(L_e)_{e\in\mathcal{E}}$ is monotone.*

Proof. Consider $u : \Sigma \to M$ satisfying $u(C_e) \subset L_e$. By assumption we have integers $N_e \in \mathbb{N}$ such that $N_e u|_{C_e}$ is contractible in M. Let $N = \prod_{e\in\mathcal{E}} N_e$, so that $Nu|_{C_e}$ is contractible for all boundary components C_e of Σ. Let $\tilde{\Sigma} \to \Sigma$ be a finite N-cover defined by a representation $\rho : \pi_1(\Sigma) \to \mathbb{Z}_N$ with $\rho([C_e]) = [N/N_e]$, so that each component of the inverse image $\tilde{C}_e$

of C_e is an N_e-fold cover. The pull-back $\tilde{u} : \tilde{\Sigma} \to M$ of $u : \Sigma \to M$ has restrictions to the boundary $\tilde{u}|_{\tilde{C}_e}$ that are homotopically trivial in M. Thus $\tilde{u}$ is homotopic to the union of some maps $v_e : (D, \partial D) \to (M, L_e)$ and a map $v : S \to M$ on a closed surface S. We can now use the closedness of ω and the monotonicity of M and each L_e to deduce

$$\begin{aligned} 2N \int u^*\omega \;=\; 2\int \tilde{u}^*\omega \;&=\; 2\int v^*\omega + \sum_{e\in\mathcal{E}} 2\int v_e^*\omega \\ &=\; 2\tau c_1(v^*TM) + \sum_{e\in\mathcal{E}} \tau I(v_e) \;=\; \tau I(\tilde{u}) \;=\; \tau N I(u), \end{aligned}$$

using properties of the Maslov index explained in [26, Appendix]. □

In the exact case, with $\omega = d\lambda$, any tuple of exact Lagrangians $(L_e)_{e\in\mathcal{E}}$, that is with $[\lambda|_{L_e}] = 0 \in H^1(L_e)$, is automatically monotone. Moreover, note that monotonicity is invariant under Hamiltonian isotopies of one or both Lagrangians.

Remark 3.1.4. Another situation in which one naturally has monotonicity is the Bohr-Sommerfeld setting, as pointed out to us by P. Seidel. Suppose that the cohomology class $[\omega]$ is integral. Let $(\mathcal{L}, \nabla) \to (M, \omega)$ be a unitary line-bundle-with-connection having curvature $(2\pi/i)\omega$. The restriction of $(\mathcal{L}, \nabla)$ to any Lagrangian $L \subset M$ is flat. L is *Bohr-Sommerfeld* if the restriction of $(\mathcal{L}, \nabla)$ to L is trivial, that is, there exists a non-zero horizontal section $\phi_L^{\mathcal{L}}$. The section $\phi_L^{\mathcal{L}}$ is unique up to a collection of phases $U(1)^{\pi_0(L)}$. Suppose that M is monotone, $[\omega] = \lambda c_1(M)$ for some $\lambda > 0$. Since $c_1(M)$ and $[\omega]$ are integral, we must have $\lambda = k/l$ for some integers $k, l > 0$. Let $\mathcal{K}^{-1} \to M$ denote the anticanonical bundle, $\mathcal{K}_m^{-1} = \Lambda_{\mathbb{C}}^{\mathrm{top}}(T_m^{0,1}M)$, which satisfies $kc_1(\mathcal{K}^{-1}) = l\frac{i}{2\pi}[\mathrm{curv}(\nabla)] = lc_1(\mathcal{L})$. We fix an isomorphism

$$\Phi : (\mathcal{K}^{-1})^{\otimes k} \to \mathcal{L}^{\otimes l}$$

Let $L \subset M$ be an oriented Lagrangian submanifold. The restriction of $\mathcal{K}^{-1}$ to L has a natural non-vanishing section $\phi_L^{\mathcal{K}}$ given by the orientation and the isomorphisms

$$\Lambda_{\mathbb{R}}^{\mathrm{top}} TL \to \Lambda_{\mathbb{C}}^{\mathrm{top}} T^{0,1}M|_L, \quad v_1 \wedge \ldots \wedge v_n \mapsto (v_1 + iJv_1) \wedge \ldots \wedge (v_n + iJv_n).$$

We say that L is *Bohr-Sommerfeld monotone* with respect to $(\mathcal{L}, \nabla, \Phi)$ if the sections $(\phi_L^{\mathcal{L}})^{\otimes l}$ and $\Phi \circ (\phi_L^{\mathcal{K}})^{\otimes k}$ are homotopic, that is, there exists a function $\psi : L \to \mathbb{R}$ such that

$$(\exp(2\pi i\psi)\phi_L^{\mathcal{L}})^{\otimes l} = \Phi \circ (\phi_L^{\mathcal{K}})^{\otimes k}. \tag{16}$$

Lemma 3.1.5. *Let $\underline{L} = (L_e)_{e\in\mathcal{E}}$ be a collection of Lagrangians such that each is Bohr-Sommerfeld monotone with respect to $(\mathcal{L}, \nabla, \Phi)$. Then $\underline{L}$ is monotone.*

Proof. Let Σ be a compact Riemann surface with boundary components $(C_e)_{e\in\mathcal{E}}$. Let $u : \Sigma \to M$ be a map with boundary $u(C_e) \subset L_e$. The index $I(u)$ is the sum of Maslov indices of the bundles $(u|_{C_e})^*TL_e$, with respect to some fixed trivialization of u^*TM. Equivalently, $I(u)$ is the sum of winding numbers of the sections $\phi_{L_e}^{\mathcal{K}}$ with respect to the induced trivialization of $u^*\mathcal{K}^{-1}$. Since each L_e is Bohr-Sommerfeld, $kI(u)$ is the sum of the winding numbers of the sections $(\phi_{L_e}^{\mathcal{L}})^{\otimes l}$, with respect to the induced trivialization of $u^*\mathcal{L}^{\otimes l}$. Write $u^*\nabla^{\otimes l} = \mathrm{d} + \alpha$ for some $\alpha \in \Omega^1(\Sigma)$ in this trivialization, so that $u^*\,\mathrm{curv}(\nabla^{\otimes l}) = \mathrm{d}\alpha$. Since the sections are horizontal, we have

$$kI(u) = (i/2\pi)\int_{\partial\Sigma} \alpha = (i/2\pi)\int_{\Sigma} u^*\,\mathrm{curv}(\nabla^{\otimes l}) = lA(u).$$

□

3.2. **Graded Floer cohomology.** Let $L_0, L_1 \subset M$ be compact Lagrangian submanifolds. For a time-dependent Hamiltonian $H \in C^\infty([0,1] \times M)$ let $(X_t)_{t\in[0,1]}$ denote the family of Hamiltonian vector fields for $(H_t)_{t\in[0,1]}$, and let $\phi_{t_0,t_1} : M \to M$ denote its flow. (That is, $\phi_{t_0,t_1}(y) = x(t_1)$, where $x : [0,1] \to M$ satisfies $\dot{x} = X_t(x)$, $x(t_0) = y$.) We will abbreviate $\phi_1 := \phi_{0,1}$ for the time 1 flow from $t_0 = 0$ to $t_1 = 1$. Let $\mathrm{Ham}(L_0, L_1)$ be the set of $H \in C^\infty([0,1] \times M)$ such that $\phi_1(L_0)$ intersects L_1 transversally. Then we have a finite set of perturbed intersection points

$$\mathcal{I}(L_0, L_1) := \big\{\gamma : [0,1] \to M \,\big|\, \gamma(t) = \phi_{0,t}(\gamma(0)),\ \gamma(0) \in L_0,\ \gamma(1) \in L_1\big\}.$$

It is isomorphic to the intersection $\phi_1(L_0) \pitchfork L_1$. If we assume that M and L_0, L_1 are graded as in (G1-2), then we obtain a degree map from Section 2.2,

$$\mathcal{I}(L_0, L_1) \to \mathbb{Z}_N, \qquad x \mapsto |x| = d(\sigma^N_{L_0}(x), \sigma^N_{L_1}(x)).$$

Since N is even the sign $(-1)^{|x|}$ is well-defined. It agrees with the usual sign in the intersection number, given by the orientations of $\phi_1(L_0)$ and L_1, which also determine the mod 2 grading by assumption.

Next, we denote the space of time-dependent ω-compatible almost complex structures by

$$\mathcal{J}_t(M, \omega) := \mathcal{C}^\infty([0,1], \mathcal{J}(M, \omega)).$$

For any $J \in \mathcal{J}_t(M, \omega)$ and $H \in \mathrm{Ham}(L_0, L_1)$ we say that a map $u : \mathbb{R} \times [0,1] \to M$ is (J, H)-*holomorphic* with Lagrangian boundary conditions if

$$\partial_s u(s,t) + J_{t,u(s,t)}(\partial_t u(s,t) - X_t(u(s,t))) = 0, \tag{17}$$

$$u(\mathbb{R}, 0) \subset L_0, \quad u(\mathbb{R}, 1) \subset L_1. \tag{18}$$

The (perturbed) energy of a solution is

$$E_H(u) := \int_{\mathbb{R}\times[0,1]} |\partial_s u|^2 = \int_{\mathbb{R}\times[0,1]} u^*\omega + \mathrm{d}(H(u)\mathrm{d}t).$$

The following exponential decay lemma will be needed later and is part of the proof of Theorem 3.2.2 below.

Lemma 3.2.1. *Let $H \in \mathrm{Ham}(L_0, L_1)$ and $J \in \mathcal{J}_t(M, \omega)$. Then for any (J, H)-holomorphic strip $u : \mathbb{R} \times [0,1] \to M$ with Lagrangian boundary conditions in L_0, L_1 the following are equivalent:*

(a) *u has finite energy $E_H(u) = \int_{\mathbb{R}\times[0,1]} |\partial_s u|^2 < \infty$;*
(b) *There exist $x_\pm \in \mathcal{I}(L_0, L_1)$ such that $u(s, \cdot)$ converges to $x_\pm$ exponentially in all derivatives as $s \to \pm\infty$.*

For any $x_\pm \in \mathcal{I}(L_0, L_1)$ we denote by

$$\mathcal{M}(x_-, x_+) := \big\{u : \mathbb{R} \times [0,1] \to M \,\big|\, (17), (18), E_H(u) < \infty, \lim_{s\to\pm\infty} u(s, \cdot) = x_\pm\big\}/\mathbb{R}$$

the space of finite energy (J, H)-holomorphic maps modulo translation in $s \in \mathbb{R}$. It is isomorphic to the moduli space of finite energy J'-holomorphic maps with boundary conditions in $\phi_1(L_0)$ and L_1, and without Hamiltonian perturbation. Here $J' \in \mathcal{J}_t(M, \omega)$ arises from J by pullback with $\phi_{t,1}$.

Suppose that the pair (L_0, L_1) is monotone, then for any $x_\pm \in \mathcal{I}(L_0, L_1)$ there exists a constant $c(x_-, x_+)$ such that for all $u \in \mathcal{M}(x_-, x_+)$ the energy-index relation holds:

$$2E_H(u) = \tau I(u) + c(x_-, x_+), \tag{19}$$

where $I(u)$ denotes the Maslov-Viterbo index of u (as in [4]). This monotonicity ensures energy bounds for the moduli spaces and thus compactness up to bubbling.

Theorem 3.2.2. *(Floer,Oh) Let $L_0, L_1 \subset M$ be a monotone pair of Lagrangian submanifolds satisfying (L1-2) and (M1-2). For any $H \subset \mathrm{Ham}(L_0, L_1)$, there exists a subset $\mathcal{J}_t^{\mathrm{reg}}(L_0, L_1; H) \subset \mathcal{J}_t(M, \omega)$ of Baire second category, such that the following holds for all $x_\pm \in \mathcal{I}(L_0, L_1)$.*

(a) *$\mathcal{M}(x_-, x_+)$ is a smooth manifold whose dimension near a nonconstant solution u is given by the formal dimension, equal to the Maslov-Viterbo index $I(u) - 1$.*
(b) *The component $\mathcal{M}(x_-, x_+)_0 \subset \mathcal{M}(x_-, x_+)$ of formal dimension zero is finite.*
(c) *Suppose that L_0 and L_1 have minimal Maslov numbers $N_{L_k} \geq 3$. Then the one-dimensional component $\mathcal{M}(x_-, x_+)_1 \subset \mathcal{M}(x_-, x_+)$ has a compactification as one-dimensional manifold with boundary*

$$\partial\overline{\mathcal{M}(x_-, x_+)_1} \cong \bigcup_{x \in \mathcal{I}(L_0, L_1)} \mathcal{M}(x_-, x)_0 \times \mathcal{M}(x, x_+)_0 \tag{20}$$

(d) *If (L_0, L_1) is relatively spin (as defined in e.g. [46]), then there exists a coherent set of orientations on $\mathcal{M}(x_-, x_+)_0, \mathcal{M}(x_-, x_+)_1$ for all $x_\pm \in \mathcal{I}(L_0, L_1)$, that is, orientations compatible with (20).*

For the proofs of (a-c) we refer to Oh's paper [29] and the clarifications [30], [21]. For the exact case see [37]. The proof of (d) is contained in [46] loosely following [7]. From (d) we obtain a map

$$\epsilon : \mathcal{M}(x_-, x_+)_0 \to \{\pm 1\}$$

defined by comparing the given orientation to the canonical orientation of a point.

Now let M be a monotone symplectic manifold satisfying (M1-2) and equipped with an N-fold Maslov covering. Let $L_0, L_1 \subset M$ be a monotone, relative spin pair of graded Lagrangian submanifolds satisfying (L1-3), and let $H \in \mathrm{Ham}(L_0, L_1)$. The *Floer cochain group* is the $\mathbb{Z}_N$-graded group

$$CF(L_0, L_1) = \bigoplus_{d \in \mathbb{Z}_N} CF^d(L_0, L_1), \qquad CF^d(L_0, L_1) = \bigoplus_{x \in \mathcal{I}(L_0, L_1), |x| = d} \mathbb{Z}\langle x \rangle,$$

and the *Floer coboundary operator* is the map of degree 1,

$$\partial^d : \; CF^d(L_0, L_1) \to CF^{d+1}(L_0, L_1),$$

defined by

$$\partial^d \langle x_- \rangle := \sum_{x_+ \in \mathcal{I}(L_0, L_1)} \Big(\sum_{u \in \mathcal{M}(x_-, x_+)_0} \epsilon(u) \Big) \langle x_+ \rangle.$$

Here we choose some $J \in \mathcal{J}_t^{\mathrm{reg}}(L_0, L_1; H)$. If an isolated trajectory $u \in \mathcal{M}(x_-, x_+)_0$ exists, then the degree identity $|x_+| = |x_-| + 1$ can be seen by concatenating the paths $\tilde{\gamma}_0, \tilde{\gamma}_1$ of graded Lagrangians in the definition of $|x_-|$ with the unique graded lifts of u^*TL_0, u^*TL_1 to obtain paths of graded Lagrangians defining $|x_+|$ (using a trivialization of u^*TM over the strip, compactified to a disk). By additivity of the Maslov index this shows $|x_+| = |x_-| + I(u^*TL_0, u^*TL_1) = |x_-| + 1$. It follows from Theorem 3.2.2 that $\partial^2 = 0$. Now the

Floer cohomology [5]

$$HF(L_0,L_1) := \bigoplus_{d\in\mathbb{Z}_N} HF^d(L_0,L_1), \qquad HF^d(L_0,L_1) := \ker(\partial^d)/\mathrm{im}(\partial^{d-1})$$

is $\mathbb{Z}_N$-graded. It is independent of the choice of H and J; a generalization of this fact is proved at the end of Section 4.3 below. If the gradings moreover satisfy (G1-2), then we have a well defined splitting

$$HF(L_0,L_1) = HF^{\mathrm{even}}(L_0,L_1) \oplus HF^{\mathrm{odd}}(L_0,L_1).$$

Remark 3.2.3. In a suitable derived sense the Floer cohomology $HF(L_1,L_0)$ for the switched pair is the dual space $\mathrm{Hom}(HF(L_0,L_1),\mathbb{Z})$, see Remark 7.2.5 below.

3.3. **Floer cohomology for sequences of Lagrangian correspondences.** Let M, M' be monotone symplectic manifolds satisfying (M1-2) with the same monotonicity constant $\tau \geq 0$, let both manifolds be equipped with an N-fold Maslov covering as in (G1), and fix background classes $b \in H^2(M,\mathbb{Z}_2)$, $b' \in H^2(M',\mathbb{Z}_2)$. Let $\underline{L}$ be a generalized Lagrangian correspondence from M to M' as in Definition 2.1.1, that is $\underline{L} = (L_{01},\ldots,L_{r(r+1)})$ is a sequence of compact Lagrangian correspondences $L_{(j-1)j} \subset M_{j-1}^- \times M_j$ between a sequence $M = M_0, M_1, \ldots, M_{r+1} = M'$ of symplectic manifolds. We assume that $\underline{L}$ satisfies (M1-2,L1-3), i.e. each M_j satisfies (M1-2) and each $L_{(j-1)j}$ satisfies (L1-3) with the fixed monotonicity constant $\tau \geq 0$. We moreover assume that $\underline{L}$ is graded (see Definition 2.3.1) and, similarly, that it is equipped with a relative spin structure in the following sense.

Definition 3.3.1. Let M, M' be symplectic manifolds and fix background classes $b \in H^2(M,\mathbb{Z}_2)$, $b' \in H^2(M',\mathbb{Z}_2)$. Let $\underline{L}$ be a generalized Lagrangian correspondence from M to M'. A *relative spin structure* on $\underline{L}$ is a collection of background classes $b_j \in H^2(M_j,\mathbb{Z}_2)$ and relative spin structures on $L_{(j-1)j}$ with background classes $-\pi_{j-1}^* b_{j-1} + \pi_j^* b_j$. Here $b_0 = b$ and $b_{r+1} = b'$ are the fixed classes.

A pair of such generalized Lagrangian correspondences with relative spin structures, $\underline{L}_1$ and $\underline{L}_2$, from M to M' (with fixed background classes b, b') defines a cyclic Lagrangian correspondence $\underline{L}_1\#(\underline{L}_2)^T$, which carries a relative spin structure in the following sense.

Definition 3.3.2. Let $\underline{L} = (L_{01},\ldots,L_{r(r+1)})$ be a cyclic generalized Lagrangian correspondence (i.e. $L_{j(j+1)} \subset M_j^- \times M_{j+1}$ for a cyclic sequence $M_0, M_1, \ldots, M_{r+1} = M_0$ of symplectic manifolds). A *relative spin structure* on $\underline{L}$ consists of a collection of background classes $b_j \in H^2(M_j,\mathbb{Z}_2)$ for $j = 0,\ldots,r+1$ and relative spin structures on $L_{j(j+1)}$ with background classes $-\pi_j^* b_j + \pi_{j+1}^* b_{j+1}$. The cyclic requirement on the background classes $b_0 \in H^2(M_0,\mathbb{Z}_2)$ and $b_{r+1} \in H^2(M_{r+1},\mathbb{Z}_2) = H^2(M_0,\mathbb{Z}_2)$ is $b_{r+1} = b_0$ for r odd and $b_{r+1} = b_0 + w_2(M_0)$ for r even.[6]

For a pair $\underline{L}_1, \underline{L}_2$ of generalized Lagrangian correspondences from M to M', satisfying the above assumptions, we will define a Floer cohomology

$$HF(\underline{L}_1,\underline{L}_2) := HF(\underline{L}_1\#(\underline{L}_2)^T)$$

[5] Note that our conventions differ from Seidel's definition of graded Floer cohomology in [38] in two points which cancel each other: The roles of x_- and x_+ are interchanged and we switched the sign of the Maslov index in the definition of the degree (9).

[6] This shift is necessary in order to fit in the canonical relative spin structure for the diagonal Δ_0, see Remark 6.6.5.

by a more general construction for cyclic Lagrangian correspondences. So from now on we consider a cyclic generalized Lagrangian correspondence $\underline{L}$ as in Definition 2.1.6, that is $\underline{L} = (L_{01}, \ldots, L_{r(r+1)})$ is a sequence of smooth Lagrangian correspondences $L_{(j-1)j} \subset M_{j-1}^- \times M_j$ between a sequence $M_0, M_1, \ldots, M_{r+1} = M_0$ of symplectic manifolds. We assume that $\underline{L}$ satisfies (M1-2,L1-3), i.e. each M_j satisfies (M1-2) and each $L_{(j-1)j}$ satisfies (L1-3) with a fixed monotonicity constant $\tau \geq 0$. We moreover assume that $\underline{L}$ is graded and equipped with a relative spin structure. For example, we could consider a non-cyclic sequence of symplectic manifolds $M_1, \ldots, M_r$ and Lagrangians $L_1 \subset M_1$, $\bigl(L_{(i-1)i} \subset M_{i-1}^- \times M_i\bigr)_{i=2,\ldots,r}$, $L_r \subset M_r^-$, which is a special case of this setup with $M_0 = \{\mathrm{pt}\}$.

Eventually, in Section 4.3, we will define the Floer homology $HF(\underline{L})$ directly, using "quilts of pseudoholomorphic strips". In this section however we define $HF(\underline{L})$ as a special case of the (monotone) Floer homology for pairs of Lagrangian submanifolds – which are constructed from the sequence $\underline{L}$ as follows. If $\underline{L}$ has even length $r+1 \in 2\mathbb{N}$ we define a pair of graded Lagrangian submanifolds,

$$\begin{array}{l} L_{(0)} := (L_{01} \times L_{23} \times \ldots \times L_{(r-1)r}) \\ L_{(1)} := (L_{12} \times L_{34} \times \ldots \times L_{r(r+1)})^T \end{array} \quad \subset \; M_0^- \times M_1 \times M_2^- \times \ldots \times M_r =: \widetilde{M}.$$

Here we denote by $M_1^- \times \ldots \times M_r^- \times M_0 \to M_0^- \times M_1 \times \ldots \times M_r$, $Z \mapsto Z^T$ the transposition of the last to the first factor, combined with an overall sign change in the symplectic form. If $\underline{L}$ has odd length $r+1 \in 2\mathbb{N}+1$ we insert the diagonal $\Delta_0 \subset M_0^- \times M_0 = M_{r+1}^- \times M_0$ (equipped with its canonical grading) into $\underline{L}$ before arranging it into a pair of Lagrangian submanifolds as above, yielding

$$\begin{array}{l} L_{(0)} = (L_{01} \times L_{23} \times \ldots \times L_{r(r+1)}) \\ L_{(1)} = (L_{12} \times L_{34} \times \ldots \times L_{(r-1)r} \times \Delta_0)^T \end{array} \quad \subset \; M_0^- \times M_1 \times \ldots \times M_r^- \times M_{r+1} = \widetilde{M}$$

In the case of a noncyclic correspondence with $M_0 = M_{r+1} = \{\mathrm{pt}\}$ the transposition as well as insertion of the diagonal are trivial operations. Note that, beyond the grading, also the assumptions (L1-3) on $\underline{L}$ transfer directly to properties (L1-3) for $L_{(0)}$ and $L_{(1)}$.[7] Similarly, a relative spin structure on $\underline{L}$ induces compatible relative spin structures on $L_{(0)}$ and $L_{(1)}$, see [46]. Moreover, we say that $\underline{L}$ is *monotone* if the pair of Lagrangians $(L_{(0)}, L_{(1)})$ is monotone. If this is the case, then a graded Floer cohomology for $\underline{L}$ can be defined by

$$HF(\underline{L}) := HF(L_{(0)}, L_{(1)}).$$

In the case of a non-cyclic sequence this specializes to

$$HF(L_1, L_{12}, \ldots, L_{(r-1)r}, L_r) = HF(L_1 \times L_{23} \times \ldots, L_{12} \times L_{34} \times \ldots).$$

Recall that the Floer complex $CF(L_{(0)}, L_{(1)})$ is generated by the perturbed intersection points $\mathcal{I}(L_{(0)}, L_{(1)}) = \phi^H(L_{(0)}) \cap L_{(1)}$, where ϕ^H is the time-one flow of a Hamiltonian $H : [0,1] \times \widetilde{M} \to \mathbb{R}$ that makes this intersection transverse. In Section 4.3 we will wish to use perturbation data of split type, that is given by a tuple of Hamiltonian functions

$$\underline{H} = \bigl(H_j \in C^\infty([0,1] \times M_j)\bigr)_{j=0,\ldots,r}.$$

We identify this tuple with the Hamiltonian $H = \sum_{j=0}^r (-1)^{j+1} H_j$ for r odd and with $H = -\frac{1}{2} H_0 + \sum_{j=1}^r (-1)^{j+1} H_j + \frac{1}{2} H_{r+1}$ for r even, where $H_{r+1} := H_0 \in \mathcal{C}^\infty([0,1] \times M_{r+1})$.

[7] This would not necessarily be true if we had defined N_L as the positive generator of $I(\pi_2(M, L))$.

The perturbed intersection points

$$\mathcal{I}(L_{(0)},L_{(1)}) = \{ y : [0,1] \to \widetilde{M} \,|\, \dot{y} = X_H(y), y(0) \in L_{(0)}, y(1) \in L_{(1)} \}$$

are in canonical one-to-one correspondence [8] with the set of tuples of Hamiltonian chords,

$$\mathcal{I}(\underline{L}) := \left\{ \underline{x} = \big(x_j : [0,1] \to M_j\big)_{j=0,\dots,r} \;\middle|\; \begin{array}{c} \dot{x}_j(t) = X_{H_j}(x_j(t)), \\ (x_j(1), x_{j+1}(0)) \in L_{j(j+1)} \end{array} \right\}.$$

(Here and below we use the index j modulo $r+1$, i.e. $x_{r+1} := x_0$ resp. $m_{r+1} := m_0$.) Moreover, $\mathcal{I}(\underline{L})$ is canonically identified with $\times_{\phi^{H_0}}\big(L_{01} \times_{\phi^{H_1}} L_{12} \ldots \times_{\phi^{H_r}} L_{r(r+1)}\big)$, the set of points

$$\big\{ (m_0,\dots,m_r) \in M_0 \times \ldots \times M_r, \,\big|\, (\phi^{H_j}(m_j), m_{j+1}) \in L_{j(j+1)} \big\},$$

where ϕ^{H_j} is the time-one flow of the Hamiltonian H_j. In this setting we can check that Hamiltonians of split type suffice to achieve transversality for the intersection points.

Proposition 3.3.3. *There is a dense open subset* $\operatorname{Ham}(\underline{L}) \subset \oplus_{k=0}^r C^\infty([0,1] \times M_k)$ *such that for every* $(H_0,\dots,H_r) \in \operatorname{Ham}(\underline{L})$ *the set* $\times_{\phi^{H_0}}\big(L_{01} \times_{\phi^{H_1}} L_{12} \ldots \times_{\phi^{H_r}} L_{r(r+1)}\big)$ *is smooth and finite, that is, the defining equations are transversal.*

Proof. By assumption $L_{j(j+1)}$ is an embedded submanifold of $M_{j(j+1)} := M_j^- \times M_{j+1}$ and so locally $L_{j(j+1)}$ is the zero set of a submersion $\psi_{j(j+1)} : M_{j(j+1)} \to \mathbb{R}^{n_j + n_{j+1}}$. The defining equations for $\times_{\phi^{H_0}}\big(L_{01} \times_{\phi^{H_1}} L_{12} \ldots \times_{\phi^{H_r}} L_{r(r+1)}\big)$ are

$$\psi_{j(j+1)}\big(\phi^{H_j}(m_j), m_{j+1}\big) = 0 \qquad \text{for all } j = 0,\dots,r. \tag{21}$$

Consider the universal moduli $\mathcal{U}$ space of data $(H_0,\dots,H_r,m_0,\dots,m_r)$ satisfying (21), where now each H_j has class C^ℓ for some $\ell \geq 2$. The linearized equations for $\mathcal{U}$ are

$$D\psi_{j(j+1)}(D\phi^{H_j}(h_j,v_j), v_{j+1}) = 0 \qquad \text{for all } j = 0,\dots,r. \tag{22}$$

for $v_j \in T_{m_j} M_j$ (with $v_{r+1} := v_0$) and $h_j \in C^\ell([0,1] \times M_j)$. The map

$$C^\ell([0,1] \times M_j) \to T_{\phi^{H_j}(m_j)} M_j, \quad h_j \mapsto D\phi^{H_j}(h_j, 0)$$

is surjective, which shows that the product of the operators on the left-hand side of (22) is also surjective. By the implicit function theorem $\mathcal{U}$ is a smooth Banach manifold, and we consider its projection to $\oplus_{k=0}^r C^\ell([0,1] \times M_k)$. By the Sard-Smale theorem, the set of regular values (the set of functions $H = (H_0,\dots,H_r)$ such that the perturbed intersection is transversal) is dense in $\oplus_{k=0}^r C^\ell([0,1] \times M_k)$. On the other hand, the set of regular values is clearly open. Hence the set of smooth functions that are regular values is open and dense. $\square$

By this Proposition we can pick the Hamiltonian H in the definition of $HF(\underline{L})$ of split form given by a tuple $(H_0,\dots,H_r) \in \operatorname{Ham}(\underline{L})$. The (graded) Floer chain group $CF(L_{(0)}, L_{(1)})$ can then be identified with

$$CF(\underline{L}) := \bigoplus_{d \in \mathbb{Z}_N} CF^d(\underline{L}), \qquad CF^d(\underline{L}) := \bigoplus_{\underline{x} \in \mathcal{I}(\underline{L}), |\underline{x}| = d} \mathbb{Z}\langle \underline{x} \rangle.$$

[8] The correspondence is by $y(t) = \big(x_0(1-t), x_1(t), x_2(1-t), \dots, x_r(t)\big)$ for r odd, and $y(t) = \big(x_0(1-\tfrac{1}{2}t), x_1(t), x_2(1-t) \dots, x_r(1-t), x_0(\tfrac{1}{2}t)\big)$ for r even.

The grading is defined as in Section 2.3,

$$\mathcal{I}(\underline{L}) \cong \mathcal{I}(L_{(0)}, L_{(1)}) \to \mathbb{Z}_N, \qquad \underline{x} \cong y \mapsto |y| = |\underline{x}|.$$

Next, the Floer differential is defined by counting finite energy (J,H)-holomorphic maps $v : \mathbb{R} \times [0,1] \to \widetilde{M}$ with boundary conditions $v(\mathbb{R},0) \subset L_{(0)}$, $v(\mathbb{R},1) \subset L_{(1)}$ and limits in $\mathcal{I}(L_{(0)}, L_{(1)})$. If $J \in \mathcal{J}_t^{reg}(L_{(0)}, L_{(1)}; H)$ is of split form, that is given by a tuple

$$\underline{J} = \big(J_j \in \mathcal{C}^\infty([0,1], \mathcal{J}(M_j, \omega_j)\big)_{j=0,\ldots,r},$$

then these moduli spaces are in one-to-one correspondence [9] with the moduli spaces of $(r+1)$-tuples of finite energy (J_j, H_j)-holomorphic maps $u_j : \mathbb{R} \times [0,1] \to M_j$ for $j = 0, \ldots, r$ (and $u_{r+1} := u_0$) with limits in $\mathcal{I}(\underline{L})$, satisfying the boundary conditions

$$(u_j(s,1), u_{j+1}(s,0)) \in L_{j(j+1)}, \qquad \text{for all } j = 0, \ldots, r, \ s \in \mathbb{R}. \tag{23}$$

Remark 3.3.4. To see that there exists a regular $\underline{J} \in \mathcal{J}_t^{reg}(L_{(0)}, L_{(1)}; H)$ of split form we fix an almost complex structure of split form and note that the unique continuation theorem [5, Theorem 4.3] applies to the interior of every single nonconstant strip $u_j : \mathbb{R} \times (0,1) \to M_j$. It implies that the set of regular points, $(s_0, t_0) \in \mathbb{R} \times (0,1)$ with $\partial_s u_j(s_0,t_0) \neq 0$ and $u_j^{-1}(u_j(\mathbb{R} \cup \{\pm\infty\}), t_0) = \{(s_0,t_0)\}$, is open and dense. These points can be used to prove surjectivity of the linearized operator for a universal moduli space of solutions with respect to split almost complex structures. (The constant solutions are automatically transverse due to the previously ensured transversality of the intersection points $\phi^H(L_{(0)}) \pitchfork L_{(1)}$.) The existence of a regular, split $\underline{J}$ then follows from the usual Sard-Smale argument as in [26].

We can thus define the differential ∂ on $CF(\underline{L})$ by counting these $(r+1)$-tuples of holomorphic strips; see Figure 1 for $r = 1$ and $r = 2$. Then $HF(\underline{L})$ is the cohomology of $(CF(\underline{L}), \partial)$. We will use this for a further reformulation in terms of "quilts" in Section 4.3.

4. Quilted pseudoholomorphic surfaces

In this section we review the construction of relative invariants for surfaces with strip-like (or cylindrical) ends analogous to [39, Section 2.4] (in the exact case) and [36] (in the case of surfaces without boundary). We then introduce quilted pseudoholomorphic surfaces and use these to construct new relative invariants associated to Lagrangian correspondences. Finally, we express the Floer cohomology for sequences of Lagrangian correspondences in terms of quilted surfaces.

4.1. Invariants for surfaces with strip-like ends.

We begin with a formal definition of surfaces with strip-like ends. To reduce notation somewhat we restrict to strip-like ends, i.e. punctures on the boundary. One could in addition allow cylindrical ends by adding punctures in the interior of the surface, see Remark 4.1.12.

Definition 4.1.1. A *surface with strip-like ends* consists of the following data:

[9]For r odd the correspondence is by $v(s,t) = \big(u_0(s,1-t), u_1(s,t), u_2(s,1-t), \ldots, u_r(s,t)\big)$ with $\underline{J} = \big(-J_0(1-t), J_1(t), -J_2(1-t), \ldots, J_r(t)\big)$, and for r even it is by $v(s,t) = \big(u_0(\tfrac12 s, 1-\tfrac12 t), u_1(s,t), u_2(s,1-t) \ldots, u_r(s,1-t), u_0(\tfrac12 s, \tfrac12 t)\big)$ with $\underline{J}(t) = \big(-J_0(1-\tfrac12 t), J_1(t), -J_2(1-t), \ldots, -J_r(1-t), J_0(\tfrac12 t)\big)$.

(a) A Riemann surface $\overline{S}$ with boundary $\partial\overline{S} = C_1 \sqcup \ldots \sqcup C_m$ and $d_n \geq 0$ distinct points $z_{n,1}, \ldots, z_{n,d_n} \in C_n$ in cyclic order on each boundary circle $C_n \cong S^1$. We will use the indices on C_n modulo d_n, index all marked points by

$$\mathcal{E} = \mathcal{E}(S) = \big\{ e = (n,l) \,\big|\, n \in \{1,\ldots,m\}, l \in \{1,\ldots,d_n\} \big\},$$

and use the notation $e \pm 1 := (n, l \pm 1)$ for the cyclically adjacent index to $e = (n,l)$. For $l = 1, \ldots, d_n$ we denote by $I_e = I_{n,l} \subset C_n$ the component of ∂S between $z_e = z_{n,l}$ and $z_{e+1} = z_{n,l+1}$.

(b) A complex structure j_S on $S := \overline{S} \setminus \{z_e \,|\, e \in \mathcal{E}\}$.

(c) A set of *strip-like ends* for S: A set of embeddings

$$\epsilon_{S,e} : \mathbb{R}^\pm \times [0,1] \to S$$

such that $\lim_{s\to\pm\infty}(\epsilon_{S,e}(s,t)) = z_e$ and on each end, and $\epsilon_{S,e}^* j_S = j_0$ is the canonical complex structure on $\mathbb{R}^\pm \times [0,1]$. We denote the set of incoming ends $\epsilon_{S,e} : (-\infty,0) \times [0,1] \to S$ by $\mathcal{E}_- = \mathcal{E}_-(S)$ and the set of outgoing ends $\epsilon_{S,e} : (0,\infty) \times [0,1] \to S$ by $\mathcal{E}_+ = \mathcal{E}_+(S)$.

(d) An ordering of the set of (compact) boundary components of $\overline{S}$ and an ordering of the sets $\mathcal{E}_\pm$ of incoming resp. outgoing ends,

$$\mathcal{E}_- = (e_1^-, \ldots, e_{N_-}^-), \qquad \mathcal{E}_+ = (e_1^+, \ldots, e_{N_+}^+),$$

where $e_i^\pm = (n_i^\pm, l_i^\pm)$ denotes the incoming or outgoing end at $z_{e_i^\pm}$.

Elliptic boundary value problems are associated to surfaces with strip-like ends as follows. Let E be a complex vector bundle over S and $F = (F_e)_{e\in\mathcal{E}(S)}$ a tuple of totally real subbundles over the boundary components I_e. Suppose that the sub-bundles F_{e-1}, F_e intersect transversally at z_e. Let

$$D_{E,F} : \Omega^0(S,E;F) \to \Omega^{0,1}(S,E)$$

be a real Cauchy-Riemann operator acting on sections with boundary values in F_e over I_e. Transversality at infinity implies that the operator $D_{E,F}$ is Fredholm. If S has no strip-like ends, and $S_0 \subset S$ denotes the union of components without boundary, we denote by $I(E,F)$ the topological index

$$I(E,F) = \deg(E|S_0) + \sum_{e\in\mathcal{E}(S)} I(F_e),$$

where $I(F_e)$ is the Maslov index of the boundary data determined from a trivialization of $E|(\overline{S}\backslash S_0) \cong (\overline{S}\backslash S_0) \times \mathbb{C}^r$. The index theorem for surfaces with boundary [26, Appendix C] implies (with $\chi(S) := \chi(\overline{S})$)

$$\operatorname{Ind} D_{E,F} = \operatorname{rank}_{\mathbb{C}}(E)\chi(S) + I(E,F). \tag{24}$$

A special case of these totally real boundary conditions will arise from Lagrangian submanifolds. We fix a compact, monotone (or noncompact, exact) symplectic manifold (M,ω) satisfying (M1-2) as in Section 3.1 and let M be equipped with an N-fold Maslov covering satisfying (G1). For every boundary component I_e let $L_e \subset M$ be a compact, monotone, graded Lagrangian submanifold satisfying (L1-2) and (G2). The grading $|x| \in \mathbb{Z}_N$ on the Floer chain groups then induces a $\mathbb{Z}_2$-grading $(-1)^{|x|}$ which only depends on the orientations of $\underline{L}$.

We say that the tuple $\underline{L} = (L_e)_{e\in\mathcal{E}(S)}$ is *relatively spin* if all Lagrangians L_e are relatively spin with respect to one fixed background class $b \in H^2(M,\mathbb{Z}_2)$, see [46] for more details.

With these preparations we can construct moduli spaces of pseudoholomorphic maps from the surface S. For each pair (L_{e-1}, L_e) choose a regular perturbation datum (H_e, J_e) as in section 2.2 such that the graded Floer cohomology $HF(L_{e-1}, L_e)$ is well defined. Let $\mathrm{Ham}(S;\underline{L})$ denote the set of $C^\infty(M)$-valued one-forms $K_S \in \Omega^1(S, C^\infty(M))$ such that $K_S|_{\partial S} = 0$ and $\epsilon_{S,e}^* K_S = H_e \mathrm{d}t$ on each strip-like end. Let $Y_S \in \Omega^1(S, \mathrm{Vect}(M))$ denote the corresponding Hamiltonian vector field valued one-form, then $\epsilon_{S,e}^* Y_S$ equals to $X_{H_e}\mathrm{d}t$ on each strip-like end. We denote by $\mathcal{J}(S;\underline{L})$ the subset of $J_S \in \mathcal{C}^\infty(S, \mathcal{J}(M,\omega))$ that equals to the given perturbation datum J_e on each strip-like end.

We denote by $\mathcal{I}_-(\underline{L})$ the set of tuples $\underline{x}^- = (x_e^-)_{e\in\mathcal{E}_-}$ with $x_e^- \in \mathcal{I}(L_e, L_{e-1})$ and by $\mathcal{I}_+(\underline{L})$ the set of tuples $\underline{x}^+ = (x_e^+)_{e\in\mathcal{E}_+}$ with $x_e^+ \in \mathcal{I}(L_{e-1}, L_e)$. For each of these tuples we denote by

$$\mathcal{M}_S(\underline{x}^-, \underline{x}^+) := \big\{ u : S \to M \,\big|\, (a) - (d) \big\}$$

the space of (J_S, K_S)-holomorphic maps with Lagrangian boundary conditions, finite energy, and fixed limits, that is

(a) $J_S(u) \circ (\mathrm{d}u - Y_S(u)) = (\mathrm{d}u - Y_S(u)) \circ j_S$,
(b) $u(I_e) \subset L_e$ for all $e \in \mathcal{E}(S)$,
(c) $E_{K_S}(u) := \int_S \big(u^*\omega + \mathrm{d}(K_S \circ u)\big) < \infty$,
(d) $\lim_{s\to\pm\infty} u(\epsilon_{S,e}(s,t)) = x_e^\pm(t)$ for all $e \in \mathcal{E}_\pm$.

Remark 4.1.2. If the tuple of Lagrangians $\underline{L}$ is monotone in the sense of Definition 3.1.2, then elements $u \in \mathcal{M}_S(\underline{x}^-, \underline{x}^+)$ satisfy the energy-index relation

$$2E_{K_S}(u) = \tau I(u^*TM, (u^*TL_e)_{e\in\mathcal{E}(S)}) + c(\underline{x}^-, \underline{x}^+)$$

analogous to (19). This is seen by gluing u with a fixed element u_0 to a map on the compact doubled surface. Since $K_S|_{\partial S} = 0$ this term drops out of the total energy.

Moreover, the index $I(u^*TM, (u^*TL_e)_{e\in\mathcal{E}(S)})$ is determined mod 2 by the limit conditions $\underline{x}^-, \underline{x}^+$. This is since different pullback bundles with fixed limits differ by a complex bundle over closed components (with index $2c_1$) and by disks with boundary loops in TL_e, which have even index by (L2) resp. (L2').

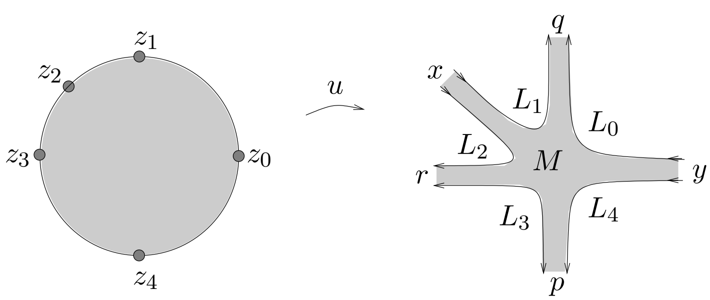


Figure 2. A holomorphic curve $u \in \mathcal{M}_S((x,y),(p,q,r))$ for a surface S with ends $\mathcal{E}_- = \{2,0\}$ and $\mathcal{E}_+ = \{4,1,3\}$

Theorem 4.1.3. *Suppose that $\underline{L} = (L_e)_{e\in\mathcal{E}(S)}$ is a monotone tuple of Lagrangian submanifolds satisfying (L1-2) and (M1-2). For any $H_S \in \mathrm{Ham}(S;\underline{L})$ there exists a subset $\mathcal{J}^{\mathrm{reg}}(S;\underline{L};H_S) \subset \mathcal{J}(S;\underline{L})$ of Baire second category such that for any tuple $(\underline{x}^-, \underline{x}^+) \in \mathcal{I}_-(\underline{L}) \times \mathcal{I}_+(\underline{L})$ the following holds.*

(a) *$\mathcal{M}_S(\underline{x}^-, \underline{x}^+)$ is a smooth manifold.*

(b) *The zero dimensional component* $\mathcal{M}_S(\underline{x}^-, \underline{x}^+)_0$ *is finite.*
(c) *The one-dimensional component* $\mathcal{M}_S(\underline{x}^-, \underline{x}^+)_1$ *has a compactification as a one-manifold with boundary*

$$\begin{aligned}\partial\overline{\mathcal{M}_S(\underline{x}^-, \underline{x}^+)_1} \cong & \bigcup_{e\in\mathcal{E}_-,\, y\in\mathcal{I}(L_e,L_{e-1})} \mathcal{M}(x_e^-, y)_0 \times \mathcal{M}_S(\underline{x}^-|_{x_e^-\to y}, \underline{x}^+)_0 \\ & \cup \bigcup_{e\in\mathcal{E}_+,\, y\in\mathcal{I}(L_{e-1},L_e)} \mathcal{M}_S(\underline{x}^-, \underline{x}^+|_{x_e^+\to y})_0 \times \mathcal{M}(y, x_e^+)_0,\end{aligned}$$

where the tuple $\underline{x}|_{x_e\to y}$ *is* $\underline{x}$ *with the intersection point* x_e *replaced by* y.
(d) *If* $\underline{L}$ *is relatively spin then there exist a coherent set of orientations* ϵ *on the zero and one-dimensional moduli spaces so that the inclusion of the boundary in (c) has the signs* $(-1)^{\sum_{f<e}|x_f^-|}$ *(for incoming trajectories) and* $-(-1)^{\sum_{f<e}|x_f^+|}$ *(for outgoing trajectories.)*

The proof is similar to that of Theorem 3.2.2. In the moduli spaces of dimension 0 and 1 the bubbling of spheres and disks is ruled out by monotonicity and (L2) resp. (L2'). The orientations are defined in [46]. We can thus define

$$C\Phi_S : \bigotimes_{e\in\mathcal{E}_-} CF(L_e, L_{e-1}) \to \bigotimes_{e\in\mathcal{E}_+} CF(L_{e-1}, L_e)$$

by

$$C\Phi_S\Big(\bigotimes_{e\in\mathcal{E}_-}\langle x_e^-\rangle\Big) := \sum_{\underline{x}^+\in\mathcal{I}_+(\underline{L})}\Big(\sum_{u\in\mathcal{M}_S(\underline{x}^-,\underline{x}^+)_0}\epsilon(u)\Big)\bigotimes_{e\in\mathcal{E}_+}\langle x_e^+\rangle.$$

By items (c),(d), the maps $C\Phi_S$ are chain maps and so descend to a map of Floer cohomologies

$$\Phi_S : \bigotimes_{e\in\mathcal{E}_-} HF(L_e, L_{e-1}) \to \bigotimes_{e\in\mathcal{E}_+} HF(L_{e-1}, L_e). \tag{25}$$

In order for the Floer cohomologies to be well defined, we have to assume in addition that all Lagrangians in $\underline{L}$ satisfy (L3).

Remark 4.1.4. The standard Floer homotopy argument shows that the maps Φ_S are in fact *relative invariants*, that is independent of the choices of perturbation data (H_S, J_S) and complex structure j_S on S. The key fact is that any two choices $(H_i, J_i, j_i)_{i=0,1}$ (of fixed form over the strip-like ends) can be connected by a homotopy $(H_\lambda, J_\lambda, j_\lambda)_{\lambda\in[0,1]}$. One then considers the universal moduli spaces consisting of pairs (λ, u) of $\lambda \in [0,1]$ and a solution u with respect to the data $(H_\lambda, J_\lambda, j_\lambda)$. For a generic homotopy, these are smooth manifolds. Their 0-dimensional components can be oriented and counted to define a map $C\Psi : \otimes_{\mathcal{E}_-}CF \to \otimes_{\mathcal{E}_+}CF$. The 1-dimensional component has boundaries corresponding to the solutions contributing to $C\Phi_0$ and $C\Phi_1$ (the chain maps defined with respect to the $\lambda = 0$ and $\lambda = 1$ data) and ends corresponding to pairs of solutions contributing to $C\Psi$ and the boundary operators $\delta_\pm$ in the Floer complexes for the ends. (Sphere and disk bubbling is excluded by monotonicity.) Counting these with orientations proves $C\Phi_0 - C\Phi_1 = \delta_+ \circ C\Psi + C\Psi \circ \delta_-$, i.e. $C\Psi$ defines a chain homotopy between $C\Phi_0$ and $C\Phi_1$, and hence $\Phi_0 = \Phi_1$ on cohomology. See [36, Chapter 5.2] for the detailed construction (including the isomorphism for different choices of perturbation data over the strip-like ends), which directly generalizes from the closed case. The orientations are given by the orientation of the determinant line bundles constructed in [46] plus a (first) $\mathbb{R}$-factor for the $[0,1]$-variable. The gluing of orientations is the same as in [46], and the signs for $C\Phi_i$ arise from the boundary orientation of $\partial[0,1] = \{0\}^- \cup \{1\}$.

Remark 4.1.5. Recall that M is equipped with an N-fold Maslov covering and each Lagrangian submanifold $L_e \subset M$ is graded. Suppose that S is connected. Then the effect of the relative invariant Φ_S on the grading is by a shift in degree of

$$|\Phi_S| = \frac{1}{2}\dim M(\#\mathcal{E}_+ - \chi(\overline{S})) \mod N.$$

That is, the coefficient of $C\Phi_S\big(\bigotimes_{e\in\mathcal{E}_-}\langle x_e^-\rangle\big)$ in front of $\bigotimes_{e\in\mathcal{E}_+}\langle x_e^+\rangle$ is zero unless the degrees $|x_e^-| = d(\sigma^N_{L_e}(x_e^-), \sigma^N_{L_{e-1}}(x_e^-))$ and $|x_e^+| = d(\sigma^N_{L_{e-1}}(x_e^+), \sigma^N_{L_e}(x_e^+))$ satisfy

$$\sum_{e\in\mathcal{E}_+}|x_e^+| - \sum_{e\in\mathcal{E}_-}|x_e^-| = \tfrac{1}{2}\dim M\big(\#\mathcal{E}_+ - \chi(\overline{S})\big) \qquad \mod N.$$

Here $\#\mathcal{E}_+$ is the number of outgoing ends of S. So, for example, Φ_S preserves the degree if S is a disk with one outgoing end and any number of incoming ends.

To check the degree identity fix paths $\tilde{\Lambda}_e : [0,1] \to \mathrm{Lag}^N(T_{x_e}M)$ from $\sigma^N_{L_{e-1}}(x_e)$ to $\sigma^N_{L_e}(x_e)$ for each end $e \in \mathcal{E}(S)$, and denote their projections by $\Lambda_e : [0,1] \to T_{x_e}M$. Let $D_{T_{x_e}M,\Lambda_e}$ be the Cauchy-Riemann operator in $T_{x_e}M$ on the disk with one incoming strip-like end and with boundary conditions Λ_e. Then Lemma 2.2.7 gives

$$|x_e^\pm| = \mathrm{Ind}(D_{T_{x_e}M,\Lambda_e^{\pm1}}), \quad e \in \mathcal{E}_\pm$$

with the reversed path $\tilde{\Lambda}_e^{-1}$ from $\sigma^N_{L_e}(x_e^-)$ to $\sigma^N_{L_{e-1}}(x_e^-)$ in case $e \in \mathcal{E}_-$. In this case we have

$$\mathrm{Ind}(D_{T_{x_e}M,\Lambda_e^{-1}}) + \mathrm{Ind}(D_{T_{x_e}M,\Lambda_e}) = \tfrac{1}{2}\dim M \qquad \mod N$$

since gluing the two disks gives rise to a Cauchy-Riemann operator on the disk with boundary conditions given by the loop $\Lambda_e^{-1}\#\Lambda_e$, which lifts to a loop in $\mathrm{Lag}^N(T_{x_e}M)$ and hence has Maslov index 0 mod N. Now consider an isolated solution $u \in \mathcal{M}_S(\underline{x}^-, \underline{x}^+)_0$. For each end $e \in \mathcal{E}(S)$ we can glue the operator $D_{T_{x_e}M,\Lambda_e^{-1}}$ on the disk to the linearized Cauchy-Riemann operator $D_{u^*TM,u^*T\underline{L}}$ on the surface S. This gives rise to a Cauchy-Riemann operator on the compact surface $\overline{S}$ with boundary conditions given by loops of Lagrangian subspaces (composed of u^*TL_e and Λ_e^{-1}) that lift to loops in $\mathrm{Lag}^N(M)$ (composed of $\sigma_{L_e}\circ u$ and $(\tilde{\Lambda}_e)^{-1}$). In a trivialization of u^*TM their Maslov indices are hence divisible by N, and so the index of the glued Cauchy-Riemann operator is

$$\begin{aligned}\tfrac{1}{2}\dim M\cdot\chi(\overline{S}) &= \mathrm{Ind}(D_{u^*TM,u^*T\underline{L}}) + \sum_{e\in\mathcal{E}(S)}\mathrm{Ind}(D_{TM,\Lambda_e^{-1}}) \qquad \mod N \\ &= 0 + \sum_{e\in\mathcal{E}_+(S)}(\tfrac{1}{2}\dim M - |x_e^+|) + \sum_{e\in\mathcal{E}_-(S)}|x_e^-| \qquad \mod N.\end{aligned}$$

Example 4.1.6. If $S_\|$ is the strip $\mathbb{R}\times[0,1]$ (i.e. the disk with one incoming and one outgoing puncture) then we can choose perturbation data that preserves the $\mathbb{R}$-invariance of the holomorphic curves. Then any nonconstant solution comes in a 1-dimensional family and hence $\Phi_{S_\|}$ is generated by the constant solutions. The same holds for the disks $S_\cap$ and $S_\cup$ with two incoming resp. two outgoing ends. With our choice of coherent orientations, see [46], we obtain

$$\Phi_\| := \Phi_{S_\|} = \mathrm{Id} : HF(L_0, L_1) \to HF(L_0, L_1),$$

which is a map of degree 0. The disks $S_\cap$ and $S_\cup$ give rise to maps of degree $-\frac{1}{2}\dim M$ and $\frac{1}{2}\dim M$ respectively,

$$\Phi_\cap := \Phi_{S_\cap} : HF(L_0, L_1) \otimes HF(L_1, L_0) \to \mathbb{Z}, \tag{26}$$

$$\Phi_\cup := \Phi_{S_\cup} : \mathbb{Z} \to HF(L_0, L_1) \otimes HF(L_1, L_0). \tag{27}$$

We write $\langle x_i \rangle_{01} \in CF(L_0, L_1)$ resp. $\langle x_i \rangle_{10} \in CF(L_1, L_0)$ for the generators corresponding to the intersection points $\mathcal{I}(L_0, L_1) \cong \mathcal{I}(L_1, L_0) = \{x_i \,|\, i = 1, \dots N\}$. Then, on the chain level, the maps are given by $C\Phi_\| : \langle x_i \rangle_{01} \mapsto \langle x_i \rangle_{01}$ and

$$C\Phi_\cap : \langle x_i \rangle_{01} \otimes \langle x_j \rangle_{10} \mapsto (-1)^{|\langle x_i \rangle_{01}|} \epsilon_i \delta_{ij}, \qquad C\Phi_\cup : 1 \mapsto \sum_i \epsilon_i \langle x_i \rangle_{01} \otimes \langle x_i \rangle_{10}$$

for some signs $\epsilon_1, \dots, \epsilon_N \in \{\pm 1\}$.[10] Here the degrees are related by $|\langle x_i \rangle_{01}| + |\langle x_i \rangle_{10}| = \frac{1}{2}\dim M$ as in Remark 4.1.5.

The relative invariants satisfy a tensor product law for disjoint union. A careful construction of the orientations (see [46]) leads to the following convention.

Lemma 4.1.7. *Let S_1, S_2 be surfaces with strip like ends and let $S_1 \sqcup S_2$ be the disjoint union, with ordering of boundary components and incoming and outgoing ends induced by the corresponding orderings on S_1, S_2. Then*

$$\Phi_{S_1 \sqcup S_2} = \Phi_{S_1} \otimes \Phi_{S_2},$$

where the graded tensor product is defined by

$$(\Phi_{S_1} \otimes \Phi_{S_2})(\langle \underline{x}_1 \rangle \otimes \langle \underline{x}_2 \rangle) = (-1)^{|\Phi_{S_2}||\underline{x}_1|} \Phi_{S_1}(\langle \underline{x}_1 \rangle) \Phi_{S_2}(\langle \underline{x}_2 \rangle). \tag{28}$$

The relative invariants satisfy a composition law for gluing along ends. Let S be a surface with strip-like ends, M a symplectic manifold, and $\underline{L}$ Lagrangian boundary conditions as in Section 4.1. Suppose that $e_+ = e^+_{i^+} \in \mathcal{E}_+(S)$ and $e_- = e^-_{i^-} \in \mathcal{E}_-(S)$ are outgoing resp. incoming ends of S such that the Lagrangians agree, $L_{e_+ - 1} = L_{e_-}$ and $L_{e_+} = L_{e_- - 1}$. Then we can algebraically define the trace of Φ_S at (e_-, e_+)

$$\mathrm{Tr}_{e_-, e_+}(\Phi_S) : \bigotimes_{e \in \mathcal{E}_-(S) \setminus \{e_-\}} HF(L_e, L_{e-1}) \;\to\; \bigotimes_{e \in \mathcal{E}_+(S) \setminus \{e_+\}} HF(L_{e-1}, L_e)$$

by

$$\begin{aligned} \mathrm{Tr}_{e_-, e_+}(\Phi_S) := &\left(\mathrm{Id}^{\mathcal{E}_+ \setminus e_+} \otimes \Phi_\cap^{e_+, e_0}\right) \circ \left(\Psi_{e_+} \otimes \mathrm{Id}^{e_0}\right) \circ \left(\Phi_S \otimes \mathrm{Id}^{e_0}\right) \\ &\circ \left(\Psi_{e_-} \otimes \mathrm{Id}^{e_0}\right) \circ \left(\mathrm{Id}^{\mathcal{E}_- \setminus e_-} \otimes \Phi_\cup^{e_-, e_0}\right), \end{aligned} \tag{29}$$

where

(a) superscripts indicate the ends (and associated Floer cohomology groups) that the maps act on,
(b) e_0 is an additional end associated to the Floer cohomology group $HF(L_{e_- - 1}, L_{e_-}) = HF(L_{e_+}, L_{e_+ - 1})$,

[10] For a fixed ordered pair (L_0, L_1) one can refine the choice of coherent orientations such that the signs in $\Phi_\cap$ are $\epsilon_i = +1$. For the reversed pair (L_1, L_0) this convention yields the same signs $\epsilon'_i = +1$ in $\Phi_\cap$ if $\frac{1}{2}\dim M$ is odd, but they vary, $\epsilon'_i = (-1)^{|\langle x_i \rangle_{10}|}$, if $\frac{1}{2}\dim M$ even. (In the latter case the signs in $\Phi_\cup$ are all positive.)

(c) $\Psi_{e_\pm}$ are the permutations of the factors in the graded tensor product needed to make the compositions well-defined,

$$\Psi_{e_i^-}: \ \Big(\bigotimes_{e\in\mathcal{E}_-\setminus\{e_i^-\}} \langle x_e\rangle\Big) \otimes \langle x_{e_i^-}\rangle \ \mapsto \ (-1)^{|x_{e_i^-}|\sum_{j=i+1}^{N_-}|x_e|} \bigotimes_{e\in\mathcal{E}_-} \langle x_e\rangle,$$

$$\Psi_{e_i^+}: \ \bigotimes_{e\in\mathcal{E}_+} \langle x_e\rangle \ \mapsto \ (-1)^{|x_{e_i^+}|\sum_{j=i+1}^{N_+}|x_e|} \Big(\bigotimes_{e\in\mathcal{E}_+\setminus\{e_i^-\}} \langle x_e\rangle\Big) \otimes \langle x_{e_i^+}\rangle.$$

Note that the trace does not depend on the choice of the signs ϵ_i in (26). On the other hand, let $\#_{e_+}^{e_-}(S)$ denote the surface obtained by gluing together the ends $e_\pm$, and choose an ordering of the boundary components and strip-like ends. (There is no canonical choice for this general gluing procedure.) The glued surface $\#_{e_+}^{e_-}(S)$ can be written as the "geometric trace"

$$\big(S_{\|}^{\mathcal{E}_+\setminus e_+} \sqcup S_{\cap}^{e_+,e_0}\big)\#\big(S_{\Psi_{e_+}} \sqcup S_{\|}^{e_0}\big)\#\big(S \sqcup S_{\|}^{e_0}\big)\#\big(S_{\Psi_{e_-}} \sqcup S_{\|}^{e_0}\big)\#\big(S_{\|}^{\mathcal{E}_-\setminus e_-} \sqcup S_{\cup}^{e_-,e_0}\big),$$

where $S_0\#S_1$ denotes the gluing of all incoming ends of S_0 to the outgoing ends of S_1 (which must be identical and in the same order). Here superscripts indicate the indexing of the ends of the surfaces, so e.g $S_{\|}^{\mathcal{E}_+\setminus e_+}$ is a product of strips $\mathbb{R}\times[0,1]$ with both incoming and outgoing ends indexed by $\mathcal{E}_+ \setminus e_+$. The surfaces $S_{\Psi_{e_\pm}}$ are the products of strips with incoming ends indexed by $(\mathcal{E}_- \setminus \{e_-\}, e_-)$ resp. $\mathcal{E}_+$ and outgoing ends indexed by $\mathcal{E}_-$ resp. $(\mathcal{E}_+ \setminus \{e_+\}, e_+)$ (in the order indicated). The relative invariants associated to the surfaces in this geometric trace are exactly the ones that we compose in the definition (29) of the algebraic trace. In fact, the standard Floer gluing construction implies the following analogue of the gluing formula [39, 2.30] in the exact case.

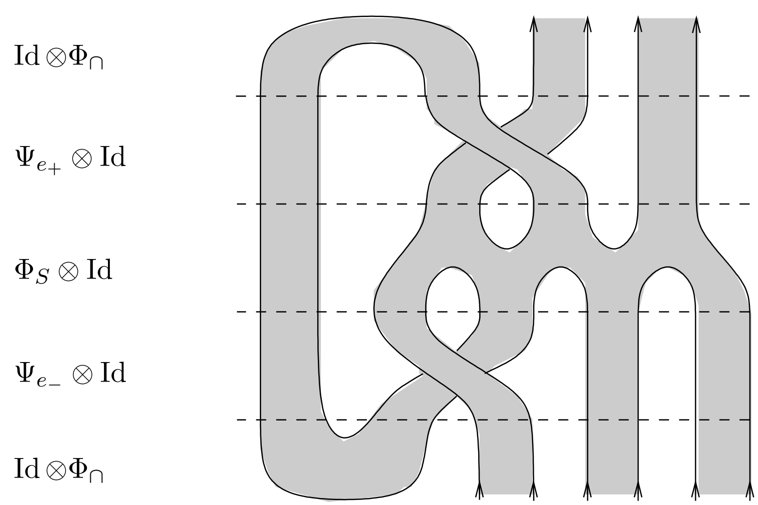


FIGURE 3. Gluing example for a connected surface S

Theorem 4.1.8. *Let S be a surface with strip-like ends and $\underline{L}$ Lagrangian boundary conditions as in Theorem 4.1.3, satisfying in addition (L3) and (G1-2). Suppose that $e_\pm \in \mathcal{E}_\pm(S)$*

such that $L_{e_+-1} = L_{e_-}$ *and* $L_{e_+} = L_{e_--1}$. *Then*

$$\Phi_{\#_{e_+}^{e_-}(S)} = (\epsilon_{S,\#_{e_+}^{e_-}(S)})^{\dim(M)/2}\,\mathrm{Tr}_{e_-,e_+}(\Phi_S),$$

where $\epsilon_{S,\#_{e_+}^{e_-}(S)} = \pm 1$ *is a universal sign depending on the surfaces, that is, the ordering of boundary components etc.*

Unfortunately it seems one cannot make the sign more precise, since there is no canonical convention for ordering the boundary components etc. of the glued surface.

Sketch of Proof: By Remark 4.1.4, the relative invariant for the geometric trace can be computed using a surface with long necks between the glued surfaces. Solutions (of both the linear and non-linear equation) on this surface are in one-to-one correspondence with pairs of solutions on the two separate surfaces; counting the latter exactly corresponds to composition. The one-to-one correspondence is proven by an implicit function theorem (using the fact that the linearized operator as well as its adjoint are surjective in the index 0 case) and a compactness result (using monotonicity to exclude bubbling). Details for the analogous closed case can be found in [36, Chapter 5.4]. Universality of the gluing sign for the case of simultaneous gluing of all ends is proved in [46]. By definition, our algebraic trace is the composition of the relative invariants of $S_{\|}^{\mathcal{E}_+\setminus e_+} \sqcup S_{\cap}^{e_+,e_0}$, $S_{\Psi_{e_+}} \sqcup S_{\|}^{e_0}$, $S \sqcup S_{\|}^{e_0}$, $S_{\Psi_{e_-}} \sqcup S_{\|}^{e_0}$, and $S_{\|}^{\mathcal{E}_-\setminus e_-} \sqcup S_{\cup}^{e_-,e_0}$, see Lemma 4.1.7. □

In [46] we determine more explicitly the gluing signs in two special cases: Gluing a surface with one outgoing end to the first incoming end of another surface, and gluing the ends of a surface with single incoming and outgoing ends (which lie on the same boundary component).

Theorem 4.1.9. *If* S *is a disjoint union* $S = S_0 \sqcup S_1$ *with* $z_{e_-} \in S_0$, $z_{e_+} \in S_1$, *and if* S_1 *has a single outgoing end* $\mathcal{E}_+(S_1) = \{e_+\}$, *we define canonical orderings as follows: Suppose that* e_- *is the last incoming end of* S_0 *and the boundary components containing* z_{e_-} *resp.* z_{e_+} *are last in* S_0 *resp. first in* S_1. *Then we order the boundary components and ends of the glued surface* $(S_0)\#_{e_+}^{e_-}(S_1) := \#_{e_+}^{e_-}(S_0 \sqcup S_1)$ *by appending the additional boundaries and incoming ends of* S_1 *to the ordering for* S_0. *With these conventions we have*

$$\Phi_{(S_0)\#_{e_+}^{e_-}(S_1)} = \epsilon \cdot \Phi_{S_0} \circ (1^{\mathcal{E}_-(S_0)\setminus e_-} \otimes \Phi_{S_1}),$$

where $\epsilon = 1$ *if* $n = \frac{1}{2}\dim M$ *is even or the number* b_1 *of boundary components of* $\overline{S}_1$ *is odd, and in general*

$$\epsilon = (-1)^{n(b_1+1)\sum_{e\in\mathcal{E}_-(S_0)\setminus\{e_-\}}(n-|x_e|)}.$$

In our concrete situations the surfaces will always have one boundary component, $b_1 = 1$, and one outgoing end, hence the gluing sign will be $\epsilon = +1$.

Theorem 4.1.10. *If* S *is connected with exactly one incoming and one outgoing end* $\mathcal{E} = \{e_+, e_-\}$ *lying on the same, first boundary component, we define canonical orderings as follows: The glued surface* $\#_{e_+}^{e_-}(S)$ *has no further ends but two new compact boundary components, which we order by taking the one labelled* L^1 *first and that labelled* L^2 *second, where* (L^1, L^2) *denote the ordered boundary conditions at the outgoing end* e_+. *(Then the* *ordered boundary conditions at* e_- *are* (L^2, L^1)*.) After these new components we order the remaining boundary components in the order induced by* S*. With that convention we have*

$$\Phi_{\#_{e_+}^{e_-}(S)} : 1 \mapsto \mathrm{Tr}(\Phi_S) = \sum_i (-1)^{|x_i|} \langle C\Phi_S(\langle x_i \rangle), \langle x_i \rangle \rangle, \tag{30}$$

the (graded) sum over the $\langle x_i \rangle$ *coefficients of* $C\Phi_S(\langle x_i \rangle)$*.*

Example 4.1.11. We compute the invariants for closed surfaces as follows:

(a) (Disk) If S is the disk with boundary condition L, then Φ_S is the number of isolated perturbed J-holomorphic disks with boundary in L. Because of the monotonicity assumption, and since we do not quotient out by automorphisms of the disk, each component of the moduli space of such disks has at least the dimension of L, hence $\Phi_S = 0$.

(b) (Annulus) Let $A = \#S_{||}$ denote the annulus, obtained by gluing along the two ends of the infinite strip $S_{||} = \mathbb{R} \times [0, 1]$ with boundary conditions L_0 and L_1. Let the boundary components be ordered like (L_0, L_1), as in Theorem 4.1.10. Then the gluing formula produces

$$\Phi_A = \mathrm{Tr}(\mathrm{Id}) = \mathrm{rank}\, HF^{\mathrm{even}}(L_0, L_1) - \mathrm{rank}\, HF^{\mathrm{odd}}(L_0, L_1).$$

The same result can be obtained by decomposing the annulus into cup and cap and computing the universal sign to $\Phi_A = \Phi_\cap \circ \Phi_\cup$.

(c) (Sphere with holes) Let S denote the sphere with $g + 1$ disks removed and boundary condition L over each component. S can be obtained by gluing together $g - 1$ copies of the surface S_0, which is obtained by removing a disk from the strip $\mathbb{R} \times [0, 1]$; see Figure 4. The latter defines an automorphism Φ_{S_0} on $HF(L, L)$, and the gluing formulas give

$$\Phi_S = \mathrm{Tr}(\Phi_{S_0}^{g-1}) = \sum (-1)^{|x_i|} \langle \Phi_{S_0}^{g-1}(\langle x_i \rangle), \langle x_i \rangle \rangle.$$

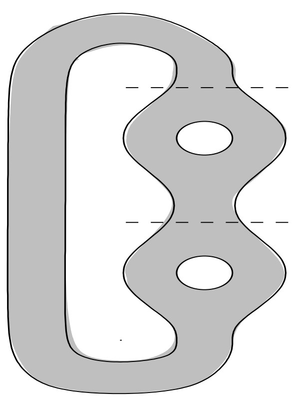


FIGURE 4. Gluing copies of S_0

Remark 4.1.12. One can also allow surfaces to have incoming or outgoing cylindrical ends (with a periodic Hamiltonian perturbation). In this case the relative invariant acts on the product of Floer cohomology groups with a number of copies of the cylindrical Floer cohomology $HF(\mathrm{Id})$, isomorphic to the quantum cohomology $QH(M)$ of M. For instance, a disk with one puncture in the interior gives rise to a canonical element $\phi_L \in HF(\mathrm{Id})$.

Splitting the annulus into two half-cylinders glued at a cylindrical end gives rise to the identity

$$\operatorname{rank} HF^{\text{even}}(L_0, L_1) - \operatorname{rank} HF^{\text{odd}}(L_0, L_1) = \langle \phi_{L_0}, \phi_{L_1} \rangle_{HF(\mathrm{Id})}.$$

More generally, by considering a disk with one interior and one boundary puncture one obtains the open-closed map $HF(L, L) \to HF(\mathrm{Id})$, see Remark 6.8.3.

4.2. **Relative invariants for quilted surfaces.** Quilted surfaces are obtained from a collection of surfaces with strip-like ends by "sewing together" certain pairs of boundary components. We give a formal definition below, again restricting to strip-like ends, i.e. punctures on the boundary. One could in addition allow cylindrical ends by adding punctures in the interior of the surface, see Remark 4.2.7.

Definition 4.2.1. A *quilted surface* $\underline{S}$ with strip-like ends consists of the following data:

(a) A collection $\underline{S} = (S_k)_{k=1,\ldots,m}$ of surfaces with strip-like ends as in Definition 4.1.1 (a)-(c). In particular, each S_k carries a complex structures j_k and has strip-like ends $(\epsilon_{k,e})_{e\in\mathcal{E}(S_k)}$ near marked points $\lim_{s\to\pm\infty} \epsilon_{k,e}(s,t) = z_{k,e} \in \partial\overline{S}_k$.

(b) With the boundary components $\partial S_k = (I_{k,e})_{e\in\mathcal{E}(S_k)}$ in cyclic order on each component, a collection $\mathcal{S}$ of pairwise disjoint 2-element subsets

$$\sigma \subset \bigcup_{k=1}^{m} \{k\} \times \mathcal{E}(S_k),$$

and for each $\sigma = \{(k_\sigma, e_\sigma), (k'_\sigma, e'_\sigma)\} \in \mathcal{S}$, an identification of the corresponding boundary components

$$\varphi_\sigma : I_{k_\sigma, e_\sigma} \xrightarrow{\sim} I_{k'_\sigma, e'_\sigma}$$

Here either $I_{k_\sigma,e_\sigma} \cong S^1 \cong I_{k'_\sigma,e'_\sigma}$ or φ_σ is compatible with the strip-like ends. The latter means that the two ends e_σ and $e'_\sigma - 1$ are either both incoming and we have $\varphi_\sigma(\epsilon_{k_\sigma,e_\sigma}(s,0)) = \epsilon_{k'_\sigma,e'_\sigma-1}(s,1)$, or they are both outgoing and $\varphi_\sigma(\epsilon_{k_\sigma,e_\sigma}(s,1)) = \epsilon_{k'_\sigma,e'_\sigma-1}(s,0)$. Similarly, the ends $e_\sigma - 1$ and e'_σ are either both incoming and $\varphi_\sigma(\epsilon_{k_\sigma,e_\sigma-1}(s,1)) = \epsilon_{k'_\sigma,e'_\sigma}(s,0)$, or they are both outgoing and $\varphi_\sigma(\epsilon_{k_\sigma,e_\sigma-1}(s,0)) = \epsilon_{k'_\sigma,e'_\sigma}(s,1)$.

(c) Orderings of the boundary components of each $\overline{S}_k$ as in Definition 4.1.1 (d). There are no orderings of ends of components but orderings $\mathcal{E}_-(\underline{S}) = (\underline{e}_1^-, \ldots, \underline{e}_{N_-(\underline{S})}^-)$ and $\mathcal{E}_+(\underline{S}) = (\underline{e}_1^+, \ldots, \underline{e}_{N_-(\underline{S})}^+)$ of the incoming and outgoing ends of $\underline{S}$. Here each end $\underline{e} \in \mathcal{E}(\underline{S}) = \mathcal{E}_-(\underline{S}) \sqcup \mathcal{E}_+(\underline{S})$ of the quilt consists of a maximal sequence of ends $\underline{e} = (k_i, e_i)_{i=1,\ldots,n_{\underline{e}}}$ with boundaries $\epsilon_{k_i,e_i}(\cdot,1) \cong \epsilon_{k_{i+1},e_{i+1}}(\cdot,0)$ identified via some seam ϕ_{σ_i}. (The sequence could be cyclic, i.e. with an additional identification $\epsilon_{k_n,e_n}(\cdot,1) \cong \epsilon_{k_1,e_1}(\cdot,0)$ via some seam ϕ_{σ_n}.) These ends are either all incoming, and hence $e_i \in \mathcal{E}_-(S_{k_i})$, or they are all outgoing, and hence $e_i \in \mathcal{E}_+(S_{k_i})$.

We call $\sigma \in \mathcal{S}$ the *seams* of $\underline{S}$. The remaining "true boundary components" of $\underline{S}$ are indexed by

$$\mathcal{B} := \bigcup_{k=1}^{m} \{k\} \times \mathcal{E}(S_k) \ \setminus \ \bigcup_{\sigma\in\mathcal{S}} \sigma.$$

This is the set of ends (k, e) whose corresponding boundary component $I_{k,e}$ is not identified with another boundary component of $\underline{S}$. A picture of a quilt is shown in Figure 5. Ends at the top resp. bottom of a picture will always be outgoing resp. incoming ends. The alternative picture is that in which the ends converge to a point and we indicate by arrows whether the ends are outgoing or incoming. Here we draw the quilted surface as the interior of a circle (in general this could be a more general surface), although this is somewhat misleading as the outer edges of the circle consist of boundary components of different surfaces. In this example the end sequences are $(2,0),(1,0)$ and $(2,1)$ for the incoming ends (at the bottom), and $(2,2),(3,0),(2,3),(1,1)$ for the outgoing end (at the top). (Our only choice here is which marked point on the boundary circles of S_i to label by 0.)

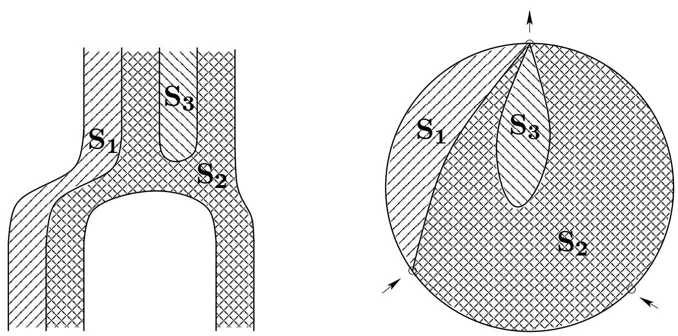


FIGURE 5. Two views of a quilted surface

Elliptic boundary value problems are associated to quilted surfaces with strip-like ends as follows. Suppose that $\underline{E} = (E_k)_{k=1,\ldots,m}$ is a collection of complex vector bundles over the components of $\underline{S}$, and $\underline{F}$ is a collection of totally real sub-bundles

$$\underline{F} = \big(F_{(k_\sigma,e_\sigma),(k'_\sigma,e'_\sigma)} \subset E^-_{k_\sigma} \times E_{k'_\sigma}\big)_{\sigma\in\mathcal{S}} \cup \big(F_{(k,e)}\big)_{(k,e)\in\mathcal{B}}.$$

Here we write E^- as short hand for the complex vector bundle with reversed complex structure, $(E,J)^- = (E,-J)$. Suppose, furthermore, that the totally real boundary conditions are transverse along each end. Let

$$\Omega^0(\underline{S},\underline{E};\underline{F}) \subset \bigoplus \Omega^0(S_k,E_k)$$

denote the subspace of collections of sections $u_k \in \Gamma(E_k)$ such that $(u_{k_\sigma}, u_{k'_\sigma})$ maps $I_{(k_\sigma,e_\sigma)} \cong I_{(k'_\sigma,e'_\sigma)}$ to $F_{(k_\sigma,e_\sigma),(k'_\sigma,e'_\sigma)}$ for every $\sigma \in \mathcal{S}$ and u_k maps $I_{(k,e)}$ to $F_{(k,e)}$ for every $(k,e) \in \mathcal{B}$. The direct sum of Cauchy-Riemann operators

$$D_{\underline{E},\underline{F}} : \Omega^0(\underline{S},\underline{E};\underline{F}) \to \Omega^{0,1}(\underline{S},\underline{E}) := \bigoplus \Omega^{0,1}(S_k,E_k)$$

maps to the direct sum of $(0,1)$-forms on the components, and we denote by $\mathrm{Ind}(\underline{E},\underline{F})$ its index. In the case that $\underline{S}$ has no strip-like ends, we define a topological index $I(\underline{E},\underline{F})$ as follows. For each component S_k with boundary we choose a complex trivialization of the bundle $E_k \cong S_k \times \mathbb{C}^{r_k}$. Each bundle

$$F_{(k,e)} \subset E_k|_{I_{(k,e)}} \cong I_{k,e} \times \mathbb{C}^{r_k}$$

has a Maslov index $I(F_{(k,e)})$ depending on the trivialization of E_k. Similarly,

$$F_{(k_\sigma,e_\sigma),(k'_\sigma,e'_\sigma)} \subset E^-_{k_\sigma}|_{I_{(k_\sigma,e_\sigma)}} \times \varphi^*_{(k_\sigma,e_\sigma),(k'_\sigma,e'_\sigma)} E_{k'_\sigma}|_{I_{(k'_\sigma,e'_\sigma)}} \cong I_{k_\sigma,e_\sigma} \times (\mathbb{C}^{r_{k_\sigma}})^- \otimes \mathbb{C}^{r_{k'_\sigma}}$$

has a Maslov index $I(F_{(k_\sigma,e_\sigma),(k'_\sigma,e'_\sigma)})$ depending on the trivializations of E_{k_σ} and $E_{k'_\sigma}$. Let $S_0 \subset \underline{S}$ be the union of components without boundary and define

$$I(\underline{E},\underline{F}) := \deg(\underline{E}|S_0) + \sum_{\sigma\in\mathcal{S}} I(F_{(k_\sigma,e_\sigma),(k'_\sigma,e'_\sigma)}) \; + \sum_{(k,e)\in\mathcal{B}} I(F_{(k,e)}),$$

where the second sum is over seams and third sum over boundaries of $\underline{S}$. We leave it to the reader to check that the sum is independent of the choice of trivializations. Both the topological and analytic index are invariant under deformation, and by deforming the boundary conditions to those of split form one obtains from (24) an index formula

$$\mathrm{Ind}(D_{\underline{E},\underline{F}}) = \sum_i \mathrm{rank}_{\mathbb{C}}(E_i)\chi(S_i) + I(\underline{E},\underline{F}).$$

We now construct moduli spaces of pseudoholomorphic quilted surfaces. Let

$$\underline{M} = \big((M_k,\omega_k)\big)_{k=1,\ldots,m}$$

be a collection of symplectic manifolds. A *Lagrangian boundary condition* for $(\underline{S},\underline{M})$ is a collection

$$\underline{L} = \big(L_{(k_\sigma,e_\sigma),(k'_\sigma,e'_\sigma)} \subset M^-_{k_\sigma}\times M_{k'_\sigma}\big)_{\sigma\in\mathcal{S}} \cup \big(L_{(k,e)}\subset M_k\big)_{(k,e)\in\mathcal{B}}$$

of Lagrangian correspondences and Lagrangian submanifolds associated to the seams and boundary components of the quilted surface. We will indicate the domains M_k, the "seam conditions" $L_{(k_\sigma,e_\sigma),(k'_\sigma,e'_\sigma)}$, and the true boundary conditions $L_{(k,e)}$ by marking the surfaces, seams and boundaries of the quilted surfaces as in figure 6.

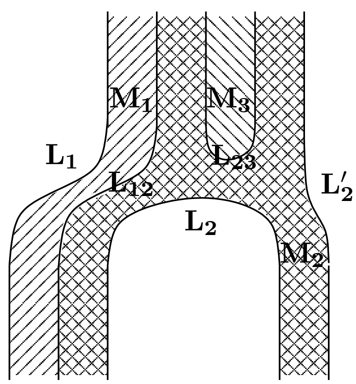


FIGURE 6. Lagrangian boundary conditions for a quilt

We say that the tuples $\underline{M}$ and $\underline{L}$ are *graded* if each M_k is equipped with an N-fold Maslov covering for a fixed $N\in\mathbb{N}$ and each Lagrangian $L_{(k_\sigma,e_\sigma),(k'_\sigma,e'_\sigma)} \subset M_{k_\sigma}\times M_{k'_\sigma}$ and $L_{(k,e)}\subset M_k$ is graded with respect to the respective Maslov covering. Moreover, we assume that the gradings are compatible with orientations in the sense of (G1-2).

We say that $\underline{L}$ is *relatively spin* if all Lagrangians in the tuple are relatively spin with respect to one fixed set of background classes $b_k\in H^2(M_k,\mathbb{Z}_2)$, see [46].

Definition 4.2.2. We say that the Lagrangian boundary condition $\underline{L}$ for $\underline{S}$ is *monotone* if the sequences $\underline{L}_{\underline{e}}$ in (31) are monotone in the sense of Section 3.3 for each end $\underline{e}\in\mathcal{E}(\underline{S})$ and the following holds: Let $\underline{S}^-\#\underline{S}$ denote the quilted surface obtained by gluing a copy $\underline{S}^-$ of $\underline{S}$ with reversed complex structure (and hence reversed ends) to $\underline{S}$ at all corresponding ends.

This quilted surface has components $(S_k^- \# S_k)_{k=1,\ldots,m}$, seams $(C_{(k_\sigma,e_\sigma),(k'_\sigma,e'_\sigma)} \cong S^1)_{\sigma\in\mathcal{S}}$, and boundary components $(C_{k,e} \cong S^1)_{(k,e)\in\mathcal{B}}$, but no strip-like ends. Then for each tuple of maps $\underline{u} : \underline{S}^- \# \underline{S} \to \underline{M}$ (that is $u_k : S_k^- \# S_k \to M_k$) that takes values in $\underline{L}$ over the seams and boundary components (that is $(u_{k_\sigma} \times u_{k'_\sigma})(C_{(k_\sigma,e_\sigma),(k'_\sigma,e'_\sigma)}) \subset L_{(k_\sigma,e_\sigma),(k'_\sigma,e'_\sigma)}$ and $u_k(C_{k,e}) \subset L_{(k,e)}$) we have the action-index relation

$$2\sum_{k=1}^m \int u_k^*\omega_k = \tau I(\underline{E}_u, \underline{F}_u)$$

for $\underline{E}_u = (u_k^* TM_k)_{k=1,\ldots,m}$ and

$$\underline{F}_u = \big((u_{k_\sigma} \times u_{k'_\sigma})^* TL_{(k_\sigma,e_\sigma),(k'_\sigma,e'_\sigma)}\big)_{\sigma\in\mathcal{S}} \cup (u_k^* TL_{(k,e)})_{(k,e)\in\mathcal{B}}.$$

Note the following analogue of Lemma 3.1.3: If each M_k is monotone in the sense of (M1) and each $L_{(k_\sigma,e_\sigma),(k'_\sigma,e'_\sigma)}$ and $L_{(k,e)}$ is monotone in the sense of (L1), all with the same constant $\tau \geq 0$, and each $\pi_1(L_{(k_\sigma,e_\sigma),(k'_\sigma,e'_\sigma)}) \to \pi_1(M_{k_\sigma}^- \times M_{k'_\sigma})$ and $\pi_1(L_{(k,e)}) \to \pi_1(M_k)$ has torsion image, then $\underline{L}$ is monotone.

For every end $\underline{e}_\pm = ((k_i,e_i))_{i=1,\ldots,n_{\underline{e}}} \in \mathcal{E}_\pm(\underline{S})$ we will consider the following sequence of Lagrangian correspondences:

$$\begin{aligned} \underline{L}_{\underline{e}_-} &:= (L_{(k_1,e_1)}, L_{(k_1,e_1-1)(k_2,e_2)}, \ldots, L_{(k_{n-1},e_{n-1}-1)(k_n,e_n)}, L_{(k_n,e_n-1)}), \\ \underline{L}_{\underline{e}_+} &:= (L_{(k_1,e_1-1)}, L_{(k_1,e_1)(k_2,e_2-1)}, \ldots, L_{(k_{n-1},e_{n-1})(k_n,e_n-1)}, L_{(k_n,e_n)}), \end{aligned} \tag{31}$$

with $n = n_{\underline{e}_\pm}$. (If the end is cyclic, then the corresponding sequence is cyclic - with the first and last entry above replaced by $L_{(k_n,e_n-1)(k_1,e_1)}$ resp. $L_{(k_n,e_n)(k_1,e_1-1)}$.) We can fix a perturbation datum $(H_{k_i,e_i})_{i=1,\ldots,n_{\underline{e}}}$ and $(J_{k_i,e_i})_{i=1,\ldots,n_{\underline{e}}}$ for each end $\underline{e} \in \mathcal{E}(\underline{S})$ such that the Floer homology $HF(\underline{L}_{\underline{e}})$ is well defined. (By Section 3.3 these can be chosen of split form.) In particular, the intersection of the Lagrangian sequence $\mathcal{I}(\underline{L}_{\underline{e}})$ is finite for each end $e \in \mathcal{E}(\underline{S})$, by Proposition 3.3.3. For $\underline{e} \in \mathcal{E}_-(\underline{S})$ (and similarly for $\underline{e} \in \mathcal{E}_+(\underline{S})$ and in the cyclic case) this intersection corresponds bijectively to

$$L_{k_1,e_1} \times_{\phi_1} L_{(k_1,e_1-1)(k_2,e_2)} \times_{\phi_2} \ldots L_{(k_{n-1},e_{n-1}-1)(k_n,e_n)} \times_{\phi_n} L_{(k_n,e_n-1)} \subset \prod_{i=1}^n M_{k_i}.$$

Here ϕ_i denotes the time 1 flow of the Hamiltonian H_{k_i,e_i} on M_{k_i}. Next, let $\mathrm{Ham}(\underline{S},\underline{L})$ denote the set of tuples

$$\underline{K} = \big(K_k \in \Omega^1(S_k, C^\infty(M_k))\big)_{k=1,\ldots,m}$$

such that $K_k|_{\partial S_k} = 0$ and on each end $\epsilon_{k,e}^* K_k = H_{k,e}\mathrm{d}t$. We denote the corresponding Hamiltonian vector field valued one-forms by $\underline{Y} \in \Omega^1(\underline{S}, \mathrm{Vect}(\underline{M}))$. These satisfy $\epsilon_{k,e}^* Y_k = X_{H_{k,e}}\mathrm{d}t$ on each strip-like end. Next, let $\mathcal{J}(\underline{S},\underline{L})$ denote the set of collections

$$\underline{J} = \big(J_k \in \mathrm{Map}(S_k, \mathcal{J}(M_k,\omega_k))\big)_{k=1,\ldots,m}$$

agreeing with the chosen almost complex structures on the ends. Now we denote by $\underline{\mathcal{I}}_-(\underline{S},\underline{L})$ resp. $\underline{\mathcal{I}}_+(\underline{S},\underline{L})$ the set of tuples $X^\pm = (\underline{x}_{\underline{e}}^\pm)_{\underline{e}\in\mathcal{E}_\pm(\underline{S})}$ consisting of one intersection tuple $\underline{x}_{\underline{e}}^\pm = (x_{k_i,e_i}^\pm)_{i=1,\ldots,n_{\underline{e}}} \in \mathcal{I}(\underline{L}_{\underline{e}})$ for each incoming resp. outgoing end $\underline{e}$. For each pair $(X^-,X^+) \in \underline{\mathcal{I}}_-(\underline{S},\underline{L}) \times \underline{\mathcal{I}}_+(\underline{S},\underline{L})$ we denote by

$$\mathcal{M}_{\underline{S}}(X^-,X^+) := \big\{\underline{u} = \big(u_k : S_k \to M_k\big)_{k=1,\ldots,m} \,\big|\, (a)-(d)\big\}$$

the space of collections of $(\underline{J}, \underline{K})$-holomorphic maps with Lagrangian boundary and seam conditions, finite energy, and fixed limits, that is

(a) $J_k(u_k) \circ (\mathrm{d}u_k - Y_k(u_k)) = (\mathrm{d}u_k - Y_k(u_k)) \circ j_k$ for $k = 1, \ldots, m$,
(b) $(u_{k_\sigma}, u_{k'_\sigma} \circ \varphi_\sigma)(I_{k_\sigma, e_\sigma}) \subset L_{(k_\sigma, e_\sigma),(k'_\sigma, e'_\sigma)}$ for all $\sigma \in \mathcal{S}$ and $u_k(I_{k,e}) \subset L_{(k,e)}$ for all $(k, e) \in \mathcal{B}$.
(c) $E_{\underline{K}}(\underline{u}) = \sum_{k=1}^m E_{K_k}(u_k) < \infty$,
(d) $\lim_{s \to \pm\infty} u_{k_i}(\epsilon_{k_i, e_i}(s, t)) = x^{\pm}_{k_i, e_i}(t)$ for all $\underline{e} = (k_i, e_i)_{i=1,\ldots,n_{\underline{e}}} \in \mathcal{E}_\pm(\underline{S})$.

Remark 4.2.3. If $\underline{L}$ is monotone, then elements $\underline{u} \in \mathcal{M}_{\underline{S}}(X^-, X^+)$ satisfy an energy-index relation

$$2E_{\underline{K}}(\underline{u}) = \tau I(\underline{u}^* T\underline{M}, \underline{u}^* T\underline{L}) + c(X^-, X^+)$$

as in Remark 4.1.2. Moreover, the index $I(\underline{u}^* T\underline{M}, \underline{u}^* T\underline{L})$ is determined mod 2 by the limit conditions X^-, X^+, due to the same discussion as before on each component of the quilted surface.

Theorem 4.2.4. *Suppose that each M_k satisfies (M1-2), $\underline{L}$ is monotone, and each Lagrangian in $\underline{L}$ satisfies (L1-2). For any $\underline{K} \in \mathrm{Ham}(\underline{S}, \underline{L})$ there exists a subset $\mathcal{J}(\underline{S}, \underline{L}, \underline{K})^{\mathrm{reg}} \subset \mathcal{J}(\underline{S}, \underline{L})$ of Baire second category such that for all $X^\pm \in \underline{\mathcal{I}}_\pm(\underline{S}, \underline{L})$*

(a) *$\mathcal{M}_{\underline{S}}(X^-, X^+)$ is a smooth manifold;*
(b) *The zero dimensional component $\mathcal{M}_{\underline{S}}(X^-, X^+)_0$ is finite;*
(c) *The one-dimensional component $\mathcal{M}_{\underline{S}}(X^-, X^+)_1$ has a compactification as a one-manifold with boundary*

$$\partial \overline{\mathcal{M}_{\underline{S}}(X^-, X^+)_1} \cong \bigcup_{\underline{e} \in \mathcal{E}_-, \underline{y} \in \mathcal{I}(\underline{L}_{\underline{e}})} \mathcal{M}(\underline{x}^-_{\underline{e}}, \underline{y})_0 \times \mathcal{M}_{\underline{S}}(X^-|_{\underline{x}^-_{\underline{e}} \to \underline{y}}, X^+)_0$$
$$\cup \bigcup_{\underline{e} \in \mathcal{E}_+, \underline{y} \in \mathcal{I}(\underline{L}_{\underline{e}})} \mathcal{M}_{\underline{S}}(X^-, X^+|_{\underline{x}^+_{\underline{e}} \to \underline{y}})_0 \times \mathcal{M}(\underline{y}, \underline{x}^+_{\underline{e}})_0,$$

where the multi-tuple $X|_{\underline{x}_{\underline{e}} \to \underline{y}}$ is X with the tuple $\underline{x}_{\underline{e}}$ replaced by $\underline{y}$.
(d) *If $\underline{L}$ is relatively spin, then there exists a coherent set of orientations on the zero and one-dimensional moduli spaces so that the inclusion of the boundary in (c) has the signs $(-1)^{\sum_{\underline{f} < \underline{e}} |\underline{x}^-_{\underline{f}}|}$ (for incoming trajectories) and $-(-1)^{\sum_{\underline{f} < \underline{e}} |\underline{x}^+_{\underline{f}}|}$ (for outgoing trajectories.)*

Sketch of Proof: The first remark is that the elliptic estimates, used in the Fredholm theory and for the compactness, are *locally constructed*. Near any seam in the quilted surface we may treat a J-holomorphic map of the quilted surface as a J-holomorphic map of the half-plane with Lagrangian boundary conditions. Similarly, in the proof of compactness, the development of sphere or disk bubbles is an entirely local phenomenon and so ruled out, for the zero and one-dimensional moduli spaces, by the energy-index relation and the assumptions (L1-2) which imply that all Lagrangians in $\underline{L}$ have minimal Maslov number at least 2. Exponential decay holds on the ends since these are simple strip-like ends (with values in a product of manifolds) as in Lemma 3.2.1. To achieve transversality we can, as in Remark 3.3.4, work with split almost complex structures on each surface of the quilt. The gluing construction, which shows the reverse inclusion in (c), is local near each end in the sense that it uses only the non-degeneracy of operator on the neck and the regularity of the linearized operators for the Floer trajectories, which hold by the choice of the perturbation data. The orientations are defined, and coherence is proven in [46]. □

Associated to the data $(\underline{S}, \underline{M}, \underline{L})$ as in Theorem 4.2.4 we construct a relative invariant $\Phi_{\underline{S}}$ as follows. Define

$$C\Phi_{\underline{S}} : \bigotimes_{\underline{e}\in\mathcal{E}_-(\underline{S})} CF(\underline{L}_{\underline{e}}) \to \bigotimes_{\underline{e}\in\mathcal{E}_+(\underline{S})} CF(\underline{L}_{\underline{e}})$$

by

$$C\Phi_{\underline{S}}\Big(\bigotimes_{\underline{e}\in\mathcal{E}_-} \langle \underline{x}_{\underline{e}}^-\rangle\Big) := \sum_{X^+\in\underline{\mathcal{I}}_+(\underline{S},\underline{L})} \Bigg(\sum_{\underline{u}\in\mathcal{M}_{\underline{S}}(X^-,X^+)_0} \epsilon(u)\Bigg) \bigotimes_{\underline{e}\in\mathcal{E}_+} \langle \underline{x}_{\underline{e}}^+\rangle,$$

where

$$\epsilon : \mathcal{M}_{\underline{S}}(X^-, X^+)_0 \to \{-1, +1\}$$

is defined by comparing the orientation given by Theorem 4.2.4 (d) to the canonical orientation of a point. By items (c),(d) of Theorem 4.2.4, the maps $C\Phi_{\underline{S}}$ are chain maps and so descend to a map of Floer cohomologies

$$\Phi_{\underline{S}} : \bigotimes_{\underline{e}\in\mathcal{E}_-(\underline{S})} HF(\underline{L}_{\underline{e}}) \to \bigotimes_{\underline{e}\in\mathcal{E}_+(\underline{S})} HF(\underline{L}_{\underline{e}}). \tag{32}$$

Here we assume in addition that all Lagrangians in $\underline{L}$ satisfy (L3) and hence the Floer cohomologies are well defined.

Floer's argument using parametrized moduli spaces (as sketched in Remark 4.1.4) carries over to this case to show that $\Phi_{\underline{S}}$ is independent of the choice of perturbation data, complex structures j_k on S_k, and the strip-like ends.

Remark 4.2.5. The effect of the relative invariant $\Phi_{\underline{S}}$ on the grading is by a shift in degree of

$$|\Phi_{\underline{S}}| = \sum_{k=1}^{m} \tfrac{1}{2}\dim M_k\big(\#\mathcal{E}_+(S_k) - \chi(\overline{S}_k)\big) \qquad \text{mod } N. \tag{33}$$

To check this we consider an isolated solution $\underline{u} \in \mathcal{M}_{\underline{S}}(X^-, X^+)_0$. We can deform the linearized seam conditions on the ends, $T_{\underline{x}_{\underline{e}}^{\pm}}\underline{L}_{\underline{e}}$ for $\underline{e} \in \mathcal{E}_\pm(\underline{S})$, to split type, $(\Lambda_{(k_1,e_1)}, \Lambda_{(k_1,e_1-1)} \times \Lambda_{(k_2,e_2)}, \ldots, \Lambda_{(k_{n-1},e_{n-1}-1)} \times \Lambda_{(k_n,e_n)}, \Lambda_{(k_n,e_n-1)})$ for $\underline{e} \in \mathcal{E}_-(\underline{S})$ resp. $(\Lambda_{(k_1,e_1-1)}, \Lambda_{(k_1,e_1)} \times \Lambda_{(k_2,e_2-1)}, \ldots, \Lambda_{(k_{n-1},e_{n-1})} \times \Lambda_{(k_n,e_n-1)}, \Lambda_{(k_n,e_n)})$ for $\underline{e} \in \mathcal{E}_+(\underline{S})$, without changing the index. (This is possible since the space of Lagrangian subspaces transverse to a given one is always connected.) Note here that the indexing $\Lambda_{(k,e)}$ is simplified since the boundary component I_e of the surface S_k occurs in two – not necessarily different – ends, but with a different linearized boundary condition $\Lambda_{(k,e)}$. The point is that Lemma 2.3.5 expresses the degree of each end as a sum,

$$|\underline{x}_{\underline{e}}^-| = \sum_{i=1}^{n} d(\tilde\Lambda_{(k_i,e_i)}, \tilde\Lambda_{(k_i,e_i-1)}), \qquad |\underline{x}_{\underline{e}}^+| = \sum_{i=1}^{n} d(\tilde\Lambda_{(k_i,e_i-1)}, \tilde\Lambda_{(k_i,e_i)}),$$

and summing this over all incoming resp. outgoing ends of $\underline{S}$ is the same as summing over all incoming resp. outgoing ends of all surfaces S_k, $k = 1, \ldots, m$. Next, we deform the linearized boundary conditions with fixed ends to split type over each seam. Again, this does not affect the index and thus results in a splitting of the index $0 = \sum_{k=1}^m \mathrm{Ind}(D_{E_k,F_k})$ into the indices of Cauchy-Riemann operators on $E_k = u_k^*TM_k$ with boundary conditions in the totally real subbundles $F_k \subset E_k|_{\partial S_k}$ that are constant equal to some $\Lambda_{(k,\cdot)}$ on the strip-like ends. From Remark 4.1.5 we have a mod N degree index identity for each surface,

$$\tfrac{1}{2}\chi(\overline{S}_k)\dim M_k = \operatorname{Ind}(D_{E_k,F_k}) + \sum_{e\in\mathcal{E}_+(S_k)} \big(\tfrac{1}{2}\dim M_k - d(\tilde{\Lambda}^0_e,\tilde{\Lambda}^1_e)\big) + \sum_{e\in\mathcal{E}_-(S_k)} d(\tilde{\Lambda}^0_e,\tilde{\Lambda}^1_e),$$

where the degree $d(\tilde{\Lambda}^0_e,\tilde{\Lambda}^1_e)$ for each end equals to $d(\tilde{\Lambda}_{(k,e-1)},\tilde{\Lambda}_{(k,e)})$ resp. $d(\tilde{\Lambda}_{(k,e)},\tilde{\Lambda}_{(k,e-1)})$ for $e\in\mathcal{E}_+(S_k)$ resp. $e\in\mathcal{E}_-(S_k)$. So, summing over all surfaces S_k we obtain as claimed

$$\sum_{k=1}^m \tfrac{1}{2}\dim M_k\big(\chi(\overline{S}_k) - \#\mathcal{E}_+(S_k)\big) = -\sum_{\underline{e}\in\mathcal{E}_+(\underline{S})} |\underline{x}^+_{\underline{e}}| + \sum_{\underline{e}\in\mathcal{E}_-(\underline{S})} |\underline{x}^-_{\underline{e}}| \qquad \operatorname{mod} N.$$

Remark 4.2.6. When working with $\mathbb{Z}$ coefficients, then changing the ordering of the patches (and hence their boundary components) in $\underline{S}$ changes the relative invariant $\Phi_{\underline{S}}$ by a universal sign. These signs are only affected by the ordering of patches S_k whose deficiency $\#\mathcal{E}_+(S_k) - b(S_k)$ is odd. Here $b(S_k)$ denotes the number of boundary components of $\overline{S}_k$. We will not mention ordering for patches of even deficiency, e.g. for patches with one boundary component and one outgoing end.

Remark 4.2.7. As in Remark 4.1.12, one can also allow the component surfaces to have incoming and outgoing cylindrical ends. In this case, the relative invariants (32) have additional factors of cylindrical Floer homologies $HF(\mathrm{Id}_{M_k})$ on either side.

The gluing theorem 4.1.8 generalize to the quilted case as follows. Let $\underline{S}$ be a quilted surface, $\underline{M}$ a collection of monotone symplectic manifolds associated to the components of $\underline{S}$, and $\underline{L}$ Lagrangian boundary conditions for $\underline{S}$ as in Theorem 4.2.4, satisfying in addition (L3) and (G1-2). Associated to $\underline{S}$ we may form an unquilted surface with strip-like ends $\underline{S}'$, by "cutting along the seams". Let $\underline{L}'$ be a collection of boundary conditions for the corresponding collection of symplectic manifolds $\underline{M}'$.

Theorem 4.2.8. *Suppose that $\underline{e}_\pm = (k_i^\pm, e_i^\pm)_{i=1,\dots,N} \in \mathcal{E}_\pm(\underline{S})$ are ends with $n_{\underline{e}_-} = n_{\underline{e}_+} = N$ and such that the data $M_{k_i^-,e_i^-} = M_{k_i^+,e_i^+}$ and $\underline{L}_{\underline{e}_-} = \underline{L}_{\underline{e}_+}$ coincide. Then we have*

$$\Phi_{\#^{\underline{e}_-}_{\underline{e}_+}(\underline{S})} = \epsilon_{\underline{S},\#^{e_-}_{e_+}(\underline{S}),\underline{M}} \operatorname{Tr}_{\underline{e}_-,\underline{e}_+}(\Phi_{\underline{S}}), \tag{34}$$

where $\#^{\underline{e}_-}_{\underline{e}_+}(\underline{S})$ is the quilted surface obtained by gluing the ends in $\underline{e}_-$ to the corresponding ends in $\underline{e}_+$. The algebraic trace $\operatorname{Tr}_{\underline{e}_-,\underline{e}_+}$ is defined by the formula (29) but using the quilted cup and cap (a union of strips with Lagrangian boundary and seam conditions given by $\underline{L}_{\underline{e}_\pm}$ as in Figure 7, but with two outgoing resp. incoming ends) to define $\Phi_\cup, \Phi_\cap$. The sign $\epsilon_{\underline{S},\#^{e_-}_{e_+}(\underline{S}),\underline{M}}$ is the gluing sign in Theorem 4.1.8 for the unquilted surface $\underline{S}'$; it depends on the orderings of the boundary components and the dimensions of the entries of $\underline{M}$.

Unfortunately, the explicit gluing sign is too complicated to write down even in the special case of a disjoint union $\underline{S} = \underline{S}_0 \sqcup \underline{S}_1$ of quilted surfaces.

4.3. **Quilted Floer cohomology.** In the following we reformulate the definition of Floer cohomology for sequences of Lagrangian correspondences in terms of quilted surfaces. As in Section 3.3 consider a cyclic generalized Lagrangian correspondence $\underline{L}$, that is, a sequence of symplectic manifolds

$$M_0, M_1, \dots, M_r, M_{r+1} \qquad \text{such that } M_0 = M_{r+1}$$

for $r \geq 0$, and a sequence of Lagrangian correspondences

$$L_{01} \subset M_0^- \times M_1, \quad L_{12} \subset M_1^- \times M_2, \quad \ldots, \quad L_{r(r+1)} \subset M_r^- \times M_{r+1}.$$

Suppose that these satisfy (M1-2) with the same value of the monotonicity constant τ, (L1-3), and the monotonicity assumption[11] for the pair $(L_{(0)}, L_{(1)})$, as in Definition 3.1.2. Proposition 3.3.3 provides a tuple of Hamiltonians $\underline{H} = (H_k : M_k \to \mathbb{R})_{k=0\ldots,r} \in \mathrm{Ham}(\underline{L})$ such that the intersection points

$$\mathcal{I}(\underline{L}) = \left\{ \underline{x} = \big(x_k : [0,1] \to M_k\big)_{k=0,\ldots,r} \;\middle|\; \begin{array}{r} \dot{x}_k(t) = X_{H_k}(x_k(t)), \\ (x_k(1), x_{k+1}(0)) \in L_{k(k+1)} \end{array} \right\}$$

(with $x_{r+1} := x_0$) are nondegenerate and hence finite. The Floer chain group is

$$CF(\underline{L}) := \bigoplus_{x \in \mathcal{I}(\underline{L})} \mathbb{Z}\langle x \rangle.$$

As seen in Remark 3.3.4 there exists a regular tuple $(J_k \in \mathcal{J}_t(M_k, \omega_k))_{k=0,\ldots,r}$ of almost complex structures such that the Floer differential can be defined by counting tuples of finite energy (J_k, H_k)-holomorphic maps $(u_k : \mathbb{R} \times [0,1] \to M_k)_{k=0,\ldots,r}$ with limits corresponding to $\underline{x}^\pm \in \mathcal{I}(\underline{L})$ and boundary conditions $(u_k(s,1), u_{k+1}(s,0)) \in L_{k(k+1)}$ (where $u_{r+1} := u_0$). These exactly correspond to the *quilted Floer trajectories*

$$\underline{u} \in \mathcal{M}_{\underline{Z}}(\underline{x}^-, \underline{x}^+) \qquad \text{with} \quad \underline{K} = (H_k \mathrm{d}t)_{k=0,\ldots,r}, \quad \underline{J} = (J_k)_{k=0,\ldots,r}.$$

Here the quilted surface is the quilted cylinder $\underline{Z} = (S_k = \mathbb{R} \times [0,1])_{k=0,\ldots,r}$ as indicated on the right in Figure 7, with the canonical ends $\epsilon_{k,e_+}(s,t) = (s, 1+t)$ and $\epsilon_{k,e_-}(s,t) = (-s, -1-t)$, seams $\sigma_k = \{(k,e_+),(k+1,e_-)\}$ for $k = 0,\ldots,r$ modulo $(r+1)$, ends $\underline{e}_- = \big((0,e_-),(1,e_-),\ldots,(r,e_-)\big)$, $\underline{e}_+ = \big((0,e_+),(1,e_+),\ldots,(r,e_+)\big)$, and no remaining boundary components. On the left in Figure 7 we indicated the special case of a noncyclic sequence of Lagrangian correspondences with $M_0 = \{\mathrm{pt}\}$.

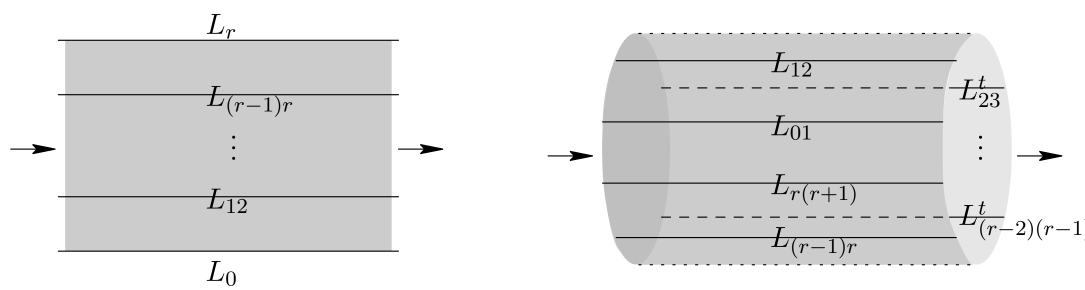


FIGURE 7. Quilted Floer trajectories for $M_0 = \{\mathrm{pt}\}$ and in general

Note however that the perturbation data $(\underline{J}, \underline{K})$ is $\mathbb{R}$-invariant and the count for the Floer differential is modulo simultaneous $\mathbb{R}$-shift of all u_k. Hence we have for all $\underline{x}^- \in \mathcal{I}(\underline{L})$

$$\partial\langle \underline{x}^- \rangle = \sum_{\underline{x}^+ \in \mathcal{I}(\underline{L})} \Big(\sum_{\underline{u} \in \mathcal{M}_{\underline{Z}}(\underline{x}^-, \underline{x}^+)_1/\mathbb{R}} \epsilon(u) \Big) \langle \underline{x}^+ \rangle,$$

[11] The monotonicity assumption can be phrased directly for $\underline{L}$ (without reference to the pair $L_{(0)}, L_{(1)}$), as in Definition 4.2.2: $\underline{L}$ is a monotone boundary condition for the quilted cylinder $\underline{Z}$ defined below. This amounts to the action-index relation $2\sum_{k=0}^r \int u_k^*\omega_k = \tau I((u_k^* TM_k), s_{k(k+1)}^* TL_{k(k+1)})$ for each tuple of maps $u_k : \mathbb{R} \times S^1 \to M_k$ with seam conditions $s_{k(k+1)}(s) := (u_k(s,1), u_{k+1}(s,0)) \in L_{k(k+1)}$.

where the sign $\epsilon(u) = \pm 1$ is given by comparing the orientation on $\mathcal{M}_{\underline{Z}}(\underline{x}^-, \underline{x}^+)_1$ with the canonical orientation induced by the $\mathbb{R}$-action. By the identification with the construction in Section 3.3 we know that $\partial \circ \partial = 0$ and $HF(\underline{L}) := \ker \partial / \operatorname{im} \partial \cong HF(L_{(0)}, L_{(1)})$ is independent up to isomorphism of the choice of perturbation data $(\underline{H}, \underline{J})$.

In the quilted setup for $HF(\underline{L})$ we can introduce further auxiliary choices by using strips $S_k = \mathbb{R} \times [0, \delta_k]$ of width $\underline{\delta} = (\delta_k > 0)_{k=0,\dots,r}$ in the quilted cylinder $\underline{Z}_{\underline{\delta}}$. This is equivalent to changing the complex structure j_k on each $S_k = \mathbb{R} \times [0,1]$ such that the (J_k, H_k)-holomorphic equation for $u_k : \mathbb{R} \times [0,1] \to M_k$ becomes

$$\partial_s u_k(s,t) + \delta_k^{-1} J_k(t, u_k(s,t)) \big(\partial_t u_k(s,t) - X_{H_k,t}(u_k(s,t)) \big) = 0.$$

By a global rescaling we could fix one width $\delta_0 = 1$ but not all of them due to the identification of the surfaces at the seams. In other words, the maps u_k cannot be rescaled independently since they are related by possibly non-split Lagrangian correspondences on the seams.

Proposition 4.3.1. *$HF(\underline{L})$ is independent, up to isomorphism of $\mathbb{Z}_N$-graded groups, of the choice of perturbation data $(\underline{H}, \underline{J})$ and widths $\underline{\delta}$ of the strips in $\underline{Z} = \underline{Z}_{\underline{\delta}}$.*

Proof. Suppose that $(\underline{H}^i, \underline{J}^i, \underline{\delta}^i)$ are two different choices for $i = 0, 1$. For $\{i, l\} = \{0, 1\}$ let $\underline{Z}_{il}$ be the quilted cylinder as before, but with complex structures j_k on each $S_k \cong \mathbb{R} \times [0,1]$ that interpolate between the two widths δ_k^i at the end (k, e_-) and δ_k^l at the end (k, e_+). Figure 8 shows the example for $r = 3$ and $M_0 = M_4 = \{\mathrm{pt}\}$. We moreover interpolate the

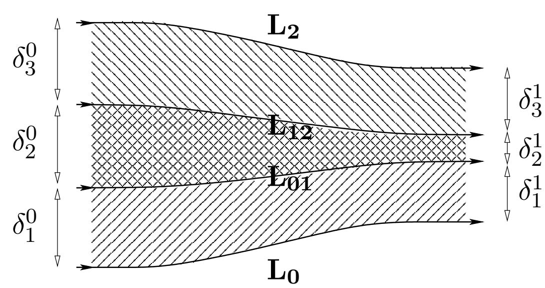


FIGURE 8. Interpolating between two widths

perturbation data on the two ends by some regular $(\underline{K}_{il}, \underline{J}_{il})$ on $\underline{Z}_{il}$. The relative invariants, constructed in Section 4.2 from the zero-dimensional moduli spaces, then provide maps between the corresponding Floer cohomology groups

$$\Phi_{\underline{Z}_{01}} : HF(\underline{L})^0 \to HF(\underline{L})^1, \qquad \Phi_{\underline{Z}_{10}} : HF(\underline{L})^1 \to HF(\underline{L})^0.$$

The surface $\underline{Z}_{01} \# \underline{Z}_{10}$ that is glued at $\{\underline{e}_-\} = \mathcal{E}_-(\underline{Z}_{01})$ and $\{\underline{e}_+\} = \mathcal{E}_+(\underline{Z}_{10})$ can be deformed to the infinite strip with translationally invariant perturbation data $(\underline{H}^1, \underline{J}^1, \underline{\delta}^1)$, hence $\Phi_{\underline{Z}_{01} \# \underline{Z}_{10}}$ is the identity on $HF(\underline{L})^1$ (and similarly for $\Phi_{\underline{Z}_{10} \# \underline{Z}_{01}}$). Then, by the gluing formula (34) we have

$$\Phi_{\underline{Z}_{01}} \circ \Phi_{\underline{Z}_{10}} = \Phi_{\underline{Z}_{01} \# \underline{Z}_{10}} = \mathrm{Id}, \quad \Phi_{\underline{Z}_{10}} \circ \Phi_{\underline{Z}_{01}} = \Phi_{\underline{Z}_{10} \# \underline{Z}_{01}} = \mathrm{Id}.$$

This proves that the Floer cohomology groups $HF(\underline{L})^0$ and $HF(\underline{L})^1$ arising from the different choices of data are isomorphic. $\square$

Remark 4.3.2. One can also allow the sequence $\underline{L}$ to have length zero (that is, the empty sequence) as a generalized correspondence from M to M; this is the case $r = -1$ in the previous notation. In this case we define $HF(\underline{L}) = HF(\mathrm{Id}_M)$, the cylindrical Floer homology. This would be the case without seams in Figure 7.

5. An isomorphism of Floer cohomologies

In this section we prove Theorem 1.0.1, more precisely stated as follows.

Theorem 5.0.3. *Let $\underline{L} = (L_{01}, \ldots, L_{r(r+1)})$ be a cyclic sequence of compact, oriented Lagrangian correspondences between symplectic manifolds $M_0, \ldots, M_{r+1} = M_0$ as in Section 3.3. Assume that*

(a) *the symplectic manifolds all satisfy (M1-2) with the same monotonicity constant τ,*
(b) *the Lagrangian correspondences all satisfy (L1-3),*
(c) *the sequence $\underline{L}$ is monotone, relatively spin, and graded in the sense of Section 3.3;*
(d) *for some $1 \le j \le r$ the composition $L_{(j-1)j} \circ L_{j(j+1)}$ is embedded in the sense of Definition 2.0.5, monotone, and has minimal Maslov number at least three;*
(e) *the modified sequence $\underline{L}' := (L_{01}, \ldots, L_{(j-1)j} \circ L_{j(j+1)}, \ldots, L_{r(r+1)})$ is monotone.*

Then with respect to the induced relative spin structure, orientation, and grading[12] *on $\underline{L}'$ there exists a canonical isomorphism of graded groups*

$$HF(\underline{L}) = HF(\ldots L_{(j-1)j}, L_{j(j+1)} \ldots) \xrightarrow{\sim} HF(\ldots L_{(j-1)j} \circ L_{j(j+1)} \ldots) = HF(\underline{L}'),$$

induced by the canonical identification of intersection points in Remark 2.3.3.

The isomorphism $HF(L_0 \times L_{12}, L_{01} \times L_2) \xrightarrow{\sim} HF(L_0 \times L_2, L_{01} \circ L_{12})$ in Theorem 1.0.1 is the special case $HF(L_0, L_{01}, L_{12}, L_2) \xrightarrow{\sim} HF(L_0, L_{01} \circ L_{12}, L_2)$ of the above theorem with $r = 3$ and $M_0 = M_4 = \{\mathrm{pt}\}$. The assumption on the minimal Maslov numbers is needed only to make the Floer cohomology well-defined, see Theorem 7.2.6 for the general case. The relative spin structures are only needed to define the Floer cohomology groups with $\mathbb{Z}$ coefficients. In the following we will prove the isomorphism with $\mathbb{Z}_2$ coefficients. The full result then follows from a comparison of signs in [46]. Similarly, the gradings on the Lagrangians can be dropped if one wants only an isomorphism of ungraded groups.

For simplicity we give the proof in the setting of Theorem 1.0.1; the general case is completely analogous. We denote $L_{02} := L_{01} \circ L_{12}$ and fix $(H_0, H_2) \in \mathrm{Ham}(L_0, L_{02}, L_2)$ such that the perturbed intersection points $\mathcal{I}(L_0, L_{02}, L_2)$ are finite and nondegenerate. Then they are canonically identified with the perturbed intersection points $\mathcal{I}(L_0, L_{01}, L_{12}, L_2)$ for $(H_0, 0, H_2) \in \mathrm{Ham}(L_0, L_{01}, L_{12}, L_2)$, i.e.

$$L_0 \times_{\phi^{H_0}} L_{02} \times_{\phi^{H_2}} L_2 \cong L_0 \times_{\phi^{H_0}} L_{01} \times_{\mathrm{Id_{M_1}}} L_{12} \times_{\phi^{H_2}} L_2. \tag{35}$$

Indeed, by assumption every point in $L_{02} := L_{01} \circ L_{12}$ has a unique lift to $L_{01} \times_{\mathrm{Id}_{M_1}} L_{12}$. Moreover, the perturbed intersection points on the right hand side are also nondegenerate, since by assumption $L_{01} \times L_{12}$ is transverse to the diagonal in M_1. So with the choices (H_0, H_2) and $(H_0, 0, H_2)$ of Hamiltonians we have a one-to-one correspondence

[12] The grading is given by (14), the orientation by the splitting (44) and the orientation of the diagonal in Remark 2.2.1 (b), and for the relative spin structure see [46].

$\mathcal{I}(L_0, L_{02}, L_2) \cong \mathcal{I}(L_0, L_{01}, L_{12}, L_2)$ which induces a natural isomorphism of the Floer chain groups

$$\begin{aligned} CF(L_0, L_{01}, L_{12}, L_2) &= CF(L_0 \times L_{12}, L_{01} \times L_2) \\ &\xrightarrow{\sim} CF(L_0 \times L_2, L_{02}) \;=\; CF(L_0, L_{02}, L_2). \end{aligned}$$

Lemma 2.3.8 asserts that this is in fact an isomorphism of graded groups. In the light of Proposition 4.3.1 it now suffices to show that this isomorphism descends to the cohomology for an appropriate choice of almost complex structures and widths on the quilted strips. We will prove this by establishing a bijection between the Floer trajectories for (L_0, L_{02}, L_2) on strips of width $(1, 1)$ and those for $(L_0, L_{01}, L_{12}, L_2)$ on strips of width $(1, \delta, 1)$ for sufficiently small width δ of the middle strip. These Floer trajectories are holomorphic quilts associated to the pictures in Figure 9.

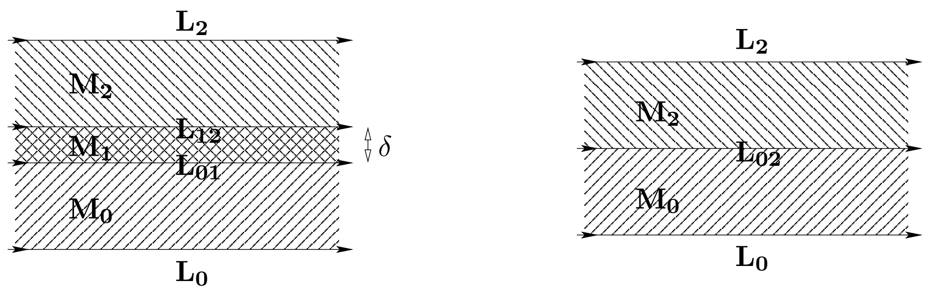


Figure 9. Shrinking the middle strip

Remark 5.0.4. The natural alternative approach to defining an isomorphism, or even just a homomorphism $HF(L_0, L_{01}, L_{12}, L_2) \to HF(L_0, L_{02}, L_2)$, is to try and interpolate the middle strip to zero width in the relative invariant of Figure 8, as indicated on the left in Figure 10. We however do not have a good analytic setup for seams running together within a quilted surface.

In the spirit of Section 6.8 we could replace this picture by one where the seams corresponding to L_{01}, L_{12}, L_{02} run into an infinite cylindrical end, as on the right in Figure 10. This picture defines a relative invariant

$$HF(L_{02}, (L_{01}, L_{12})) \otimes HF((L_0, L_{01}, L_{12}), L_2) \to HF((L_0, L_{02}), L_2),$$

which in the notation of Section 6.8 is given by $T \otimes f \mapsto \Phi_T(L_0) \circ f$. Here $\Phi_T : \Phi(L_{02}) \to \Phi(L_{01}, L_{12})$ is a natural transformation of functors associated to sequences of Lagrangian correspondences, see Figure 38. The isomorphism in Theorem 5.0.3 can alternatively be described by this relative invariant, where $T \in HF(L_{02}, (L_{01}, L_{12}))$ is the morphism corresponding to the identity $1_{L_{02}} \in HF(L_{02}, L_{02})$, see Corollary 5.4.3.

In the present setup, we will use the definition of Floer cohomology via quilted strips of variable widths as in Section 4.3. For that purpose we fix regular Hamiltonians $(H_0, H_2) \in \mathrm{Ham}(L_0, L_{02}, L_2)$ and almost complex structures $(J_0, J_2) \in \mathcal{J}_t^{\mathrm{reg}}(L_0, L_{02}, L_2; H_0, H_2)$, and we pick an almost complex structure $J_1 \in \mathcal{J}(M_1, \omega_1)$ (and the Hamiltonian $H_1 \equiv 0$ on M_1). In order to transfer to the unperturbed equations we replace the Lagrangians L_0 and L_2 with their Hamiltonian translates $\phi_{0,1}^{H_0}(L_0)$ and $\phi_{1,0}^{H_2}(L_2)$, while replacing the almost complex structures J_0 and J_2 with $(\phi_{t,1}^{H_0})_* J_0$ and $(\phi_{t,0}^{H_2})_* J_2$. So from now on we work with

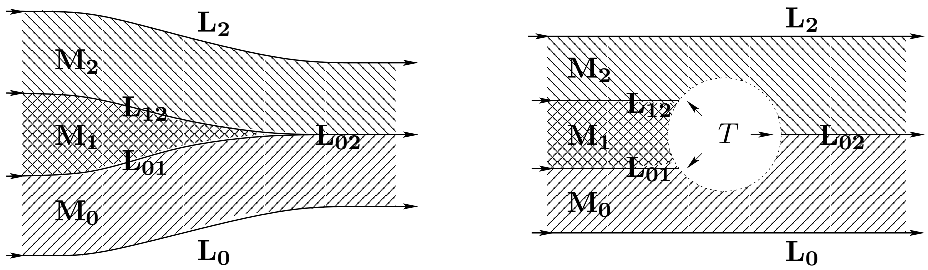


FIGURE 10. Alternative approaches to a homomorphism

the unperturbed equations for (by abuse of notation) the transformed Lagrangians L_0, L_2 and almost complex structures $(J_0, J_2) \in \mathcal{J}_t^{\rm reg}(L_0, L_{02}, L_2; 0, 0)$, and with the same L_{01}, L_{12} and J_1 as before. In this setup the t-dependent almost complex structures J_0 and J_2 can be chosen constant near $t = 0$ and $t = 1$. (This freedom of choice suffices to achieve transversality of the moduli spaces for (L_0, L_{02}, L_2).) The Floer chain groups are now generated by the unperturbed intersection points

$$\mathcal{I} := L_0 \times_{{\rm Id}_{M_0}} L_{02} \times_{{\rm Id}_{M_2}} L_2 \cong L_0 \times_{{\rm Id}_{M_0}} L_{01} \times_{{\rm Id}_{M_1}} L_{12} \times_{{\rm Id}_{M_2}} L_2.$$

For any $x^-, x^+ \in \mathcal{I}$ we denote by $\widetilde{\mathcal{M}}_0^1(x^-, x^+)$ the one dimensional (i.e. index 1) component of the space of quilted Floer trajectories for (L_0, L_{02}, L_2) with perturbation data $(H_0, H_2), (J_0, J_2)$ and widths $(1, 1)$ of the strips. For $\delta > 0$ we denote by $\widetilde{\mathcal{M}}_\delta^1(x^-, x^+)$ the index 1 component of the space of quilted Floer trajectories for $(L_0, L_{01}, L_{12}, L_2)$ with perturbation data $(H_0, 0, H_2), J_\delta$ and widths $(1, \delta, 1)$. Here roughly $J_\delta = (J_0, J_1, J_2)$, but more precisely we will be using $J_{0,\delta} \in \mathcal{J}_t(M_0, \omega_0)$ and $J_{2,\delta} \in \mathcal{J}_t(M_2, \omega_2)$ that converge to J_0 and J_2 in the $\mathcal{C}^\infty$-topology as $\delta \to 0$. Define

$$\mathcal{M}_0^1(x^-, x^+) := \widetilde{\mathcal{M}}_0^1(x^-, x^+)/\mathbb{R}, \qquad \mathcal{M}_\delta^1(x^-, x^+) := \widetilde{\mathcal{M}}_\delta^1(x^-, x^+)/\mathbb{R},$$

then our task is to prove the following.

Theorem 5.0.5. *For all sufficiently small $\delta > 0$ the moduli spaces $\mathcal{M}_\delta^1(x^-, x^+)$ are regular and zero dimensional, and there is a bijection*

$$\mathcal{T}_\delta : \mathcal{M}_0^1(x^-, x^+) \to \mathcal{M}_\delta^1(x^-, x^+).$$

We now describe the strategy of proof and introduce the relevant notations. First we use the assumption that $L_{01} \circ L_{12}$ is embedded by π_{02}. Consider a solution $u = (u_0, u_2) \in \widetilde{\mathcal{M}}_0^1(x^-, x^+)$, that is a pair $u_0 : \mathbb{R} \times [0, 1] \to M_0$, $u_2 : \mathbb{R} \times [0, 1] \to M_2$ of index 1, with limits $\lim_{s\to\pm\infty}(u_0, u_2)(s, \cdot) = x^\pm$, and satisfying

$$\begin{aligned} &\overline{\partial}_{J_0} u_0 = 0, \qquad \overline{\partial}_{J_2} u_2 = 0, \\ &u_0|_{t=0} \in L_0, \qquad (u_0|_{t=1}, u_2|_{t=0}) \in L_{02}, \qquad u_2|_{t=1} \in L_2. \end{aligned}$$

We can identify (u_0, u_2) with the map $u_{02} : \mathbb{R} \times [0, 1] \to M_0 \times M_2$ given by $u_{02}(s, t) = (u_0(s, 1-t), u_2(s, t))$, which satisfies $\lim_{s\to\pm\infty} u_{02}(s, \cdot) = x^\pm$ and

$$\overline{\partial}_{J_{02}} u_{02} = 0, \qquad u_{02}|_{t=0} \in L_{02}, \qquad u_{02}|_{t=1} \in L_0 \times L_2.$$

Here we denoted $J_{02}(s, t) := (-J_0(s, 1-t), J_2(s, t))$. We will also use the notation $\bar{J}_{02} := J_{02}|_{t=0}$ and $\bar{u}_{02} := u_{02}|_{t=0} : \mathbb{R} \to L_{02}$. Finally, we will denote by $(L_{01} \times L_{12})^T \subset M_0 \times M_2 \times M_1 \times M_1$ the obvious transposition of factors in the Lagrangian submanifold. Since

$\pi_{02} : L_{01} \times_{M_1} L_{12} \to L_{02} \subset M_0 \times M_2$ is transversal and embedded (see Remark 2.0.6), there exists a unique, smooth solution $\bar{u}_1 = \ell_1 \circ \bar{u}_{02} : \mathbb{R} \to M_1$ to

$$\bar{u}(s) := (\bar{u}_{02}(s), \bar{u}_1(s), \bar{u}_1(s)) \in (L_{01} \times L_{12})^T.$$

We also denote by $\bar{u} := (\bar{u}_{02}, \bar{u}_1, \bar{u}_1)$ the extension $\mathbb{R} \times [0, \delta] \to M_0 \times M_2 \times M_1 \times M_1$ that is constant along $[0, \delta]$. Given δ these choices are unique, so we can identify u with the pair $(u_{02}, \bar{u})$. In the same spirit we find unique points $x_1^{\pm} \in M_1$ such that $(x^{\pm}, x_1^{\pm}) \in (L_0 \times L_{12}) \cap (L_{01} \times L_2) \subset M_0 \times M_1 \times M_2$. In this notation we have the limit $\lim_{s\to\pm\infty} \bar{u}_1(s) = x_1^{\pm}$. Given a solution $u \in \widetilde{\mathcal{M}}^1_0(x^-, x^+)$ as above and $\delta > 0$ we wish to find a nearby solution $(v_0, v_1, v_2) \in \widetilde{\mathcal{M}}^1_\delta(x^-, x^+)$, that is a triple $v_0 : \mathbb{R} \times [0, 1] \to M_0$, $v_1 : \mathbb{R} \times [0, \delta] \to M_1$, $v_2 : \mathbb{R} \times [0, 1] \to M_2$ with limits $\lim_{s\to\pm\infty}(v_0, v_2)(s, \cdot) = x^{\pm}$, $\lim_{s\to\pm\infty} v_1(s, \cdot) = x_1^{\pm}$, and satisfying

$$\begin{aligned} &\overline{\partial}_{J_0} v_0 = 0, \qquad \overline{\partial}_{J_1} v_1 = 0, \qquad \overline{\partial}_{J_2} v_2 = 0, \\ &v_0(s, 0) \in L_0, \quad (v_0(s, 1), v_1(s, 0)) \in L_{01}, \quad (v_1(s, \delta), v_2(s, 0)) \in L_{12}, \quad v_2(s, 1) \in L_2. \end{aligned}$$

To solve this problem we use the assumption that $L_{01} \times L_{12}$ is transversal to the diagonal. This is best done in the following reformulation of the δ-moduli spaces.

Let $\bar{\delta} := \delta/(2 - \delta)$ (or equivalently $\delta = 2\bar{\delta}/(1 + \bar{\delta})$). Instead of the triple strip we consider a quadruple of maps $v = (v_{02}, v'_{02}, v_1, v'_1)$ with $v_{02} \in \mathcal{C}^\infty(\mathbb{R} \times [0, 1], M_0 \times M_2)$, $v'_{02} \in \mathcal{C}^\infty(\mathbb{R} \times [0, \bar{\delta}], M_0 \times M_2)$, $v_1, v'_1 \in \mathcal{C}^\infty(\mathbb{R} \times [0, \bar{\delta}], M_1)$ that have limits $\lim_{s\to\pm\infty} v_{02}(s, \cdot) = \lim_{s\to\pm\infty} v'_{02}(s, \cdot) = x^{\pm}$, $\lim_{s\to\pm\infty} v_1(s, \cdot) = \lim_{s\to\pm\infty} v_1(s, \cdot) = x_1^{\pm}$, and satisfy

$$\begin{aligned} &\overline{\partial}_{J_{02}} v_{02} = 0, \qquad \overline{\partial}_{-\bar{J}_{02}} v'_{02} = 0, \qquad \overline{\partial}_{-J_1} v'_1 = 0, \qquad \overline{\partial}_{J_1} v_1 = 0, \\ &(v'_{02}, v_{02})|_{t=0} \in \Delta_0 \times \Delta_2, \qquad (v'_1, v_1)|_{t=0} \in \Delta_1, \\ &(v'_{02}, v'_1, v_1)|_{t=\bar{\delta}} \in (L_{01} \times L_{12})^T, \qquad v_{02}|_{t=1} \in L_0 \times L_2. \end{aligned} \tag{36}$$

For notational convenience we will also group these quadruples of maps as $v = (v_{02}, \hat{v})$ with $\hat{v} = (v'_{02}, v_1, v'_1)$. Then we can abbreviate $J = (J_{02}, \hat{J})$ with $\hat{J} := (-\bar{J}_{02}, -J_1, J_1)$, and reformulate (36) as

$$\begin{aligned} &\overline{\partial}_J v := \big(\overline{\partial}_{J_{02}} v_{02}\, ,\, \overline{\partial}_{\hat{J}} \hat{v}\big) = 0, \\ &(v_{02}, \hat{v})|_{t=0} \in \Delta_0 \times \Delta_2 \times \Delta_1, \qquad \hat{v}_{t=\bar{\delta}} \in (L_{01} \times L_{12})^T, \qquad v_{02}|_{t=1} \in L_0 \times L_2. \end{aligned}$$

We denote the moduli space of such solutions $v = (v_{02}, \hat{v})$ by $\widehat{\mathcal{M}}^1_{\bar{\delta}}(x^-, x^+)$. It is in one-to-one correspondence to $\widetilde{\mathcal{M}}^1_\delta(x^-, x^+)$ as follows: Given $v = (v_{02}, v'_{02}, v'_1, v_1) \in \widehat{\mathcal{M}}^1_{\bar{\delta}}(x^-, x^+)$ we obtain $\bar{v} = (v_0, v_1, v_2) \in \widetilde{\mathcal{M}}^1_\delta(x^-, x^+)$ from

$$\big(v_0(s, 1 - t), v_2(s, t)\big) = \begin{cases} v'_{02}((1 + \bar{\delta})s, \bar{\delta} - (1 + \bar{\delta})t) & \text{for } 0 \le t \le \frac{1}{2}\delta, \\ v_{02}((1 + \bar{\delta})s, (1 + \bar{\delta})t - \bar{\delta}) & \text{for } \frac{1}{2}\delta \le t \le 1, \end{cases}$$

$$v_1(s, t) = \begin{cases} v'_1((1 + \bar{\delta})s, \bar{\delta} - (1 + \bar{\delta})t) & \text{for } 0 \le t \le \frac{1}{2}\delta, \\ v_1((1 + \bar{\delta})s, (1 + \bar{\delta})t - \bar{\delta}) & \text{for } \frac{1}{2}\delta \le t \le \delta. \end{cases}$$

The two different formulations for double and triple strips each are indicated in Figure 11. Strictly speaking, this triple strip (v_0, v_1, v_2) only satisfies $\overline{\partial}_{J_{i,\delta}} v_i = 0$ for $i = 0, 2$ with almost complex structures $J_{i,\delta}$ that are given by rescaling J_0 to $[0, 1 - \delta/2]$ and J_2 to $[\delta/2, 1]$, and extending them constantly by $J_0(1)$ and $J_2(0)$ respectively. Note however that $J_{i,\delta} \to J_i$ in the $\mathcal{C}^\infty$-topology as $\delta \to 0$ since J_i is smooth and constant near $t = 0, 1$. So we can use the almost complex structure $J_\delta = (J_{0,\delta}, J_1, J_{2,\delta})$ for the definition of the moduli spaces

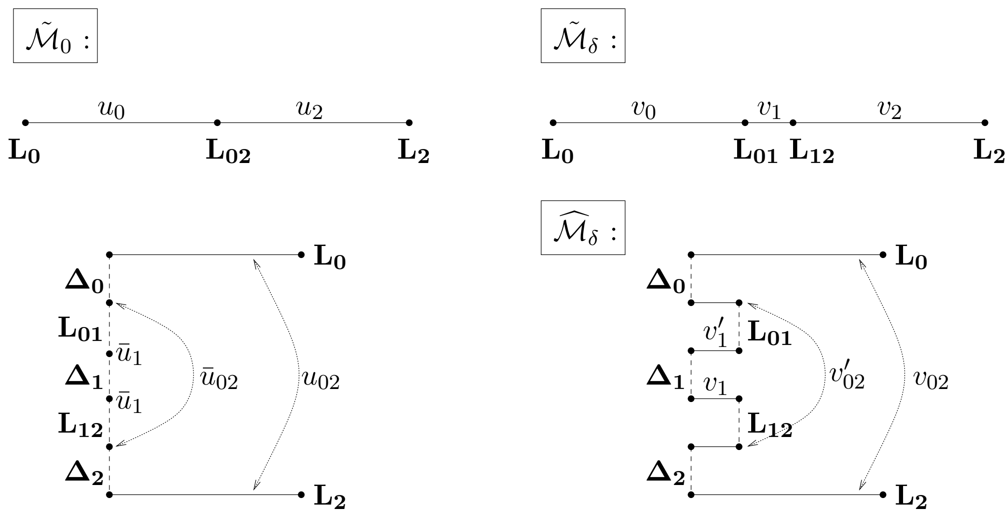


FIGURE 11. Double and triple strips

$\widetilde{\mathcal{M}}^1_\delta(x^-,x^+)$. The bijection $\mathcal{T}_\delta$ to the moduli space $\widetilde{\mathcal{M}}^1_0(x^-,x^+)$ can then be established via a bijection

$$\mathcal{T}_{\bar{\delta}} : \mathcal{M}^1_0(x^-,x^+) \to \mathcal{M}^1_{\bar{\delta}}(x^-,x^+) := \widehat{\mathcal{M}}^1_{\bar{\delta}}(x^-,x^+)/\mathbb{R}. \tag{37}$$

This map will be constructed by the implicit function theorem 5.1.1. We prove injectivity in corollary 5.1.6, and the surjectivity will follow from the compactness theorem 5.3.1.

5.1. Implicit function theorem.

The purpose of this section is to construct the map $\mathcal{T}_\delta : \mathcal{M}^1_0(x^-,x^+) \to \mathcal{M}^1_\delta(x^-,x^+)$ of Theorem 5.0.5. We will do this by constructing the map (37), with $\bar{\delta}$ replaced by δ, from the following implicit function theorem.

Theorem 5.1.1. *There exist constants C_0, $\epsilon > 0$, and $\delta_0 > 0$ such that the following holds for every $\delta \in (0,\delta_0]$. For every $u \in \widetilde{\mathcal{M}}^1_0(x^-,x^+)$ there exists a unique $v_u \in \widehat{\mathcal{M}}^1_\delta(x^-,x^+)$ such that $v_u = e_u(\xi)$ with $\xi \in \Gamma_{1,\delta}(\epsilon) \cap K_0$. The solution moreover satisfies*

$$\|\xi\|_{H^2_{1,\delta}} \le C_0\sqrt{\delta}. \tag{38}$$

Here $e_u(\xi) := (v_{02}, v'_{02}, v'_1, v_1)$ is given in terms of $u = (u_{02}, \bar{u})$ and $\xi = (\xi_{02}, \hat{\xi})$ with $\xi_{02} \in \Gamma(u^*_{02}T(M_0 \times M_2))$ and $\hat{\xi} = (\xi'_{02}, \xi'_1, \xi_1) \in \Gamma(\bar{u}^*T(M_0 \times M_2 \times M_1 \times M_1))$. The precise definitions of the exponential map e_u, the ϵ-ball $\Gamma_{1,\delta}(\epsilon)$, the $H^2_{1,\delta}$-norm, and the local slice K_0 of the $\mathbb{R}$-shift symmetry will be given in the process of the proof.

To prove the theorem we fix a solution $u \in \widetilde{\mathcal{M}}^1_0(x^-,x^+)$, and in the following will allow all constants to depend on u up to translation in $\mathbb{R}$. (Since $\mathcal{M}^1_0(x^-,x^+)$ is finite we can then easily find uniform constants C_0 and $\delta_0 > 0$.) We will then roughly solve $\overline{\partial}_J e_u(\xi) = 0$ for sections $\xi = (\xi_{02}, \hat{\xi})$, $\hat{\xi} = (\xi'_{02}, \xi'_1, \xi_1)$ satisfying the boundary conditions

$$\begin{aligned} &(\xi'_{02}, \xi_{02})|_{t=0} \in T_{(\bar{u}_{02},\bar{u}_{02})}\Delta_{M_0\times M_2}, \qquad (\xi'_1, \xi_1)|_{t=0} \in T_{(\bar{u}_1,\bar{u}_1)}\Delta_1, \\ &\hat{\xi}|_{t=\delta} = (\xi'_{02}, \xi'_1, \xi_1)|_{t=\delta} \in T_{\bar{u}}(L_{01}\times L_{12})^T, \qquad \xi_{02}|_{t=1} \in T_{u_{02}}(L_0 \times L_2). \end{aligned} \tag{39}$$

The exponential map $e_u(\xi)$ will then be constructed such that the nonlinear Lagrangian boundary conditions are satisfied automatically. The index of the new solution v_u will coincide with that of the given solution u due to Lemma 2.3.10. Here we identified v_u with a solution $\bar{v}_u \in \widetilde{\mathcal{M}}^1_{\tilde{\delta}}(x^-, x^+)$, $\tilde{\delta} = 2\delta/(1+\delta)$. Then the homotopy between $v_u = e_u(\xi)$ and $(u_{02}, \bar{u})$ induces a homotopy $\bar{v}_u \cong (u_0, \bar{u}_1, u_2)$.

To set up the implicit function theorem we introduce the space of H^k-sections over $(u_{02}, \bar{u})$ for $k \in \mathbb{N}_0$,

$$H^k_{1,\delta} := \left\{ (\eta_{02}, \eta'_{02}, \eta'_1, \eta_1) \,\middle|\, \begin{array}{l} \eta_{02} \in H^k(\mathbb{R}\times[0,1], u_{02}^* T(M_0\times M_2)), \\ \eta'_{02} \in H^k(\mathbb{R}\times[0,\delta], \bar{u}_{02}^* T(M_0\times M_2)), \\ \eta'_1, \eta_1 \in H^k(\mathbb{R}\times[0,\delta], \bar{u}_1^* TM_1) \end{array} \right\}.$$

We also write these sections as $\eta = (\eta_{02}, \hat{\eta}) \in H^k_{1,\delta}$, where the subscripts indicate the width of the domains of η_{02} and $\hat{\eta} = (\eta'_{02}, \eta'_1, \eta_1) \in H^k(\mathbb{R}\times[0,\delta], \bar{u}^* T(M_0 \times M_2 \times M_1 \times M_1))$. The corresponding H^k-norm on this space is

$$\begin{aligned} \big\|(\eta_{02}, \eta'_{02}, \eta'_1, \eta_1)\big\|^2_{H^k_{1,\delta}} &:= \big\|\eta_{02}\big\|^2_{H^k(\mathbb{R}\times[0,1])} + \big\|\hat{\eta}\big\|^2_{H^k(\mathbb{R}\times[0,\delta])} \\ &= \big\|\eta_{02}\big\|^2_{H^k(\mathbb{R}\times[0,1])} + \big\|\eta'_{02}\big\|^2_{H^k(\mathbb{R}\times[0,\delta])} + \big\|\eta'_1\big\|^2_{H^k(\mathbb{R}\times[0,\delta])} + \big\|\eta_1\big\|^2_{H^k(\mathbb{R}\times[0,\delta])}. \end{aligned}$$

We denote the space of H^2-sections satisfying the boundary conditions by

$$\Gamma_{1,\delta} := \big\{ \xi \in H^2_{1,\delta} \,\big|\, (39) \big\}$$

and equip this space with the norm

$$\big\|\xi\big\|_{\Gamma_{1,\delta}} := \big\|\xi\big\|_{H^2_{1,\delta}} + \big\|\nabla\xi\big\|_{L^4_{1,\delta}},$$

with the L^4-norm $\|\nabla(\xi_{02}, \hat{\xi})\|_{L^4_{1,\delta}} := \big(\|\nabla\xi_{02}\|^4_{L^4_{1,\delta}(\mathbb{R}\times[0,1])} + \|\nabla\hat{\xi}\|^4_{L^4_{1,\delta}(\mathbb{R}\times[0,\delta])} \big)^{1/4}$ on the multistrip. We denote the ϵ-ball in $\Gamma_{1,\delta}$ by

$$\Gamma_{1,\delta}(\epsilon) := \big\{ \xi \in H^2_{1,\delta} \,\big|\, \|\xi\|_{\Gamma_{1,\delta}} < \epsilon, (39) \big\}.$$

We equip the target space $\Omega_{1,\delta} := H^1_{1,\delta}$ with the norm

$$\|\eta\|_{\Omega_{1,\delta}} := \|\eta\|_{H^1_{1,\delta}} + \|\eta\|_{L^4_{1,\delta}}.$$

The reason for adding the L^4-norms in domain and target is that we do not have uniform Sobolev embeddings on the strips of varying width. Instead, we build the necessary Sobolev multiplication properties into the norms.

Next, we make some preparations for defining an exponential map that is compatible with the boundary conditions (39).

Lemma 5.1.2. *(Existence of compatible quadratic corrections) There exists $\epsilon_0 > 0$ and smooth families of maps (defined on the ϵ_0-balls)*

$$Q_s : T_{\bar{u}(s)}\big(M_0 \times M_2 \times M_1 \times M_1\big) \supset B_{\epsilon_0} \to T_{\bar{u}(s)}\big(M_0 \times M_2 \times M_1 \times M_1\big), \qquad \forall s \in \mathbb{R},$$

$$Q^{02}_{s,t} : T_{u_{02}(s,t)}(M_0 \times M_2) \supset B^{02}_{\epsilon_0} \to T_{u_{02}(s,t)}(M_0 \times M_2) \qquad \forall (s,t) \in \mathbb{R}\times[0,1],$$

that are a diffeomorphism onto their image and have the following properties:

(Quadratic): *$Q_s(0) = 0$, $dQ_s(0) \equiv 0$, $Q^{02}_{s,t}(0) = 0$, and $dQ^{02}_{s,t}(0) \equiv 0$ for all $(s,t) \in \mathbb{R}\times[0,1]$. In particular, there is a constant C_Q such that for all $\hat{\xi} \in B_{\epsilon_0}$ and $\xi_{02} \in B^{02}_{\epsilon_0}$*

$$|Q_s(\hat{\xi})| \le C_Q|\hat{\xi}|^2, \qquad |Q^{02}_{s,t}(\xi_{02})| \le C_Q|\xi_{02}|^2. \tag{40}$$

(Linearizing $\mathbf{L_{01}} \times \mathbf{L_{12}}$): $\exp_{\bar u(s)} \circ (1+Q_s)$ *maps* $T_{\bar u(s)}(L_{01}\times L_{12})^T \cap B_{\epsilon_0}$ *to* $(L_{01}\times L_{12})^T$.

(Linearizing $\mathbf{M_0}\times\mathbf{M_2}\times\boldsymbol{\Delta_1}$): $\exp_{\bar u(s)} \circ (1+Q_s)$ *maps* $T_{\bar u(s)}(M_0\times M_2\times\Delta_1)\cap B_{\epsilon_0}$ *to* $M_0\times M_2\times\Delta_1$.

(Linearizing $\mathbf{L_{02}}$): $\exp_{u_{02}(s,1)} \circ (1+Q^{02}_{s,1})$ *maps* $T_{u_{02}(s,1)}L_{02}\cap B^{02}_{\epsilon_0}$ *to* L_{02}.

(Compatible): *Restricting* Q_s *to* $T_{\bar u}(M_0\times M_2\times\Delta_1)$ *and composing it with the projection* $\mathrm{Pr}_{02} : T_{(\bar u_{02},\bar u_1,\bar u_1)}(M_0\times M_2\times M_1\times M_1)\to T_{\bar u_{02}}(M_0\times M_2)$ *yields a map that is independent of the* $T_{(\bar u_1,\bar u_1)}\Delta_1$*-component. The resulting family*

$$Q^{02}_s : T_{\bar u_{02}(s)}(M_0\times M_2) \supset B^{02}_{\epsilon_0} \to T_{\bar u_{02}(s)}(M_0\times M_2)$$

coincides with $Q^{02}_{s,0}$.

Proof. We fix $s\in\mathbb{R}$ and restrict the exponential map $\exp_{\bar u(s)}$ to a geodesic ball around 0. The subsequent constructions will depend smoothly on $s\in\mathbb{R}$, which we drop from now on. By assumption the submanifold $\mathcal{L}_{0211} := \exp_{\bar u}^{-1}(L_{01}\times L_{12})^T$ in the vector space $X := T_{\bar u}(M_0\times M_2\times M_1\times M_1)$ is transverse to the subspace $\Delta := T_{\bar u}(M_0\times M_2\times\Delta_1)$. Their intersection $\hat{\mathcal{L}}_{02} := \mathcal{L}_{0211}\cap\Delta$ is diffeomorphic to the submanifold $\mathcal{L}_{02} := \exp_{\bar u_{02}}^{-1}(L_{02}) \subset T_{\bar u_{02}}(M_0\times M_2)$ by a map $(m_0,m_2)\mapsto(m_0,m_2,m_1,m_1)$ with uniquely determined $m_1 = m_1(m_0,m_2)$. So we have a direct sum decomposition

$$\Delta = T_{\bar u_{02}}(M_0\times M_2)\times T_{(\bar u_1,\bar u_1)}\Delta_1 = T_0\hat{\mathcal{L}}_{02} \oplus \big((T_0\mathcal{L}_{02})^\perp\times\{0\}\big)\oplus\big(\{0\}\times T_{(\bar u_1,\bar u_1)}\Delta_1\big).$$

As a submanifold we can now write $\hat{\mathcal{L}}_{02}\subset\Delta$ as the graph of a map ψ over a sufficiently small ϵ-ball,

$$\psi = \psi_{02}^\perp\times\psi_{11} : T_0\hat{\mathcal{L}}_{02}\supset B_\epsilon \to \big(T_0\mathcal{L}_{02}\big)^\perp\times T_{(\bar u_1,\bar u_1)}\Delta_1$$

with $\psi(0)=0$ and $d\psi(0)\equiv 0$. We moreover pick a complement C of $T_0\hat{\mathcal{L}}_{02}\subset T_0\mathcal{L}_{0211}$,

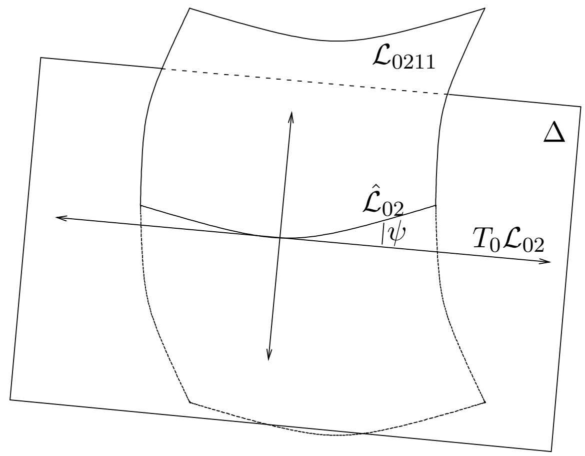
$$T_0\mathcal{L}_{0211} = C\oplus T_0\hat{\mathcal{L}}_{02},$$

then the transversality $X = T_0\mathcal{L}_{0211}+\Delta$ implies the splitting

$$X = C\oplus T_0\hat{\mathcal{L}}_{02}\oplus\big(T_0\mathcal{L}_{02}\big)^\perp\times\{0\}\oplus\{0\}\times T_{(\bar u_1,\bar u_1)}\Delta_1. \tag{41}$$

We write $X\ni x = x_C + x_{02} + (x_{02}^\perp,0)+(0,x_{11})$ in this splitting and define a map $\Psi : X\supset B_\epsilon\to X$ by

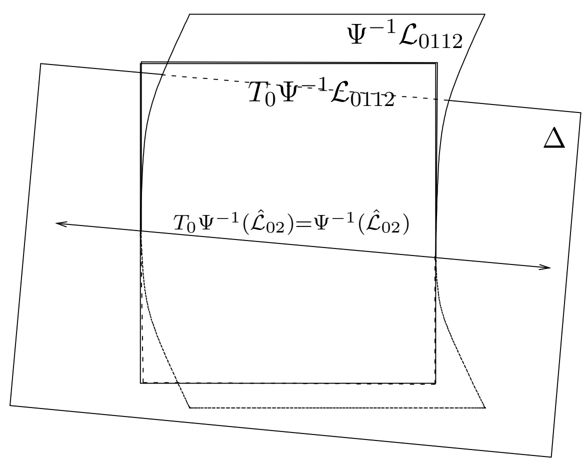
$$\begin{aligned}\Psi(x) &:= x + (\psi_{02}^\perp(x_{02}),0)+(0,\psi_{11}(x_{02}))\\ &= x_C + x_{02} + (x_{02}^\perp+\psi_{02}^\perp(x_{02}),0)+(0,x_{11}+\psi_{11}(x_{02})).\end{aligned}$$



This map linearizes the intersection, $\Psi(T_0\hat{\mathcal{L}}_{02}) = \hat{\mathcal{L}}_{02}$, and we have $\Psi(0) = 0$ and $d\Psi(0) = \mathrm{Id}$. In order to linearize the entire Lagrangian $\mathcal{L}_{0211}$ we remark that $T_0\big(\Psi^{-1}(\mathcal{L}_{0211})\big) = d\Psi(0)^{-1}T_0\mathcal{L}_{0211} = T_0\mathcal{L}_{0211}$. So we can write $\Psi^{-1}(\mathcal{L}_{0211})$ as graph of a map

$$\phi = \phi_{02}^{\perp} \times \phi_{11} : T_0\mathcal{L}_{0211} \supset B_\epsilon \to \big(T_{\bar{u}_{02}}\mathcal{L}_{02}\big)^{\perp} \times T_{(\bar{u}_1,\bar{u}_1)}\Delta_1$$

with $\phi(0) = 0$, $d\phi(0) \equiv 0$, and by the previous construction $\phi|_{T_0\hat{\mathcal{L}}_{02}} \equiv 0$.



Finally we define the entire linearization $\Phi : X \supset B_\epsilon \to X$ by

$$\Phi(x) := \Psi\big(x + (\phi_{02}^{\perp}(x_C + x_{02}), 0) + (0, \phi_{11}(x_C + x_{02}))\big)$$

for $x = x_C + x_{02} + (x_{02}^{\perp}, 0) + (0, x_{11})$ in the splitting (41). Now $Q_s := \Phi - \mathrm{Id}$ is quadratic and linearized $(L_{01} \times L_{12})^T$ by construction. Explicitly, we have

$$Q_s(x) = \big(\psi_{02}^{\perp}(x_{02}) + \phi_{02}^{\perp}(x_C + x_{02}), \psi_{11}(x_{02}) + \phi_{11}(x_C + x_{02})\big). \tag{42}$$

The construction moreover ensures that Q_s linearizes $M_0 \times M_2 \times \Delta_1$, that is $\Phi(\Delta) \subset \Delta$, since $x \in \Delta = \{x_C = 0\}$ is mapped to $\Phi(x) = x + \big(\psi_{02}^{\perp}(x_{02}), \psi_{11}(x_{02})\big) \in \Delta$.

To construct Q_s^{02} compatible with Q_s note that for $x = (m_0, m_2, m_1, m_1) \in T_{\bar{u}}(M_0 \times M_2 \times \Delta_1) \subset X$ we have a splitting

$$x = (m_0, m_2, 0, 0) + (0, 0, m_1, m_1) = x_C + x_{02} + (x_{02}^{\perp}, 0) + (0, x_{11} + (m_1, m_1)),$$

where $x_C, x_{02}, x_{02}^\perp, x_{11}$ only depend on (m_0, m_2). With this we can see in (42) that indeed $Q_s(m_0, m_2, m_1, m_1)$ is independent of m_1. We then simply define $Q_{s,0}^{02}(m_0, m_2) := \mathrm{Pr}_{02} Q_s(m_0, m_2, 0, 0)$. Moreover, a graph construction as above provides a map $Q_{s,1}^{02} : T_{u_{02}(s,1)}(M_0 \times M_2) \supset B_\epsilon^{02} \to T_{u_{02}(s,1)}(M_0 \times M_2)$ that is quadratic and linearizes L_{02}. Now the two families $Q_{s,0}^{02}$ and $Q_{s,1}^{02}$ can easily be interpolated by the smooth family $Q_{s,t}^{02} := (1-t)Q_{s,0}^{02} + tQ_{s,1}^{02}$ of quadratic maps. □

With these quadratic corrections we can now define the exponential map e_u by $e_u(\xi) := (e_{u_{02}}(\xi_{02}), e_{\bar{u}}(\hat{\xi}))$ for $\xi = (\xi_{02}, \hat{\xi}) \in \Gamma_{1,\delta}(\epsilon)$, where

$$e_{u_{02}}(\xi_{02}) := \exp_{u_{02}} \circ (1 + Q^{02})(\xi_{02}), \qquad e_{\bar{u}}(\hat{\xi}) := \exp_{\bar{u}} \circ (1 + Q)(\hat{\xi}). \tag{43}$$

Note that we have the usual properties of an exponential map,

$$e_u(0) = (u_{02}, \bar{u}), \qquad de_u(0) = \mathrm{Id}.$$

To define e_u on $\Gamma_{1,\delta}(\epsilon)$ the $\epsilon > 0$ should be chosen such that $\|\xi_{02}\|_{\mathcal{C}^0}$ and $\|\hat{\xi}\|_{\mathcal{C}^0}$ are sufficiently small for the quadratic corrections in Lemma 5.1.2 to be defined. Lemma 5.1.4 below ensures that we can pick a uniform $\epsilon > 0$ for all $\delta > 0$. Now solutions $v_u \in \widehat{\mathcal{M}}_\delta^1(x^-, x^+)$ in a neighborhood of u correspond to zeroes of the map $\mathcal{F}_u : \Gamma_{1,\delta}(\epsilon) \to \Omega_{1,\delta}$ given by

$$\mathcal{F}_u(\xi) := \big(\, \Phi_{u_{02}}(\xi_{02})^{-1}(\overline{\partial}_{J_{02}} e_{u_{02}}(\xi_{02}))\, ,\, \Phi_{\bar{u}}(\hat{\xi})^{-1}(\overline{\partial}_{\hat{J}} e_{\bar{u}}(\hat{\xi}))\, \big).$$

Here $\Phi_u(\xi)$ denotes the parallel transport $T_u M \to T_{e_u(\xi)} M$ along the path $\tau \mapsto e_u(\tau\xi)$. For $\Phi_{u_{02}}$ this parallel transport on $T(M_0 \times M_2)$ can simply use the Levi-Civita connection. In the definition of $\Phi_{\bar{u}}$ we however use a Hermitian connection $\tilde{\nabla}$ on the tangent bundle $T(M_0 \times M_2 \times M_1 \times M_1)$ that leaves $\hat{J}$ invariant. This can be done by the same construction as in [26, Proposition 3.1.1], which brings the linearized operator into simple form.

Next, we introduce projections related to the various Lagrangians:

$$\pi_{0211}^\perp \in \mathrm{Aut}\big(\mathcal{C}^\infty(\mathbb{R}, \bar{u}^* T(M_0 \times M_2 \times M_1 \times M_1))\big),$$
$$\pi_{02}\, ,\ \pi_{02}^\perp\ \in \mathrm{Aut}\big(\mathcal{C}^\infty(\mathbb{R}, \bar{u}_{02}^* T(M_0 \times M_2))\big)$$

are linear operators, given by pointwise orthogonal projection onto the subspaces $(T(L_{01} \times L_{12})^T)^\perp \subset T(M_0 \times M_2 \times M_1 \times M_1)$ resp. $TL_{02}, (TL_{02})^\perp \subset T(M_0 \times M_2)$. The following lemma contains the estimates resulting from the transversality assumption.

Lemma 5.1.3. *(Quantitative transversality) There exists a constant C such that the following holds.*

(a) *For every $s \in \mathbb{R}$ and $\hat{\xi} = (\xi'_{02}, \xi'_1, \xi_1) \in T_{\bar{u}(s)}(M_0 \times M_2 \times M_1 \times M_1)$*

$$|\hat{\xi}| \le C\big(|\pi_{02}\xi'_{02}| + |\xi'_1 - \xi_1| + |\pi_{0211}^\perp \hat{\xi}|\big),$$
$$|\pi_{02}^\perp \xi'_{02}| \le C\big(|\pi_{0211}^\perp \hat{\xi}| + |\xi'_1 - \xi_1|\big).$$

(b) *For every $\hat{\xi} \in \mathcal{C}^\infty(\mathbb{R}, \bar{u}^* T(M_0 \times M_2 \times M_1 \times M_1))$*

$$\|\hat{\xi}\|_{H^1(\mathbb{R})} \le C\big(\|\pi_{02}\xi'_{02}\|_{H^1(\mathbb{R})} + \|\xi'_1 - \xi_1\|_{H^1(\mathbb{R})} + \|\pi_{0211}^\perp \hat{\xi}\|_{H^1(\mathbb{R})}\big),$$

and the same holds with H^1 replaced by $\mathcal{C}^1$ or L^p for any $p \ge 1$. Moreover,

$$\|\pi_{02}^\perp \xi'_{02}\|_{L^2(\mathbb{R})} \le C\big(\|\pi_{0211}^\perp \hat{\xi}\|_{L^2(\mathbb{R})} + \|\xi'_1 - \xi_1\|_{L^2(\mathbb{R})}\big),$$
$$\|\pi_{02}^\perp \xi'_{02}\|_{H^1(\mathbb{R})} \le C\big(\|\pi_{0211}^\perp \hat{\xi}\|_{H^1(\mathbb{R})} + \|\xi'_1 - \xi_1\|_{H^1(\mathbb{R})} + \big\||\partial_s \bar{u}| \cdot |\hat{\xi}|\big\|_{L^2(\mathbb{R})}\big).$$

Proof. The Lagrangian $L_{01} \times L_{12}$ intersects $M_0 \times \Delta_1 \times M_2$ transversally in $\hat{L}_{02}$, which injects to $L_{02} \subset M_0 \times M_2$. So at every point of $\hat{L}_{02}$ we have a decomposition $T(M_0 \times M_2 \times M_1 \times M_1) = T\hat{L}_{02} \oplus (T\hat{L}_{02})^\perp$, where we can change the first factor to $TL_{02} \times \{0\}$. On the other hand, the transverse intersection implies

$$(T\hat{L}_{02})^\perp = \big(\{0\} \times (T\Delta_1)^\perp\big) \oplus T(L_{01} \times L_{12})^\perp, \tag{44}$$

so we obtain a splitting

$$T(M_0 \times M_2 \times M_1 \times M_1) = \big(TL_{02} \times \{0\}\big) \oplus \big(\{0\} \times (T\Delta_1)^\perp\big) \oplus T(L_{01} \times L_{12})^\perp. \tag{45}$$

This means that the product of the three orthogonal projections onto the factors defines an isomorphism. The norm of this isomorphism is bounded at each $\bar{u}(s) \in \hat{L}_{02}$, so for every $\hat{\xi} = (\xi'_{02}, \xi'_1, \xi_1) \in T_{\bar{u}(s)}(M_0 \times M_2 \times M_1 \times M_1)$ we have

$$|\hat{\xi}| \leq C\big(|\pi_{02}\xi'_{02}| + \big|\xi'_1 - \xi_1\big| + \big|\pi_{0211}^\perp \hat{\xi}\big|\big)$$

with a uniform constant C as claimed in (a). (Here the projection onto $(T\Delta_1)^\perp$ is given by $(\xi'_{02}, \xi'_1, \xi_1) \mapsto \frac{1}{2}(\xi'_1 - \xi_1, \xi_1 - \xi'_1)$.) Moreover, the splitting (45) commutes with

$$T(M_0 \times M_2) = TL_{02} \oplus (TL_{02})^\perp$$

via the canonical projection on the left hand side, and on the right hand side the identity on TL_{02} combined with a bounded map $\big(\{0\} \times (T\Delta_1)^\perp\big) \oplus T(L_{01} \times L_{12})^\perp \to TL_{02} \oplus (TL_{02})^\perp$. This implies that

$$|\pi_{02}^\perp \xi'_{02}| \leq C\big(|\pi_{0211}^\perp \hat{\xi}| + |\xi'_1 - \xi_1|\big)$$

with another uniform constant C. This proves (a). For $\hat{\xi} \in \mathcal{C}^\infty(\mathbb{R}, \bar{u}^* T(M_0 \times M_2 \times M_1 \times M_1))$ we can then apply the pointwise estimates to $\hat{\xi}(s)$ and integrate over $s \in \mathbb{R}$ to obtain for any $p \geq 1$ including $p = \infty$

$$\begin{aligned} \|\hat{\xi}\|_{L^p(\mathbb{R})} &\leq C\big(\|\pi_{02}\xi'_{02}\|_{L^p(\mathbb{R})} + \|\xi'_1 - \xi_1\|_{L^p(\mathbb{R})} + \|\pi_{0211}^\perp \hat{\xi}\|_{L^p(\mathbb{R})}\big), \\ \|\pi_{02}^\perp \xi'_{02}\|_{L^p(\mathbb{R})} &\leq C\big(\|\pi_{0211}^\perp \hat{\xi}\|_{L^p(\mathbb{R})} + \|\xi'_1 - \xi_1\|_{L^p(\mathbb{R})}\big). \end{aligned} \tag{46}$$

In order to prove the H^1- and $\mathcal{C}^1$-estimates we also apply the pointwise estimates to $\nabla_s \hat{\xi}(s)$,

$$\begin{aligned} |\nabla_s \hat{\xi}| &\leq C\big(|\pi_{02}(\nabla_s \xi'_{02})| + \big|\nabla_s \xi'_1 - \nabla_s \xi_1\big| + \big|\pi_{0211}^\perp(\nabla_s \hat{\xi})\big|\big), \\ |\pi_{02}^\perp(\nabla_s \xi'_{02})| &\leq C\big(|\pi_{0211}^\perp(\nabla_s \hat{\xi})| + |\nabla_s \xi'_1 - \nabla_s \xi_1|\big). \end{aligned}$$

Here we will need the inequalities

$$\begin{aligned} |\pi_{02}(\nabla_s \xi'_{02})| &\leq C\big(|\nabla_s(\pi_{02}(\xi'_{02}))| + |\hat{\xi}|\big), \\ |\pi_{0211}^\perp(\nabla_s \hat{\xi})| &\leq C\big(|\nabla_s(\pi_{0211}^\perp(\hat{\xi}))| + |\partial_s \bar{u}| \cdot |\hat{\xi}|\big), \\ |\nabla_s(\pi_{02}^\perp(\xi'_{02}))| &\leq C\big(|\pi_{02}^\perp(\nabla_s \xi'_{02})| + |\partial_s \bar{u}| \cdot |\hat{\xi}|\big). \end{aligned}$$

The first inequality (and similarly the others) can be seen by writing ξ'_{02} in a local orthonormal frame given by $(\gamma_i(s))_{i=1,\dots,k} \in \bar{u}_{02}(s)^* TL_{02}$ and $(\eta_i(s))_{i=1,\dots,K} \in \bar{u}_{02}(s)^*(TL_{02})^\perp$. Writing $\hat{\xi} = \sum \lambda^i \gamma_i + \sum \mu^i \eta_i$ we have

$$\begin{aligned} \big|\pi_{02}(\nabla_s \xi'_{02}) - \nabla_s(\pi_{02}(\xi'_{02}))\big| &= \Big|\sum \lambda^i \big(\pi_{02}(\nabla_s \gamma_i) - \nabla_s \gamma_i\big) + \sum \mu^i \pi_{02} \nabla_s(\eta_i)\Big| \\ &\leq C|\partial_s \bar{u}_{02}| \cdot |\xi'_{02}|. \end{aligned}$$

Note here that $\nabla_s\gamma_i = \nabla_{\partial_s\bar u_{02}}\gamma_i$ and $\nabla_s\eta_i = \nabla_{\partial_s\bar u_{02}}\eta_i$ are uniformly bounded. Putting things together we obtain the first estimate in (b) with an extra $\|\hat\xi\|_{L^2(\mathbb{R})}$ or $\|\hat\xi\|_{\mathcal{C}^0(\mathbb{R})}$ on the right hand side, for which we can use (46). For the last estimate in (b) we obtain

$$\|\nabla_s(\pi_{02}^\perp\xi'_{02})\|_{L^2(\mathbb{R})} \le C\big(\|\nabla_s(\pi_{0211}^\perp\hat\xi)\|_{L^2(\mathbb{R})} + \|\nabla_s\xi'_1 - \nabla_s\xi_1\|_{L^2(\mathbb{R})} + \big\||\partial_s\bar u|\cdot|\hat\xi|\big\|_{L^2(\mathbb{R})}\big).$$

This finishes the proof of (b). □

The following lemma contains a Sobolev estimate with a constant independent of the width δ of the middle strip; here the transversality assumption is used in a crucial way.

Lemma 5.1.4. *(Uniform Sobolev Estimate) There is a constant C_S such that for all $\delta\in(0,1]$ and $\xi=(\xi_{02},\hat\xi)\in H^2_{1,\delta}$*

$$\begin{aligned}&\|\xi_{02}\|_{\mathcal{C}^0([0,1],H^1(\mathbb{R}))} + \|\hat\xi\|_{\mathcal{C}^0([0,\delta],H^1(\mathbb{R}))}\\ &\le C_S\big(\|\xi\|_{H^2_{1,\delta}} + \|(\xi_{02}-\xi'_{02})|_{t=0}\|_{H^1(\mathbb{R})} + \|(\xi_1-\xi'_1)|_{t=0}\|_{H^1(\mathbb{R})} + \|\pi_{0211}^\perp\hat\xi|_{t=\delta}\|_{H^1(\mathbb{R})}\big).\end{aligned}$$

In particular, for all $p>2$ including $p=\infty$ and for $\xi\in\Gamma_{1,\delta}$ satisfying the boundary conditions (39),

$$\|\xi_{02}\|_{L^p(\mathbb{R}\times[0,1])} + \|\hat\xi\|_{L^p(\mathbb{R}\times[0,\delta])} \le C_S\|\xi\|_{H^2_{1,\delta}}.$$

Proof. The $\mathcal{C}^0$- and L^p-estimates will follow from the continuous embeddings $H^1(\mathbb{R})\hookrightarrow\mathcal{C}^0(\mathbb{R})$ and $H^1(\mathbb{R})\hookrightarrow L^p(\mathbb{R})$ for $p\ge 2$. So it suffices to suppose by contradiction that there are sequences $\delta^\nu>0$ and $\xi^\nu\in H^2_{1,\delta^\nu}$ with $\|\xi^\nu_{02}\|_{\mathcal{C}^0([0,1],H^1(\mathbb{R}))} + \|\hat\xi^\nu\|_{\mathcal{C}^0([0,\delta^\nu],H^1(\mathbb{R}))} = 1$ but $\|\xi^\nu\|_{H^2_{1,\delta^\nu}} + \|(\xi^\nu_{02}-\xi'^\nu_{02})|_{t=0}\|_{H^1(\mathbb{R})} + \|(\xi^\nu_1-\xi'^\nu_1)|_{t=0}\|_{H^1(\mathbb{R})} + \|\pi_{0211}^\perp\hat\xi^\nu|_{t=\delta^\nu}\|_{H^1(\mathbb{R})}\to 0$. By the standard Sobolev embedding

$$H^2([0,1]\times\mathbb{R})\subset H^1([0,1],X)\hookrightarrow\mathcal{C}^0([0,1],X)\qquad\text{with } X=H^1(\mathbb{R})$$

this implies $\|\xi^\nu_{02}\|_{\mathcal{C}^0([0,1],H^1(\mathbb{R}))}\to 0$, and so

$$\|\xi'^\nu_{02}|_{t=0}\|_{H^1(\mathbb{R})} \le \|\xi^\nu_{02}|_{t=0}\|_{H^1(\mathbb{R})} + \|(\xi^\nu_{02}-\xi'^\nu_{02})|_{t=0}\|_{H^1(\mathbb{R})}\to 0. \tag{47}$$

We can moreover integrate for all $t_0\in[0,\delta^\nu]$ to obtain

$$\|\hat\xi^\nu|_{t=t_0} - \hat\xi^\nu|_{t=\delta^\nu}\|^2_{H^1(\mathbb{R})} \le \delta^\nu\int_0^{\delta^\nu}\|\nabla_t\hat\xi^\nu\|^2_{H^1(\mathbb{R})} \le \delta^\nu\|\hat\xi^\nu\|^2_{H^2(\mathbb{R}\times[0,\delta^\nu])}\to 0. \tag{48}$$

Using Lemma 5.1.3 we then obtain

$$\begin{aligned}&\|\hat\xi^\nu|_{t=\delta^\nu}\|_{H^1(\mathbb{R})}\\ &\le C\big(\|\pi_{02}\xi'^\nu_{02}|_{t=\delta^\nu}\|_{H^1(\mathbb{R})} + \|(\xi^\nu_1-\xi'^\nu_1)|_{t=\delta^\nu}\|_{H^1(\mathbb{R})} + \|\pi_{0211}^\perp\hat\xi^\nu|_{t=\delta^\nu}\|_{H^1(\mathbb{R})}\big)\\ &\le C\big(\|\pi_{02}(\xi'^\nu_{02}|_{t=\delta^\nu} - \xi'^\nu_{02}|_{t=0})\|_{H^1(\mathbb{R})} + \|\pi_{02}(\xi'^\nu_{02})|_{t=0}\|_{H^1(\mathbb{R})}\\ &\qquad + \|(\xi^\nu_1-\xi'^\nu_1)|_{t=0}\|_{H^1(\mathbb{R})} + 2\|\hat\xi^\nu|_{t=\delta^\nu} - \hat\xi^\nu|_{t=0}\|_{H^1(\mathbb{R})} + \|\pi_{0211}^\perp\hat\xi^\nu|_{t=\delta^\nu}\|_{H^1(\mathbb{R})}\big)\\ &\to 0\end{aligned}$$

with uniform constants C, C' by (39), (47), (48), and a bound on the operator norm of π_{02}. Now combining $\|\hat\xi^\nu|_{t=\delta^\nu}\|_{H^1(\mathbb{R})}\to 0$ with (48) proves

$$\|\hat\xi^\nu\|_{\mathcal{C}^0([0,\delta^\nu],H^1(\mathbb{R}))}\to 0$$

in contradiction to the assumption. □

The solution u of the 0-equation corresponds to $\xi = 0$, which is an almost zero of $\mathcal{F}_u$. This and a quadratic estimate for $d\mathcal{F}_u$ near 0 is the content of the next lemma. For later purposes we also compare $d\mathcal{F}_u(\xi)$ with the linearized operator $D_{e_u(\xi)}$ of $\overline{\partial}_J = (\overline{\partial}_{J_{02}}, \overline{\partial}_{\hat{J}})$ at $e_u(\xi)$. To state the comparison we will need the pointwise linear operator

$$E_u(\xi)\eta := \tfrac{d}{d\tau} e_u(\xi + \tau\eta)|_{\tau=0}.$$

It satisfies $E_u(0) = \mathrm{Id}$, and since e_u maps $\Gamma_{1,\delta}$ to the space of maps satisfying the boundary conditions in (36), the linearization $E_u(\xi)$ maps $\Gamma_{1,\delta}$ to the space of sections $\zeta \in \Gamma(v_{02}^* TM_{02}) \times \Gamma(\hat{v}^* TM_{0211})$ over $v = (v_{02}, \hat{v}) := e_u(\xi)$, that satisfy the linearized boundary conditions

$$(\zeta_{02}, \zeta)|_{t=0} \in T_v(\Delta_0 \times \Delta_2 \times \Delta_1), \quad \hat{\zeta}|_{t=\delta} \in T_{\hat{v}}(L_{01} \times L_{12}), \quad \zeta_{02}|_{t=1} \in T_{v_{02}}(L_0 \times L_2).$$

The linearized operator D_v acts on this space of sections and is given by

$$D_v\zeta = \tilde{\nabla}_\tau \overline{\partial}_J e_v(\tau\zeta)|_{\tau=0}$$

with respect to the connection $\tilde{\nabla}$ introduced on page 56. In this notation we have $D_{e_u(0)} = d\mathcal{F}_u(0)$.

Lemma 5.1.5. *(Uniform quadratic and error estimates) There are uniform constants $\epsilon > 0$ and C_1, C_2, C_3 such that for all $\delta \in (0,1]$ and $\xi \in \Gamma_{1,\delta}(\epsilon)$, $\eta \in \Gamma_{1,\delta}$*

$$\begin{aligned} \|\mathcal{F}_u(0)\|_{\Omega_{1,\delta}} &\le C_1\sqrt{\delta}, \\ \|d\mathcal{F}_u(\xi)\eta - d\mathcal{F}_u(0)\eta\|_{\Omega_{1,\delta}} &\le C_2 \|\xi\|_{\Gamma_{1,\delta}} \|\eta\|_{\Gamma_{1,\delta}}, \\ \|d\mathcal{F}_u(\xi)\eta - \Phi_u(\xi)^{-1} D_{e_u(\xi)} E_u(\xi)\eta\|_{\Omega_{1,\delta}} &\le C_3 \|\xi\|_{\Gamma_{1,\delta}} \|\eta\|_{\Gamma_{1,\delta}}. \end{aligned}$$

Proof. To estimate $\mathcal{F}_u(0)$ we recall that u_{02} is holomorphic and $\bar{u}$ is constant in t, so

$$\|\mathcal{F}_u(0)\|^2_{\Omega_{1,\delta}} = \|(0, \partial_s \bar{u})\|^2_{H^1_{1,\delta}} = \delta\big(\|\partial_s u_{02}|_{t=0}\|^2_{H^1(\mathbb{R})} + 2\|\partial_s \bar{u}_1\|^2_{H^1(\mathbb{R})}\big) = C_1^2 \delta.$$

Since $\partial_s u_{02} \to 0$ converges exponentially as $s \to \pm\infty$, so does $\partial_s \bar{u}_1 = d\ell_1(\partial_s \bar{u}_{02})$ (see Remark 2.0.6), and hence the above constant C_1 is finite. For the third estimate we differentiate as in [26, p.68] the identity $\Phi_u(\xi + \tau\eta)\mathcal{F}_u(\xi + \tau\eta) = \overline{\partial}_J(e_u(\xi + \tau\eta))$ to obtain

$$\Phi_u(\xi) d\mathcal{F}_u(\xi)\eta - D_{e_u(\xi)} E_u(\xi)\eta = -\Psi_u(\xi, \eta, \mathcal{F}_u(\xi)), \tag{49}$$

where the estimate for the right hand side

$$\Psi_u(\xi, \eta, \zeta) := \tilde{\nabla}_\tau(\Phi_u(\xi + \tau\eta)\zeta)|_{\tau=0}$$

is part of the estimates below. The first component of $\mathcal{F}_u$ is independent of δ, so the quadratic estimates for it simply follow from the continuous differentiability of $\mathcal{F}_u$. For the second component we follow the argument in [26, Prop.3.5.3.] to obtain a uniform constant for all $\delta \in (0,1]$. We need to consider

$$\mathcal{F}_{\bar{u}}(\hat{\xi}) := \Phi_{\bar{u}}(\hat{\xi})^{-1}(\overline{\partial}_{\hat{J}} e_{\bar{u}}(\hat{\xi})),$$

where $e_{\bar{u}}(\hat{\xi}) = \exp_{\bar{u}}(\hat{\xi} + Q(\hat{\xi}))$ is the exponential map with quadratic correction defined in (43). Note that our parallel transport $\Phi_{\bar{u}}(\hat{\xi})$ is defined with respect to the path $\tau \mapsto e_{\bar{u}}(\tau\hat{\xi})$ and the Hermitian connection $\tilde{\nabla}$ on $T(M_0 \times M_2 \times M_1 \times M_1)$ that leaves $\hat{J}$ invariant. Since $e_{\bar{u}}(0) = \bar{u}$ and $de_{\bar{u}}(0) = \mathrm{Id}$, the same path can be used in the definition of $\nabla_{\hat{\xi}}$ instead of the geodesic. Now let $\xi, \eta \in \Gamma_{1,\delta}$ with $\|\xi\|_{H^2_{1,\delta}} \le \epsilon$. Then by Lemma 5.1.4

$$\|\hat{\xi}\|_{\mathcal{C}^0} \le C_S \|\xi\|_{H^2_{1,\delta}} \le C_S \epsilon =: c_0, \qquad \|\hat{\eta}\|_{\mathcal{C}^0} \le C_S \|\eta\|_{H^2_{1,\delta}}$$

with a uniform constant C_S thus a uniform constant c_0 that only depends on ϵ. In the following, all constants will be uniform in the sense that they only depend on c_0 and hence ϵ. Next, we consider

$$E_{\bar u}(\hat\xi)\hat\eta := \tfrac{d}{d\tau}e_{\bar u}(\hat\xi+\tau\hat\eta)|_{\tau=0}, \qquad \Psi_{\bar u}(\hat\xi,\hat\eta,\zeta) := \tilde\nabla_\tau(\Phi_{\bar u}(\hat\xi+\tau\hat\eta)\zeta)|_{\tau=0}.$$

Note that $E_{\bar u}(0)=\mathrm{Id}$ and that $\Psi(0,\hat\eta,\zeta)=0$ since the covariant derivative exactly uses the parallel transport $\Phi_{\bar u}(\tau\hat\eta)$. Moreover, these maps are linear in $\hat\eta$ and ζ, and they depend smoothly on $\hat\xi$. So given ϵ and thus $|\hat\xi|\le c_0$ we have linear bounds

$$|E_{\bar u}(\hat\xi)|\le c_1, \qquad |\nabla(E_{\bar u}(\hat\xi))|\le c_1\big(|\nabla\hat\xi|+|d\bar u||\hat\xi|\big), \qquad |\Psi_{\bar u}(\hat\xi,\hat\eta,\zeta)|\le c_1|\hat\xi||\hat\eta||\zeta|$$

with a uniform constant c_1. With these preparations we calculate from (49), using the notation of [26, Prop.3.5.3.],

$$\begin{aligned}
&\Phi_{\bar u}(\hat\xi)\big(d\mathcal{F}_{\bar u}(\hat\xi)\hat\eta - d\mathcal{F}_{\bar u}(0)\hat\eta\big)\\
&= -\Psi_{\bar u}(\hat\xi,\hat\eta,\mathcal{F}_{\bar u}(\hat\xi)) + \big(\nabla(E_{\bar u}(\hat\xi))\hat\eta\big)^{0,1} + \big(\big(E_{\bar u}(\hat\xi)-\Phi_{\bar u}(\hat\xi)\big)\nabla\hat\eta\big)^{0,1}\\
&\quad - \tfrac12\hat J(e_{\bar u}(\hat\xi))\big(\big((\nabla_{(E_{\bar u}(\hat\xi)\hat\eta-\Phi_{\bar u}(\hat\xi)\hat\eta)}\hat J)(e_{\bar u}(\hat\xi))\big)\Phi_{\bar u}(\hat\xi)d\bar u\big)^{0,1}\\
&\quad - \tfrac12\hat J(e_{\bar u}(\hat\xi))\big(\big((\nabla_{\Phi_{\bar u}(\hat\xi)\hat\eta}\hat J)(e_{\bar u}(\hat\xi)) - \Phi_{\bar u}(\hat\xi)(\nabla_{\hat\eta}\hat J)(\bar u)\Phi_{\bar u}(\hat\xi)^{-1}\big)\Phi_{\bar u}(\hat\xi)d\bar u\big)^{0,1}\\
&\quad - \tfrac12\hat J(e_{\bar u}(\hat\xi))\big((\nabla_{E_{\bar u}(\hat\xi)\hat\eta}\hat J)(e_{\bar u}(\hat\xi))(d(e_{\bar u}(\hat\xi)) - \Phi_{\bar u}(\hat\xi)d\bar u)\big)^{0,1}.
\end{aligned}$$

We then use the uniform bounds on $\|\nabla\hat J\|_\infty$, $|d\bar u|$, $|\hat\xi|$, and the estimates

$$\begin{aligned}
&|\mathcal{F}_{\bar u}(\hat\xi)|\le|d(e_{\bar u}(\hat\xi))|\le c_2\big(|\nabla\hat\xi|+|d\bar u|\big), \qquad |d(e_{\bar u}(\hat\xi))-\Phi_{\bar u}(\hat\xi)d\bar u|\le c_2\big(|\nabla\hat\xi|+|d\bar u||\hat\xi|\big),\\
&\big|E_{\bar u}(\hat\xi)-\Phi_{\bar u}(\hat\xi)\big|\le c_2|\hat\xi|, \qquad \big|(\nabla_{\Phi_{\bar u}(\hat\xi)\hat\eta}\hat J)(e_{\bar u}(\hat\xi)) - \Phi_{\bar u}(\hat\xi)(\nabla_{\hat\eta}\hat J)(\bar u)\Phi_{\bar u}(\hat\xi)^{-1}\big|\le c_2|\hat\xi||\hat\eta|
\end{aligned}$$

with a uniform constant c_2 to obtain with a further uniform constant c_3

$$\big|d\mathcal{F}_{\bar u}(\hat\xi)\hat\eta - d\mathcal{F}_{\bar u}(0)\hat\eta\big|\le c_3\big(|\hat\xi||\hat\eta|+|\hat\eta||\nabla\hat\xi|+|\hat\xi||\nabla\hat\eta|\big).$$

So far these pointwise estimates were standard calculations. Now we have to check that they actually lead to uniform bounds in the δ-dependent norms. The zeroth order part of the $\Omega_{1,\delta}$-norm over $\mathbb{R}\times[0,\delta]$ can be estimated with the help of Lemma 5.1.4 by

$$\begin{aligned}
\big\|d\mathcal{F}_{\bar u}(\hat\xi)\hat\eta - d\mathcal{F}_{\bar u}(0)\hat\eta\big\|_{L^2} &\le c_3\big(\|\hat\xi\|_{L^4}\|\hat\eta\|_{L^4}+\|\hat\eta\|_{\mathcal{C}^0}\|\nabla\hat\xi\|_{L^2}+\|\hat\xi\|_{\mathcal{C}^0}\|\nabla\hat\eta\|_{L^2}\big)\\
&\le c_3(C_S^2+2C_S)\|\xi\|_{H^2_{1,\delta}}\|\eta\|_{H^2_{1,\delta}},\\
\big\|d\mathcal{F}_{\bar u}(\hat\xi)\hat\eta - d\mathcal{F}_{\bar u}(0)\hat\eta\big\|_{L^4} &\le c_3\big(\|\hat\xi\|_{L^8}\|\hat\eta\|_{L^8}+\|\hat\eta\|_{\mathcal{C}^0}\|\nabla\hat\xi\|_{L^4}+\|\hat\xi\|_{\mathcal{C}^0}\|\nabla\hat\eta\|_{L^4}\big)\\
&\le c_3(C_S^2+2C_S)\big(\|\xi\|_{H^2_{1,\delta}}+\|\xi\|_{L^4_{1,\delta}}\big)\big(\|\eta\|_{H^2_{1,\delta}}+\|\nabla\eta\|_{L^4_{1,\delta}}\big).
\end{aligned}$$

For the first order part of the $\Omega_{1,\delta}$-norm one differentiates the above identity and uses further bounds on $\|\nabla^2\hat J\|_\infty$ and $|\nabla d\bar u|$ to find a pointwise bound

$$\begin{aligned}
\big|\nabla\big(d\mathcal{F}_{\bar u}(\hat\xi)\hat\eta - d\mathcal{F}_{\bar u}(0)\hat\eta\big)\big| &\le c_4\big(|\hat\xi|+|\nabla\hat\xi|\big)\big(|\hat\eta|+|\nabla\hat\eta|\big)\\
&\quad + c_4\big(|\nabla^2\hat\xi||\hat\eta|+|\nabla\hat\xi|^2|\hat\eta|+|\nabla\hat\xi||\nabla\hat\eta|+|\hat\xi||\nabla^2\hat\eta|\big).
\end{aligned}$$

Then we again use Lemma 5.1.4 and $\|\nabla\hat{\xi}\|_{L^2} \leq \epsilon$ to obtain with a final uniform constant c_5

$$
\begin{aligned}
&\big\|\nabla\big(d\mathcal{F}_{\bar{u}}(\hat{\xi})\hat{\eta} - d\mathcal{F}_{\bar{u}}(0)\hat{\eta}\big)\big\|_{L^2} \\
&\leq c_4\big(\|\hat{\xi}\|_{L^4} + \|\nabla\hat{\xi}\|_{L^4}\big)\big(\|\hat{\eta}\|_{L^4} + \|\nabla\hat{\eta}\|_{L^4}\big) \\
&\quad + c_4\big(\|\nabla^2\hat{\xi}\|_{L^2}\|\hat{\eta}\|_{\mathcal{C}^0} + \|\nabla\hat{\xi}\|_{L^2}\|\nabla\hat{\xi}\|_{L^4}\|\hat{\eta}\|_{L^4} + \|\nabla\hat{\xi}\|_{L^4}\|\nabla\hat{\eta}\|_{L^4} + \|\hat{\xi}\|_{\mathcal{C}^0}\|\nabla^2\hat{\eta}\|_{L^2}\big) \\
&\leq c_5\big(\|\xi\|_{H^2_{1,\delta}} + \|\nabla\xi\|_{L^4_{1,\delta}}\big)\big(\|\eta\|_{H^2_{1,\delta}} + \|\nabla\eta\|_{L^4_{1,\delta}}\big).
\end{aligned}
$$

□

Theorem 5.1.1 now follows from the implicit function theorem [26, A.3.4] if we can establish surjectivity and a uniform bound on the right inverse for the linearized operator

$$
D^\delta := d\mathcal{F}_u(0) : \Gamma_{1,\delta} \to \Omega_{1,\delta}, \qquad D^\delta\xi = \big(D_{u_{02}}\xi_{02}\,,\, D_{\bar{u}}\hat{\xi}\big) \tag{50}
$$

with

$$
\begin{aligned}
D_{u_{02}}\xi_{02} &= \nabla_s\xi_{02} + J(u_{02})\nabla_t\xi_{02} + \nabla_{\xi_{02}}J_{02}(u_{02})\partial_t u_{02}, \\
D_{\bar{u}}\hat{\xi} &= \nabla_s\hat{\xi} + \hat{J}(\bar{u})\nabla_t\hat{\xi} + \tfrac{1}{2}\nabla_{\hat{\xi}}\hat{J}(\bar{u})\hat{J}(\bar{u})\partial_s\bar{u}.
\end{aligned}
$$

Here $D_{u_{02}}$ and $D_{\bar{u}}$ are the linearized operators of $\overline{\partial}_{J_{02}}$ at u_{02} (which is holomorphic) and of $\overline{\partial}_{\hat{J}}$ at $\bar{u}$ (which satisfies $\partial_t\bar{u} = 0$) respectively. (See [26, Prop.3.1.1.] for an explicit calculation of the linearized operators, and note that we identify $\Omega^{0,1}(\mathbb{R}\times[0,1], u^*TM)$ with sections of u^*TM by $\gamma ds + J\gamma dt \mapsto \gamma$.) We can identify the cokernel of D^δ with $(\operatorname{im} D^\delta)^\perp \subset (H^1_{1,\delta})^*$. By elliptic regularity any element in this cokernel can be represented by the L^2-inner product $\langle\eta, \operatorname{im} D^\delta\rangle = 0$ with a smooth section η. Partial integration then shows that $\eta \in \Gamma_{1,\delta}$ satisfies the boundary conditions (39) and lies in the kernel of the formal adjoint operator, $(D^\delta)^*\eta = 0$. Note that $(D^\delta)^*$ is given by $\big(-\nabla_s + J_{02}(u_{02})\nabla_t, -\nabla_s + \hat{J}(\bar{u})\nabla_t\big)$ plus lower order terms. So $(D^\delta)^*$ has the same analytic properties as D^δ, and we will prove the surjectivity of D^δ by establishing injectivity for $(D^\delta)^*$.

By our assumptions on the index and regularity of $(u_0, u_2) \in \widetilde{\mathcal{M}}^1_0(x^-, x^+)$ we know that the operator $D_{u_{02}} \oplus \pi^\perp_{02}$ on the space of sections in $H^2(u^*_{02}T(M_0\times M_2))$ with boundary conditions at $t=1$ in $T(L_0\times L_2)$ (where $\pi^\perp_{02}$ is the projection at $t=0$) is surjective and has a one dimensional kernel $\ker(D_{u_{02}} \oplus \pi^\perp_{02})$. This is not a subspace of $\Gamma_{1,\delta}$, but we will fix a complement for every $\delta > 0$ in the following sense,

$$
K_0 := \big\{\xi = (\xi_{02}, \hat{\xi}) \in \Gamma_{1,\delta} \,\big|\, \langle\xi_{02}, \ker(D_{u_{02}} \oplus \pi^\perp_{02})\rangle_{L^2} \equiv 0\big\}.
$$

Here we used the L^2-inner product on $H^2(\mathbb{R}\times[0,1], u^*_{02}T(M_0\times M_2))$.

Combining the uniform linear estimates Lemma 5.2.1 and Lemma 5.2.2 we can choose $\delta_0 := \frac{1}{16}c_1^2c_2^2 > 0$ such that for all $\delta \in (0,\delta_0)$ and $\xi \in \Gamma_{1,\delta}$

$$
\begin{aligned}
(1 + c_2^{-1})\|(D^\delta)^*\xi\|_{\Omega_{1,\delta}} &\geq \tfrac{1}{2}\|(D^\delta)^*\xi\|_{H^1_{1,\delta}} + \tfrac{1}{2}\|(D^\delta)^*\xi\|_{L^4_{1,\delta}} + c_2^{-1}\|D^*_{u_{02}}\xi_{02}\|_{H^1(\mathbb{R}\times[0,1])} \\
&\geq \tfrac{1}{2}c_1\|\xi\|_{\Gamma_{1,\delta}} - c_2^{-1}\sqrt{\delta}\|\nabla_t\hat{\xi}\|_{H^1(\mathbb{R}\times[0,\delta])} \;\geq\; \tfrac{1}{4}c_1\|\xi\|_{\Gamma_{1,\delta}},
\end{aligned}
$$

and similarly for all $\xi \in \Gamma_{1,\delta} \cap K_0$

$$
\|D^\delta\xi\|_{\Omega^1_{1,\delta}} \geq \frac{c_1c_2}{4(c_2+1)}\|\xi\|_{\Gamma_{1,\delta}}. \tag{51}
$$

The first estimate shows that $(D^\delta)^*$ is injective and hence D^δ is surjective. The second estimate shows that its right inverse is uniformly bounded. It remains to check that D^δ

stays surjective when restricted to K_0. This follows from the fact that both $D_{u_{02}}$ with boundary conditions in $(L_{02}, L_0 \times L_2)$ and $D^\delta = (D_{u_{02}}, D_{\bar{u}})$ with boundary conditions (39) are surjective and have the same index 1 by Lemma 2.3.10 and the identification $\widetilde{\mathcal{M}}^1_{\delta}(x^-, x^+) \cong \widehat{\mathcal{M}}^1_{\delta}(x^-, x^+)$. So D^δ has a 1-dimensional kernel, which is transversal to K_0 by the last estimate, and hence $D^\delta|_{K_0}$ must be surjective. This finishes the proof of theorem 5.1.1. Here $\epsilon > 0$ is fixed such that the exponential map e_u is defined on $\Gamma_{1,\delta}(\epsilon)$ and such that Lemma 5.1.5 holds.

Corollary 5.1.6. *There exists $\delta_0 > 0$ such that the map $\mathcal{T}_\delta : \mathcal{M}^1_0(x^-, x^+) \to \mathcal{M}^1_\delta(x^-, x^+)$ given by $\mathcal{T}_\delta([u]) := [v_u]$ is well defined and injective for all $\delta \in (0, \delta_0]$.*

Proof. We choose $\delta_0 \le \epsilon^2 C_0^{-2}$ such that Theorem 5.1.1 applies. Then let $v_u = e_u(\xi)$ be the solution constructed from $u \in \widetilde{\mathcal{M}}^1_0(x^-, x^+)$ and consider a shifted 0-solution $\tilde{u} = u(\cdot + \sigma) \in [u]$. Then $\tilde{\xi} := \xi(\cdot + \sigma)$ satisfies $\|\tilde{\xi}\| = \|\xi\| \le C_0\sqrt{\delta} \le \epsilon$, $\mathcal{F}_u(\tilde{\xi}) = 0$, and the orthogonality condition to $\ker(D_{\tilde{u}_{02}} \oplus \pi_{02}^\perp)$. Hence $v_{\tilde{u}} = e_{u(\cdot+\sigma)}(\xi(\cdot+\sigma)) = v_u(\cdot+\sigma) \in [v_u]$, so $\mathcal{T}_\delta([u]) = [v_u]$ is well defined.

The injectivity of $\mathcal{T}_\delta$ follows from the fact that $\mathcal{M}^1_0(x^-, x^+)$ consists of isolated points, so the $\mathcal{C}^0$-distance $d_{\mathcal{C}^0}([u], [u']) > \Delta_0$ is bounded below by some $\Delta_0 > 0$ for all $[u] \ne [u']$. On the other hand, $d_{\mathcal{C}^0}([\bar{u}], \mathcal{T}_\delta([u]) \le C_0 C_S(1 + C_Q)\sqrt{\delta}$ by (38), (40), and Lemma 5.1.4. So if we had $\mathcal{T}_\delta([u]) = \mathcal{T}_\delta([u'])$ then $d_{\mathcal{C}^0}([u], [u']) \le d_{\mathcal{C}^0}([\bar{u}], [\bar{u}']) \le 2C_0 C_S(1 + C_Q)\sqrt{\delta}$. This implies $[u] = [u']$ whenever $\delta \le \delta_0$, where we choose $\delta_0 \le (2C_0 C_S(1 + C_Q))^{-2}\Delta_0^2$. □

5.2. **Uniform estimates.** In this section we establish the uniform linear and nonlinear estimates that are used in Sections 5.1 and 5.3. We will work in the setup of section 5.1 and fix a solution $u \in \widetilde{\mathcal{M}}^1_0(x^-, x^+)$. For convenience we denote the target spaces by $M_{02} := M_0 \times M_2$ and $M_{0211} := M_0 \times M_2 \times M_1 \times M_1$ and the symplectic structures by $\omega_{02} = (-\omega_0) \oplus \omega_2$ and $\omega_{0211} = \omega_0 \oplus (-\omega_2) \oplus (-\omega_1) \oplus \omega_1$ respectively. The nonlinear equation for $v = (v_{02}, \hat{v})$, $v_{02} : \mathbb{R} \times [0, 1] \to M_{02}$, $\hat{v} : \mathbb{R} \times [0, \delta] \to M_{0211}$ is

$$\overline{\partial}_J v := \partial_s v + J(v)\partial_t v := \big(\partial_s v_{02} + J_{02}(v_{02})\partial_t v_{02}\, ,\, \partial_s \hat{v} + \hat{J}(\hat{v})\partial_t \hat{v}\big).$$

We will need uniform estimates for the nonlinear operator $\xi \mapsto \overline{\partial}_J e_u(\xi)$ on $\xi \in \Gamma_{1,\delta}(\epsilon)$ and the linearized operator D^δ. For that purpose we use the Levi-Civita connection on $M = M_{02}$ and $M = M_{0211}$ respectively to identify $T_u M \times T_u M \cong T_\xi T_u M$ for every $\xi \in T_u M$. With this we decompose $Te(u, \xi) : T_\xi T_u M \to T_{e_u \xi} M$ as

$$Te(u, \xi)(X, \eta) = \partial_1 e(u, \xi)X + de_u(\xi)\eta \qquad \forall \xi, X, \eta \in T_u M.$$

We denote the pullback almost complex structure on $H^2_{1,\delta}$ under $de_u(\xi)$ by

$$\begin{aligned} J(\xi) &:= (J_{02}(\xi_{02}), \hat{J}(\hat{\xi})) \\ &:= \big((de_{u_{02}}(\xi_{02}))^{-1} J_{02}(e_{u_{02}}(\xi_{02})) de_{u_{02}}(\xi_{02}), (de_{\bar{u}}(\hat{\xi}))^{-1} \hat{J}(e_{\bar{u}}(\hat{\xi})) de_{\bar{u}}(\hat{\xi})\big) \end{aligned}$$

for $\xi = (\xi_{02}, \hat{\xi}) \in \Gamma_{1,\delta}(\epsilon)$. With this we can express

$$\overline{\partial}_J(e_u(\xi)) = de_u(\xi)\big(\nabla_s \xi + J(\xi)\nabla_t \xi\big) + \partial_1 e(u, \xi)\partial_s u + J(u)\partial_1 e(u, \xi)\partial_t u \tag{52}$$

in terms of the nonlinear operator on $H^2_{1,\delta}$,

$$\nabla_s \xi + J(\xi)\nabla_t \xi := \big(\nabla_s \xi_{02} + J_{02}(\xi_{02})\nabla_t \xi_{02}\, ,\, \nabla_s \hat{\xi} + \hat{J}(\hat{\xi})\nabla_t \hat{\xi}\big).$$

Note that $J(0) = (J_{02}, \hat{J})$ is the usual almost complex structure, so we can express the linearized operator (50) as

$$D^\delta \xi = \nabla_s \xi + J(0)\nabla_t \xi + \big(\nabla_{\xi_{02}} J_{02}(u_{02})\partial_t u_{02}\, ,\, \tfrac{1}{2}\nabla_{\hat{\xi}} \hat{J}(\bar{u})\hat{J}(\bar{u})\partial_s \bar{u}\big).$$

The following lemma provides uniform elliptic estimates.

Lemma 5.2.1.

(a) *There is a constant C_1 such that for all $\delta \in (0,1]$ and $\xi \in \Gamma_{1,\delta}$*

$$\left|\int_{\{1\}\times\mathbb{R}} \omega_{02}(\xi_{02}, \nabla_s\xi_{02})\right| + \left|\int_{\{\delta\}\times\mathbb{R}} \omega_{0211}(\hat{\xi}, \nabla_s\hat{\xi})\right| \le C_1\big(\|\xi_{02}|_{t=1}\|_{H^0(\mathbb{R})} + \|\hat{\xi}|_{t=\delta}\|_{H^0(\mathbb{R})}\big)^2,$$

$$\left|\int_{\{1\}\times\mathbb{R}} \omega_{02}(\nabla_s\xi_{02}, \nabla_s^2\xi_{02})\right| + \left|\int_{\{\delta\}\times\mathbb{R}} \omega_{0211}(\nabla_s\hat{\xi}, \nabla_s^2\hat{\xi})\right| \le C_1\big(\|\xi_{02}|_{t=1}\|_{H^1(\mathbb{R})} + \|\hat{\xi}|_{t=\delta}\|_{H^1(\mathbb{R})}\big)^2.$$

(b) *There is a constant $\epsilon > 0$ and for every $c_0 > 0$ there is a constant C_1 such that for all $\delta \in (0,1]$ and $\xi, \zeta \in H^2_{1,\delta}$ with $\|\zeta\|_\infty \le \epsilon$, $\|\nabla\zeta\|_\infty \le c_0$*

$$\|\xi\|_{H^1_{1,\delta}} \le C_1\Bigg(\|\nabla_s\xi + J(\zeta)\nabla_t\xi\|_{H^0_{1,\delta}} + \|\xi\|_{H^0_{1,\delta}} + \left|\int_{\{\delta\}\times\mathbb{R}} \omega_{0211}(\hat{\xi}, \nabla_s\hat{\xi})\right|^{1/2} + \left|\int_{\{1\}\times\mathbb{R}} \omega_{02}(\xi_{02}, \nabla_s\xi_{02})\right|^{1/2}\Bigg),$$

$$\|\xi\|_{H^2_{1,\delta}} \le C_1\Bigg(\|\nabla_s\xi + J(\zeta)\nabla_t\xi\|_{H^1_{1,\delta}} + \|\xi\|_{H^0_{1,\delta}} + \left|\int_{\{\delta\}\times\mathbb{R}} \omega_{0211}(\hat{\xi}, \nabla_s\hat{\xi})\right|^{1/2} + \left|\int_{\{\delta\}\times\mathbb{R}} \omega_{0211}(\nabla_s\hat{\xi}, \nabla_s^2\hat{\xi})\right|^{1/2} + \left|\int_{\{1\}\times\mathbb{R}} \omega_{02}(\xi_{02}, \nabla_s\xi_{02})\right|^{1/2} + \left|\int_{\{1\}\times\mathbb{R}} \omega_{02}(\nabla_s\xi_{02}, \nabla_s^2\xi_{02})\right|^{1/2}\Bigg),$$

$$\|\nabla\xi\|_{L^4_{1,\delta}} \le C_1\Big(\|\xi\|_{H^2_{1,\delta}} + \|\nabla_s\xi + J(\zeta)\nabla_t\xi\|_{L^4_{1,\delta}} + \|\hat{\xi}|_{t=\delta}\|_{H^1(\mathbb{R})}\Big).$$

(c) *There is a constant $c_1 > 0$ such that for all $\delta \in (0,1]$ and $\xi \in \Gamma_{1,\delta}$*

$$c_1\|\xi\|_{H^2_{1,\delta}} \le \|D^\delta\xi\|_{H^1_{1,\delta}} + \|\xi\|_{H^0_{1,\delta}} + \|\hat{\xi}|_{t=\delta}\|_{H^1(\mathbb{R})} + \|\xi_{02}|_{t=1}\|_{H^1(\mathbb{R})},$$

$$c_1\|\nabla\xi\|_{L^4_{1,\delta}} \le \|D^\delta\xi\|_{H^1_{1,\delta}} + \|D^\delta\xi\|_{L^4_{1,\delta}} + \|\xi\|_{H^0_{1,\delta}} + \|\hat{\xi}|_{t=\delta}\|_{H^1(\mathbb{R})} + \|\xi_{02}|_{t=1}\|_{H^1(\mathbb{R})},$$

and the same holds with D^δ replaced by $(D^\delta)^*$.

Proof. We prove (a) in general for $\int_\mathbb{R} \omega(\xi, \nabla_s\xi)$ and $\int_\mathbb{R} \omega(\nabla_s\xi, \nabla_s^2\xi)$ with a Lagrangian section $\xi : \mathbb{R} \to u^*TL$ over a path $u : \mathbb{R} \to L$. These expressions vanish if L is totally geodesic. To estimate them in general we pick a smooth family of orthonormal frames $(\gamma_i(s))_{i=1,\dots,k} \in u(s)^*TL$, then

$$\xi = \sum \lambda^i\gamma_i, \quad \nabla_s\xi = \sum\Big(\partial_s\lambda^i\gamma_i + \lambda^i\nabla_s\gamma_i\Big), \quad \nabla_s^2\xi = \sum\Big(\partial_s^2\lambda^i\gamma_i + 2\partial_s\lambda^i\nabla_s\gamma_i + \lambda^i\nabla_s^2\gamma_i\Big)$$

with $\lambda : \mathbb{R} \to \mathbb{R}^k$. By the orthonormality we have $|\lambda(s)| = |\xi(s)|$, and using $(\gamma, J\gamma)$ as a trivialization for the definition of Sobolev norms on u^*TM we obtain $\|\lambda\|_{H^s(\mathbb{R})} = \|\xi\|_{H^s(\mathbb{R})}$.

We now use the identities $\omega(\gamma_i,\gamma_j)=0$ to obtain

$$\begin{aligned}\left|\int_{\mathbb{R}}\omega(\xi,\nabla_s\xi)\right| &\leq \left|\int_{\mathbb{R}} C|\xi(s)||\lambda(s)|ds\right| = C\|\xi\|^2_{L^2(\mathbb{R})},\\ \left|\int_{\mathbb{R}}\omega(\nabla_s\xi,\nabla_s^2\xi)\right| &\leq \left|\int_{\mathbb{R}} C\bigl(|\nabla_s\xi||\lambda| + |\nabla_s\xi||\partial_s\lambda| + |\partial_s\lambda|^2 + |\lambda|^2\bigr)\right| \leq 4C\|\xi\|^2_{H^1(\mathbb{R})},\end{aligned}$$

where the constant C only depends on γ (that is on $u:\mathbb{R}\to L$) up to third derivatives. Here we used partial integration

$$\int_{\mathbb{R}}\sum_{i,j}\lambda^i\partial_s^2\lambda^j\omega(\nabla_s\gamma_i,\gamma_j) = -\int_{\mathbb{R}}\sum_{i,j}\Bigl(\partial_s\lambda^i\partial_s\lambda^j\omega(\nabla_s\gamma_i,\gamma_j) + \lambda^i\partial_s\lambda^j\partial_s\omega(\nabla_s\gamma_i,\gamma_j)\Bigr).$$

To prove (c) we can replace D^δ by $\nabla_s\xi + J(0)\nabla_t\xi$ since the difference of the operators is bounded in the different components and norms by

$$\begin{aligned}&\bigl\|\nabla_{\xi_{02}}J_{02}(u_{02})\partial_t u_{02}\bigr\|_{H^0(\mathbb{R}\times[0,1])} + \bigl\|\nabla_{\hat\xi}\hat J(\bar u)J(\bar u)\partial_s\bar u\bigr\|_{H^0(\mathbb{R}\times[0,\delta])} \leq C\bigl\|\xi\bigr\|_{H^0_{1,\delta}},\\ &\bigl\|\nabla_{\xi_{02}}J_{02}(u_{02})\partial_t u_{02}\bigr\|_{L^4(\mathbb{R}\times[0,1])} \leq C\bigl\|\nabla_{\xi_{02}}J_{02}(u_{02})\partial_t u_{02}\bigr\|_{H^1(\mathbb{R}\times[0,1])} \leq C\bigl\|\xi\bigr\|_{H^1_{1,\delta}},\\ &\bigl\|\nabla_{\hat\xi}\hat J(\bar u)J(\bar u)\partial_s\bar u\bigr\|_{H^1(\mathbb{R}\times[0,\delta])} \leq C\bigl\|\xi\bigr\|_{H^1_{1,\delta}},\\ &\bigl\|\nabla_{\hat\xi}\hat J(\bar u)J(\bar u)\partial_s\bar u\bigr\|_{L^4(\mathbb{R}\times[0,\delta])} \leq C\|\nabla\hat J\|_\infty\|\partial_s\bar u\|_\infty\|\hat\xi\|_{L^4(\mathbb{R}\times[0,\delta])} \leq C\bigl\|\xi\bigr\|_{H^2_{1,\delta}},\end{aligned} \tag{53}$$

where C denotes any uniform constant. The extra terms on the right hand side will fit into the proof and will be recalled for the relevant estimates. The proof for $(D^\delta)^*$ is completely analogous. We will use the notation $\nabla_s\xi + J(\sigma\zeta)\nabla_t\xi$ to make partial integration calculations for the nonlinear ($\sigma=1$) and linear ($\sigma=0$) operator at the same time. In the nonlinear case the almost complex structure $J(\zeta)$ is not skew-adjoint. In order to restore this property we work with the $L^2_{1,\delta}(\sigma\zeta)$-metric, which uses the pullback metric $g_{\sigma\zeta} = \langle\cdot,\cdot\rangle_{\sigma\zeta}$ under $de_{u_{02}}(\sigma\zeta_{02})$ on M_{02} and $de_{\bar u}(\sigma\zeta)$ on M_{0211} respectively. In the linear case $\sigma=0$ nothing has happened; in the nonlinear case we can pick $\epsilon>0$ and hence $\|\zeta\|_\infty$ sufficiently small such that $de_u(\zeta)$ is $\mathcal{C}^0$-close to the identity, and hence the induced $L^2_{1,\delta}(\zeta)$-norm is uniformly equivalent to the standard $L^2_{1,\delta}$-norm. With this in mind we start by calculating for any $\zeta,\eta\in H^2_{1,\delta}$ with $\|\zeta\|_\infty\leq\epsilon$ (unless otherwise specified integrals are over two infinite strips of width δ and 1)

$$\begin{aligned}&\|\nabla_s\eta + J(\sigma\zeta)\nabla_t\eta\|^2_{L^2_{1,\delta}(\sigma\zeta)}\\ &= \int\Bigl(|\nabla_s\eta|^2_{\sigma\zeta} + |\nabla_t\eta|^2_{\sigma\zeta} + \langle\nabla_s\eta, J(\sigma\zeta)\nabla_t\eta\rangle_{\sigma\zeta} - \langle\nabla_t\eta, J(\sigma\zeta)\nabla_s\eta\rangle_{\sigma\zeta}\Bigr)\end{aligned}$$

$$
\begin{aligned}
&= \|\nabla\eta\|^2_{L^2_{1,\delta}(\sigma\zeta)} - \int \Big(\nabla_s g_{\sigma\zeta}\big(\eta, J(\sigma\zeta)\nabla_t\eta\big) - \nabla_t g_{\sigma\zeta}\big(\eta, J(\sigma\zeta)\nabla_s\eta\big)\Big) \\
&\quad - \int \Big(\langle \eta, \big(\nabla_s\big(J(\sigma\zeta)\nabla_t\eta\big) - \nabla_t\big(J(\sigma\zeta)\nabla_s\eta\big)\big)\rangle_{\sigma\zeta}\Big) \\
&\quad - \lim_{S\to\infty}\int_{\{s=-S\}} \langle \eta, J(\sigma\zeta)\nabla_t\eta\rangle_{\sigma\zeta} + \lim_{S\to\infty}\int_{\{s=S\}} \langle \eta, J(\sigma\zeta)\nabla_t\eta\rangle_{\sigma\zeta} \\
&\quad + \int_{\{0\}\times\mathbb{R}} \langle \eta, J(\sigma\zeta)\nabla_s\eta\rangle_{\sigma\zeta} - \int_{\{1\}\times\mathbb{R}} \langle \eta_{02}, J_{02}(\sigma\zeta_{02})\nabla_s\eta_{02}\rangle_{\sigma\zeta_{02}} - \int_{\{\delta\}\times\mathbb{R}} \langle \hat\eta, \hat J(\sigma\hat\zeta)\nabla_s\hat\eta\rangle_{\sigma\hat\zeta} \\
&\geq \|\nabla\eta\|^2_{L^2_{1,\delta}(\sigma\zeta)} - \int C\big((1+\sigma c_0)|\eta||\nabla\eta| + |\eta|^2\big) - \Omega_{02}(\eta_{02}|_{t=1}) - \Omega_{0211}(\hat\eta|_{t=\delta}),
\end{aligned}
$$

where we abbreviated

$$
\Omega_{02}(\eta_{02}|_{t=1}) := \left|\int_{\{1\}\times\mathbb{R}} \omega_{02}(\eta_{02}, \nabla_s\eta_{02})\right|, \qquad \Omega_{0211}(\hat\eta|_{t=\delta}) := \left|\int_{\{\delta\}\times\mathbb{R}} \omega_{0211}(\hat\eta, \nabla_s\hat\eta)\right|.
$$

These boundary terms occur on the right hand side of (c) and they will be estimated by (a) to prove (b). The boundary term at $t=0$ vanishes by the diagonal boundary conditions, and the boundary terms at $S\to\pm\infty$ vanish since $\eta|_{\{s\in[S,S+1]\}}\to 0$ in the $H^2_{1,\delta}$-norm. The error term can be estimated by

$$
\int C\big((1+\sigma c_0)|\eta||\nabla\eta| + |\eta|^2\big) \leq C\|\eta\|^2_{L^2_{1,\delta}(\sigma\zeta)} + \frac{1}{2}\|\nabla\eta\|^2_{L^2_{1,\delta}(\sigma\zeta)} + \frac{1}{2}C^2(1+\sigma c_0)^2\|\eta\|^2_{L^2_{1,\delta}(\sigma\zeta)},
$$

where the highest order term $\|\nabla\eta\|$ can be absorbed on the right hand side. From now on C will denote any uniform constant (which is allowed to depend on c_0 in the nonlinear case $\sigma=1$). In summary, the estimates for $\eta=\xi$ and $\eta=\nabla_s\xi$ are

$$
\begin{aligned}
\tfrac{1}{C}\|\nabla\xi\|^2_{L^2_{1,\delta}} &\leq \big\|\nabla_s\xi + J(\sigma\xi)\nabla_t\xi\big\|^2_{L^2_{1,\delta}} + \|\xi\|^2_{L^2_{1,\delta}} + \Omega_{02}(\xi_{02}|_{t=1}) + \Omega_{0211}(\hat\xi|_{t=\delta}), \\
\tfrac{1}{C}\|\nabla\nabla_s\xi\|^2_{L^2_{1,\delta}} &\leq \big\|\nabla_s\big(\nabla_s\xi + J(\sigma\xi)\nabla_t\xi\big)\big\|^2_{L^2_{1,\delta}} + \|\nabla\xi\|^2_{L^2_{1,\delta}} \\
&\qquad + \Omega_{02}(\nabla_s\xi_{02}|_{t=1}) + \Omega_{0211}(\nabla_s\hat\xi|_{t=\delta}).
\end{aligned}
$$

This already proves the first estimate in (b). We can moreover use the identity $\nabla_t\xi = J(\sigma\zeta)\nabla_s\xi - J(\sigma\zeta)(\nabla_s\xi + J(\sigma\zeta)\nabla_t\xi)$ to obtain

$$
\|\nabla\nabla_t\xi\|_{L^2_{1,\delta}} \leq \|\nabla\nabla_s\xi\|_{L^2_{1,\delta}} + \|\nabla(\nabla_s\xi + J(\sigma\zeta)\nabla_t\xi)\|_{L^2_{1,\delta}} + C\|\nabla\xi\|_{L^2_{1,\delta}} + \sigma C c_0\|\nabla\xi\|_{L^2_{1,\delta}}.
$$

In the linear case (c) these estimates combined with (a) and (53) to prove the first estimate:

$$
c_1\|\xi\|_{H^2_{1,\delta}} \leq \big\|D^\delta\xi\big\|_{H^1_{1,\delta}} + \|\xi_{02}|_{t=1}\|_{H^1(\mathbb{R})} + \|\hat\xi|_{t=\delta}\|_{H^1(\mathbb{R})} + \|\xi\|_{L^2_{1,\delta}}
$$

with a uniform constant $c_1>0$. In the nonlinear case (b) we obtain similarly

$$
\begin{aligned}
C_1^{-1}\|\xi\|_{H^2_{1,\delta}} &\leq \big\|\nabla_s\xi + J(\zeta)\nabla_t\xi\big\|_{H^1_{1,\delta}} + \|\xi\|_{L^2_{1,\delta}} + \Omega_{02}(\xi_{02}|_{t=1}) + \Omega_{0211}(\hat\xi|_{t=\delta}) \\
&\qquad + \Omega_{02}(\nabla_s\xi_{02}|_{t=1}) + \Omega_{0211}(\nabla_s\hat\xi|_{t=\delta})
\end{aligned}
$$

with a constant C_1 that depends on $\|\nabla\xi\|_\infty \leq c_0$.

The L^4-estimate for the linear and nonlinear operators will arise by rescaling from the following basic estimate. Here $\hat u : \mathbb{R}\times[0,1]\to M_{0211}$ will be given by $\hat u(s,t) = \bar u(\delta s)$ for any $\delta\in(0,1]$. Then for every $\hat\eta\in H^1(\mathbb{R}\times[0,1], \hat u^*TM_{0211})$

$$
\|\hat\eta\|_{L^4(\mathbb{R}\times[0,1])} \leq C_0\big(\|\hat\eta|_{t=1}\|_{L^2(\mathbb{R})} + \|\nabla\hat\eta\|_{L^2(\mathbb{R}\times[0,1])}\big).
$$

This simply follows from the Sobolev embedding $H^1(\mathbb{R}\times[0,1]) \hookrightarrow L^4(\mathbb{R}\times[0,1])$ and

$$\|\hat\eta\|^2_{L^2(\mathbb{R}\times[0,1])} \le \int_0^1 \left\| \hat\eta(\cdot,1) - \int_t^1 \nabla_t\hat\eta(\cdot,\tau)d\tau \right\|^2_{L^2(\mathbb{R})} dt \le 2\|\hat\eta|_{t=1}\|^2_{L^2(\mathbb{R})} + 2\|\nabla_t\hat\eta\|^2_{L^2(\mathbb{R}\times[0,1])}.$$

When applying this to $\hat\eta(s,t) := \nabla_s\hat\xi(\delta s,\delta t)$ we encounter the following terms:

$$\begin{aligned}
\|\hat\eta\|^2_{L^4(\mathbb{R}\times[0,1])} &= \left(\int_{\mathbb{R}\times[0,1]} |\nabla_s\hat\xi(\delta s,\delta t)|^4 dsdt\right)^{1/2} = \delta^{-1}\|\nabla_s\hat\xi\|^2_{L^4(\mathbb{R}\times[0,\delta])},\\
\|\hat\eta|_{t=1}\|^2_{L^2(\mathbb{R})} &= \int_{\mathbb{R}} |\nabla_s\hat\xi(\delta s,\delta)|^2 ds = \delta^{-1}\|\nabla_s\hat\xi|_{t=\delta}\|^2_{L^2(\mathbb{R})},\\
\|\nabla\hat\eta\|^2_{L^2(\mathbb{R}\times[0,1])} &= \int_{\mathbb{R}\times[0,1]} \delta^2|\nabla\nabla_s\hat\xi(\delta s,\delta t)|^2 dsdt = \|\nabla\nabla_s\hat\xi\|^2_{L^2(\mathbb{R}\times[0,\delta])}.
\end{aligned}$$

Putting this together we find that

$$\|\nabla_s\hat\xi\|_{L^4(\mathbb{R}\times[0,\delta])} \le C_0\big(\|\nabla_s\hat\xi|_{t=\delta}\|_{L^2(\mathbb{R})} + \|\nabla\nabla_s\hat\xi\|_{H^2(\mathbb{R}\times[0,\delta])}\big) \le C_0\big(\|\hat\xi|_{t=\delta}\|_{H^1(\mathbb{R})} + \|\xi\|_{H^2_{1,\delta}}\big),$$

where the estimate for $\|\xi\|_{H^2_{1,\delta}}$ is already established. The L^4-estimate for $\nabla\xi_{02}$ simply follows from the Sobolev embedding $H^1(\mathbb{R}\times[0,1]) \hookrightarrow L^4(\mathbb{R}\times[0,1])$, and for the last component we have

$$\|\nabla_t\hat\xi\|_{L^4(\mathbb{R}\times[0,\delta])} \le \|\nabla_s\hat\xi + \hat J(\sigma\hat\zeta)\nabla_t\hat\xi\|_{L^4(\mathbb{R}\times[0,\delta])} + \|\nabla_s\hat\xi\|_{L^4(\mathbb{R}\times[0,\delta])}.$$

This finishes the proof of the second estimate, where we allow $\|\nabla_s\xi + J(\sigma\zeta)\nabla_t\xi\|_{L^4_{1,\delta}}$ on the right hand side, and the constant in the nonlinear case depends on $\|\nabla\zeta\|_\infty \le c_0$. In the linear case the difference to $\|D^\delta\xi\|_{L^4_{1,\delta}}$ in (53) is bounded by the previously established estimate. □

The lemma below gives control of the lower-order terms appearing in Lemma 5.2.1 and in particular will be used to prove surjectivity of the linearized operator.

Lemma 5.2.2. (a) *There is a constant $\epsilon > 0$ and for every $c_0 > 0$ there is a constant C_2 such that for all $\delta \in (0,1]$ and $\xi,\zeta \in H^2_{1,\delta}$ with $\|\zeta\|_\infty \le \epsilon$, $\|\nabla\zeta\|_\infty \le c_0$ we have*

$$\begin{aligned}
&\|\hat\xi|_{t=\delta}\|_{H^1(\mathbb{R})} + \|\xi_{02}|_{t=1}\|_{H^1(\mathbb{R})}\\
&\le C_2\big(\|\nabla_s\xi_{02} + J_{02}(\zeta_{02})\nabla_t\xi_{02}\|_{H^1(\mathbb{R}\times[0,1])} + \sqrt{\delta}\|\nabla_t\hat\xi\|_{H^1(\mathbb{R}\times[0,\delta])} + \|\pi^\perp_{0211}\hat\xi|_{t=\delta}\|_{H^1(\mathbb{R})}\\
&\qquad + \|\xi_{02}\|_{L^2(\mathbb{R}\times[0,1])} + \|(\xi'_1-\xi_1)|_{t=0}\|_{H^1(\mathbb{R})} + \|(\xi'_{02}-\xi_{02})|_{t=0}\|_{H^1(\mathbb{R})}\big).
\end{aligned}$$

(b) *There is a constant $c_2 > 0$ such that for all $\delta \in (0,1]$ and $\xi \in \Gamma_{1,\delta}$*

$$c_2\big(\|\hat\xi|_{t=\delta}\|_{H^1(\mathbb{R})} + \|\xi_{02}|_{t=1}\|_{H^1(\mathbb{R})} + \|\xi\|_{H^0_{1,\delta}}\big) \le \|D^*_{u_{02}}\xi_{02}\|_{H^1(\mathbb{R}\times[0,1])} + \sqrt{\delta}\|\nabla_t\hat\xi\|_{H^1(\mathbb{R}\times[0,\delta])},$$

and for all $\xi \in \Gamma_{1,\delta}\cap K_0$

$$c_2\big(\|\hat\xi|_{t=\delta}\|_{H^1(\mathbb{R})} + \|\xi_{02}|_{t=1}\|_{H^1(\mathbb{R})} + \|\xi\|_{H^0_{1,\delta}}\big) \le \|D_{u_{02}}\xi_{02}\|_{H^1(\mathbb{R}\times[0,1])} + \sqrt{\delta}\|\nabla_t\hat\xi\|_{H^1(\mathbb{R}\times[0,\delta])}.$$

Proof. The constant $\epsilon > 0$ in case (a) is chosen such that $e_{u_{02}}(\zeta_{02})$ and thus $J_{02}(\zeta_{02})$ is defined. To prove (a) (and similar for (b)) we assume by contradiction that we have sequences $\delta^\nu > 0$ and $\xi^\nu,\zeta^\nu \in H^2_{1,\delta^\nu}$ such that $\|\hat\xi^\nu|_{t=\delta^\nu}\|_{H^1(\mathbb{R})} + \|\xi^\nu_{02}|_{t=1}\|_{H^1(\mathbb{R})} = 1$ (in case (b) add $\|\xi^\nu\|_{H^0_{1,\delta}}$ here), but the right hand sides converges to zero. For technical reasons we assume in addition $\|\xi_{02}^\nu\|_{H^1(\mathbb{R}\times[0,1])}\le 1$, which we will also disprove (i.e. we actually prove a stronger estimate with this term on the left hand side). First we integrate for all $t\in[0,\delta^\nu]$

$$\|\hat\xi^\nu|_{t=t_0}-\hat\xi^\nu|_{t=\delta^\nu}\|_{H^1(\mathbb{R})}\le\int_0^{\delta^\nu}\|\nabla_t\hat\xi^\nu\|_{H^1(\mathbb{R})}\le\sqrt{\delta^\nu}\|\nabla_t\hat\xi^\nu\|_{H^1(\mathbb{R}\times[0,\delta^\nu])}\to 0. \tag{54}$$

Next, Lemma 5.1.3 implies

$$\begin{aligned}\|\pi_{02}^\perp\xi_{02}^\nu|_{t=0}\|_{L^2(\mathbb{R})}&\le\|\pi_{02}^\perp\xi_{02}'^\nu|_{t=\delta^\nu}\|_{L^2(\mathbb{R})}+\|\xi_{02}'^\nu|_{t=0}-\xi_{02}'^\nu|_{t=\delta^\nu}\|_{L^2(\mathbb{R})}+\|(\xi_{02}'^\nu-\xi_{02}^\nu)|_{t=0}\|_{L^2(\mathbb{R})}\\&\le C\big(\|\pi_{0211}^\perp\hat\xi^\nu|_{t=\delta^\nu}\|_{L^2(\mathbb{R})}+\|\hat\xi^\nu|_{t=0}-\hat\xi^\nu|_{t=\delta^\nu}\|_{L^2(\mathbb{R})}\\&\qquad+\|(\xi_1'^\nu-\xi_1^\nu)|_{t=0}\|_{L^2(\mathbb{R})}+\|(\xi_{02}'^\nu-\xi_{02}^\nu)|_{t=0}\|_{L^2(\mathbb{R})}\big)\to 0,\end{aligned}$$

(55)

$$\begin{aligned}\|\pi_{02}^\perp\xi_{02}^\nu|_{t=0}\|_{H^1(\mathbb{R})}&\le\|\pi_{02}^\perp\xi_{02}'^\nu|_{t=\delta^\nu}\|_{H^1(\mathbb{R})}+\|\xi_{02}'^\nu|_{t=0}-\xi_{02}'^\nu|_{t=\delta^\nu}\|_{H^1(\mathbb{R})}+\|(\xi_{02}'^\nu-\xi_{02}^\nu)|_{t=0}\|_{H^1(\mathbb{R})}\\&\le C\big(\|\pi_{0211}^\perp\hat\xi^\nu|_{t=\delta^\nu}\|_{H^1(\mathbb{R})}+\|\hat\xi^\nu|_{t=0}-\hat\xi^\nu|_{t=\delta^\nu}\|_{H^1(\mathbb{R})}+\|(\xi_1'^\nu-\xi_1^\nu)|_{t=0}\|_{H^1}\\&\qquad+\|(\xi_{02}'^\nu-\xi_{02}^\nu)|_{t=0}\|_{H^1(\mathbb{R})}+\big\||\partial_s\bar u|\cdot|\hat\xi^\nu|_{t=\delta^\nu}|\big\|_{L^2(\mathbb{R})}\big).\end{aligned}$$

In the two cases of (b) we use the boundary conditions for $\xi^\nu\in\Gamma_{1,\delta}$ here. In all three cases the hardest step is now to prove that $\big\||\partial_s\bar u|\cdot|\hat\xi^\nu|_{t=\delta^\nu}|\big\|_{L^2(\mathbb{R})}\to 0$. Here we exploit the assumption that $\|\xi_{02}^\nu\|_{H^1(\mathbb{R}\times[0,1])}$ is bounded. This implies a bound on $\|\xi_{02}^\nu|_{t=0}\|_{L^2(\mathbb{R})}$. Now we find a convergent subsequence $\xi_{02}^\nu\to\xi_{02}^\infty\in H^1(\mathbb{R}\times[0,1],u_{02}^*TM_{02})$ in the weak H^1-topology, and at the same time $\xi_{02}^\nu|_{t=0}\to\xi_{02}^\infty|_{t=0}$ in the L^2-norm on every compact set. (The Sobolev embedding $H^1(\Omega)\hookrightarrow L^2(\partial\Omega)$) is compact for compact domains $\Omega\subset\mathbb{R}\times[0,1]$ with smooth boundary $\partial\Omega$, see e.g. [1, Theorem 6.3].) In case (a) the limit has to be $\xi_{02}^\infty=0$ since $\|\xi_{02}^\infty\|_{L^2(\mathbb{R}\times[0,1])}\le\liminf_{\nu\to\infty}\|\xi_{02}^\nu\|_{L^2(\mathbb{R}\times[0,1])}=0$. This also holds in case (b) since the limit satisfies with $D=D_{u_{02}}$ or $D=D_{u_{02}}^*$

$$\begin{aligned}\|D\xi_{02}^\infty\|_{L^2(\mathbb{R}\times[0,1])}&\le\liminf_{\nu\to\infty}\|D\xi_{02}^\nu\|_{L^2(\mathbb{R}\times[0,1])}=0,\\\|\pi_{02}^\perp\xi_{02}^\infty|_{t=0}\|_{L^2(\mathbb{R})}&\le\liminf_{\nu\to\infty}\|\pi_{02}^\perp\xi_{02}^\nu|_{t=0}\|_{L^2(\mathbb{R})}=0.\end{aligned}$$

Since u_{02} is assumed regular, $D_{u_{02}}^*\oplus\pi_{02}^\perp$ is injective, and in the second part of case (b) we have in addition $\xi_{02}^\infty\in\ker(D_{u_{02}}\oplus\pi_{02}^\perp)^\perp$. So in all three cases we obtain

$$\|\xi_{02}^\nu|_{t=0}\|_{L^2(\mathbb{R})}\le C\qquad\text{and}\qquad\|\xi_{02}^\nu|_{t=0}\|_{L^2([-T,T])}\to 0\qquad\text{for all } T>0.$$

The same holds for $\hat\xi^\nu|_{t=\delta^\nu}$ since we can apply Lemma 5.1.3 on the interval $(-T,T)$ for any $T\in(0,\infty]$ to obtain

$$\begin{aligned}\|\hat\xi^\nu|_{t=\delta^\nu}\|_{L^2}&\le C\big(\|\pi_{02}\xi_{02}'^\nu|_{t=\delta^\nu}\|_{L^2}+\|\pi_{0211}^\perp\hat\xi^\nu|_{t=\delta^\nu}\|_{L^2}+\|(\xi_1'^\nu-\xi_1^\nu)|_{t=\delta^\nu}\|_{L^2}\big)\\&\le C'\big(\|\xi_{02}^\nu|_{t=0}\|_{L^2}+\|(\xi_{02}'^\nu-\xi_{02}^\nu)|_{t=0}\|_{L^2}+\|\hat\xi^\nu|_{t=0}-\hat\xi^\nu|_{t=\delta^\nu}\|_{L^2}\\&\qquad+\|\pi_{0211}^\perp\hat\xi^\nu|_{t=\delta^\nu}\|_{L^2}+\|(\xi_1'^\nu-\xi_1^\nu)|_{t=0}\|_{L^2}\big).\end{aligned}$$

This together with the fact that $\sup_{|s|\ge T}|\partial_s\bar u(s)|\to 0$ as $T\to\infty$ implies that $\big\||\partial_s\bar u|\cdot|\hat\xi^\nu|_{t=\delta^\nu}|\big\|_{L^2(\mathbb{R})}\to 0$ and hence $\|\pi_{02}^\perp\xi_{02}^\nu|_{t=0}\|_{H^1(\mathbb{R})}\to 0$ by (55). From this we will move on to prove that

$$\|\xi_{02}^\nu\|_{H^{3/2}(\mathbb{R}\times[0,1])}\to 0. \tag{56}$$

For that purpose we denote by D any of the three operators $\nabla_s+J_{02}(\zeta_{02})\nabla_t$ in case (a) and $D_{u_{02}}^*$ or $D_{u_{02}}$ in case (b). Then we use the fact that in all three cases the operator

$D \oplus \pi_{02}^\perp$ is Fredholm on the space of sections η that satisfy the boundary conditions $\eta|_{t=1} \in T_{u_{02}}(L_0 \times L_2)$, see e.g. [15, Theorem 20.1.2] for compact domains. The corresponding estimates add up to

$$\|\xi_{02}^\nu\|_{H^{3/2}(\mathbb{R}\times[0,1])} \leq C\big(\|D\xi_{02}^\nu\|_{H^1(\mathbb{R}\times[0,1])} + \|\pi_{02}^\perp\xi_{02}^\nu|_{t=0}\|_{H^1(\mathbb{R})} + \|\xi_{02}^\nu\|_{H^0(\mathbb{R}\times[0,1])}\big). \tag{57}$$

In the nonlinear case (a) the constant in this estimate depends continuously on $J_{02}(\zeta_{02})$ in the $\mathcal{C}^1$-topology, see e.g. [26, Appendix B]. In this case the above estimate already implies the claim (56) since we assumed $\|\xi_{02}^\nu\|_{L^2} \to 0$. In the linear cases we need to use the injectivity of the operators to remove the last term from the right hand side of (57). Since $H^{3/2}(\mathbb{R}) \hookrightarrow H^0((-T,T))$ is compact only for $T<\infty$, we first have to achieve a lower order term on a compact domain:

Consider the operator $D_{x^\pm} = \partial_s - A$, where $A := -J(x^\pm)\partial_t$ (or $A := J(x^\pm)\partial_t$ in the case $D = D_{u_{02}}^*$) is self-adjoint and invertible on its constant domain $H^1([0,1], T_{x^\pm}M_{02})$ with boundary conditions $\eta|_{t=0} \in T_{x^\pm}L_{02}$, $\eta|_{t=1} \in T_{x^\pm}(L_0 \times L_2)$. Then abstract theory (e.g. [34, Lemma 3.9, Proposition 3.14]) implies the Fredholm property and bijectivity,

$$\|\eta\|_{H^1(\mathbb{R}\times[0,1])} \leq C\|D_{x^\pm}\eta\|_{H^0(\mathbb{R}\times[0,1])}.$$

In order to apply this estimate to ξ_{02}^ν we first find an extension $\zeta \in H^1(\mathbb{R}\times[0,1])$ of $\zeta|_{t=0} = \pi_{02}^\perp\xi_{02}^\nu|_{t=0}$ such that $\|\zeta\|_{H^1} \leq C\|\pi_{02}^\perp\xi_{02}^\nu|_{t=0}\|_{H^{1/2}}$. We moreover fix a cutoff function $h \in \mathcal{C}_0^\infty(\mathbb{R},[0,1])$ with $h|_{\{|s|\leq T-1\}} \equiv 0$ and $h|_{\{|s|\geq T\}} \equiv 1$, where we fix $T>1$ sufficiently large such that $u_{02}|_{\mathrm{supp}(h)} = e_{x^\pm}(\vartheta_{02})$ for some smooth map $\vartheta_{02} : \{\pm s \geq (T-1)\} \to T_{x^\pm}M_{02}$. Then we can apply the estimate to $\eta := \Phi_{x^\pm}(\vartheta_{02})^{-1}\big(h(\xi_{02}^\nu - \zeta)\big)$, where $\Phi_{x^\pm}(\vartheta_{02})$ denotes parallel transport along the path $[0,1] \ni \tau \mapsto e_{x^\pm}(\tau\vartheta_{02})$. We obtain, denoting all uniform constants by C,

$$\begin{aligned}
&\|h\xi_{02}^\nu\|_{H^1(\mathbb{R}\times[0,1])} \\
&\leq C\|\eta\|_{H^1(\mathbb{R}\times[0,1])} + \|h\zeta\|_{H^1(\mathbb{R}\times[0,1])} \\
&\leq C\big(\|\big(D_{x^\pm} - D\circ\Phi_{x^\pm}(\vartheta_{02})\big)\eta\|_{H^0(\mathbb{R}\times[0,1])} + \|D(h\xi_{02}^\nu)\|_{H^0(\mathbb{R}\times[0,1])} + \|h\zeta\|_{H^1(\mathbb{R}\times[0,1])}\big) \\
&\leq C\big(\|\big(D_{x^\pm} - D\circ\Phi_{x^\pm}(\vartheta_{02})\big)\big|_{\{|s|>T-1\}}\| \cdot \|h(\xi_{02}^\nu - \zeta)\|_{H^1(\mathbb{R}\times[0,1])} + \|D\xi_{02}^\nu\|_{H^0(\mathbb{R}\times[0,1])} \\
&\qquad + \|\xi_{02}^\nu\|_{H^0([-T,T]\times[0,1])} + \|\pi_{02}^\perp\xi_{02}^\nu|_{t=0}\|_{H^{1/2}(\mathbb{R})}\big).
\end{aligned}$$

Here the difference of the operators goes to zero for $T\to\infty$ since $u_{02}|_{\{|s|\geq T-1\}} \to x^\pm$ with all derivatives, see Lemma 3.2.1. Thus for sufficiently large $T>0$ we can absorb the first term into the left hand side and $\|h\zeta\|_{H^1} \leq C\|\pi_{02}^\perp\xi_{02}^\nu|_{t=0}\|_{H^{1/2}}$. After all this we can finally replace the last term in (57) by $\|\xi_{02}^\nu\|_{H^0([-T,T]\times[0,1])}$.

Now in the first case of (b) we can deduce (56) from the fact that $D_{u_{02}}\oplus\pi_{02}^\perp$ is surjective by assumption and hence $D_{u_{02}}^*\oplus\pi_{02}^\perp$ is injective. So the compact embedding $H^{3/2}(\mathbb{R}\times[0,1]) \hookrightarrow H^0([-T,T]\times[0,1])$ allows the removal of the lower order term. Similarly, in the second case of (b) we can employ the injectivity of the operator on $\ker(D_{u_{02}}\oplus\pi_{02}^\perp)^\perp \ni \xi_{02}^\nu$ to deduce (56).

Next, (56) and the Sobolev trace theorem provide $\|\xi^\nu_{02}|_{t=0}\|_{H^1(\mathbb{R})} + \|\xi^\nu_{02}|_{t=1}\|_{H^1(\mathbb{R})} \to 0$, and again using Lemma 5.1.3 we can deduce that

$$
\begin{aligned}
&\|\hat\xi^\nu|_{t=\delta^\nu}\|_{H^1(\mathbb{R})}\\
&\leq C\big(\|\pi_{02}\xi'^\nu_{02}|_{t=\delta^\nu}\|_{H^1(\mathbb{R})} + \|\pi^\perp_{0211}\hat\xi^\nu|_{t=\delta^\nu}\|_{H^1(\mathbb{R})} + \|(\xi'^\nu_1 - \xi^\nu_1)|_{t=\delta^\nu}\|_{H^1(\mathbb{R})}\big)\\
&\leq C\big(\|\xi^\nu_{02}|_{t=0}\|_{H^1(\mathbb{R})} + \|(\xi'^\nu_{02} - \xi^\nu_{02})|_{t=0}\|_{H^1(\mathbb{R})} + \|\pi^\perp_{0211}\hat\xi^\nu|_{t=\delta^\nu}\|_{H^1(\mathbb{R})}\\
&\qquad\qquad + \|\hat\xi^\nu|_{t=0} - \hat\xi^\nu|_{t=\delta^\nu}\|_{H^1(\mathbb{R})} + \|(\xi'^\nu_1 - \xi^\nu_1)|_{t=0}\|_{H^1(\mathbb{R})}\big) \to 0.
\end{aligned}
$$

Finally, combining this with (54) in case (b) implies

$$\|\hat\xi^\nu\|_{L^2(\mathbb{R}\times[0,\delta^\nu])} \to 0$$

in contradiction to the assumption. □

Finally, we establish uniform exponential decay for the solutions of Floer's equation (36) on the triple strip. For that purpose we introduce the following notation for integration over finite strips,

$$\int_{[0,1]\sqcup[0,\delta]} |\partial_s v(s,t)|^2\,dt := \int_0^1 |\partial_s v_{02}(s,t)|^2\,dt + \int_0^\delta |\partial_s \hat v(s,t)|^2\,dt,$$

and similarly for the $\mathcal{C}^0$-norm

$$
\begin{aligned}
\|\partial_s v\|_{\mathcal{C}^0_{1,\delta}([s_0,s_1])} &:= \|\partial_s v_{02}\|_{L^\infty([s_0,s_1]\times[0,1])} + \|\partial_s \hat v\|_{L^\infty([s_0,s_1]\times[0,\delta])},\\
d_{\mathcal{C}^0_{1,\delta}([s_0,s_1])}(v,x^\pm) &:= \sup_{(s,t)\in[s_0,s_1]\times[0,1]} d_{M_{02}}(v_{02}(s,t),x^\pm),\\
&\qquad + \sup_{(s,t)\in[s_0,s_1]\times[0,\delta]} d_{M_{0211}}(\hat v(s,t),(x^\pm,x^\pm_1,x^\pm_1)).
\end{aligned}
$$

Lemma 5.2.3. *There are constants $\hbar,\Delta>0$ and C such that the following holds for every $\delta\in(0,1]$. If $v\in\widehat{\mathcal{M}}_\delta(x^-,x^+)$ is a smooth solution of (36) satisfying*

$$\int_0^\infty \int_{[0,1]\sqcup[0,\delta]} |\partial_s v(s,t)|^2\,dtds \;<\; \hbar, \tag{58}$$

then for every $S\geq 3$

$$d_{\mathcal{C}^0_{1,\delta}([S,\infty))}(v,x^+)^2 + \|\partial_s v\|^2_{\mathcal{C}^0_{1,\delta}([S,\infty))} \leq Ce^{-\Delta S}\int_0^2\int_{[0,1]\sqcup[0,\delta]} |\partial_s v(s,t)|^2\,dtds,$$

and the analogous statement holds on $(-\infty,0]$ for the convergence to x^-.

Proof. **Step 1:** *For every $\kappa>0$ there is an $\epsilon_\kappa>0$ such that the following holds for all $\delta\in(0,1]$. If $v\in\widehat{\mathcal{M}}_\delta(x^-,x^+)$ satisfies (58) with $\hbar=\epsilon_\kappa$, then*

$$\|\partial_s v\|_{\mathcal{C}^0_{1,\delta}([\frac12,\infty))} \leq \kappa. \tag{59}$$

Assume by contradiction that this is wrong. Then there exist $\kappa>0$ and sequences $\delta^\nu\in(0,1]$ and $v^\nu\in\widehat{\mathcal{M}}_{\delta^\nu}(x^-,x^+)$ such that

$$\lim_{\nu\to\infty}\int_0^\infty\int_{[0,1]\sqcup[0,\delta^\nu]} |\partial_s v^\nu(s,t)|^2\,dtds = 0, \tag{60}$$

but the assertion fails. So after a time-shift we can assume that

$$\|\partial_s v^\nu\|_{\mathcal{C}^0_{1,\delta^\nu}([\frac12,1])} > \tfrac12\kappa.$$

The equation $\overline{\partial}_J v^\nu = 0$ together with (60) implies that $dv^\nu|_{s\geq 0} \to 0$ in the L^2-norm. If δ^ν is bounded away from zero, then the standard compactness for holomorphic curves with Lagrangian boundary conditions implies that $dv^\nu|_{s>0} \to 0$ in $\mathcal{C}^\infty$ on every compact set (for a subsequence), in contradiction to the assumption. In the case $\delta^\nu \to 0$ the standard compactness theory still implies $dv^\nu_{02}|_{(0,1]\times(0,\infty)} \to 0$ in $\mathcal{C}^\infty$ on every compact set. For $\hat{v}$ and v_{02} near the boundary $t=0$ we obtain a $\mathcal{C}^1$-bound from Lemma 5.3.2. So we obtain $\mathcal{C}^0$-convergence of a subsequence $v^\nu_{02} \to x_{02}$, $\hat{v}^\nu \to (x_{02}, x_1, x_1)$ to constants $x_{02} \in L_0 \times L_2$, $x_1 \in M_1$ such that $(x_{02}, x_1, x_1) \in L_{01} \times L_{12}$. Now we can use the same compactness arguments as in the proof of Lemma 5.3.2 (step 2, using a cutoff function only in s) to deduce that $dv^\nu|_{s\in[\frac{1}{2},1]} \to 0$ in the $\mathcal{C}^0$-norm. This again is a contradiction.

Step 2: *There are constants $\epsilon_1 > 0$ and C_1 such that the following holds for all $\delta \in (0,1]$. If $v \in \widehat{\mathcal{M}}_\delta(x^-, x^+)$ satisfies (58) with $\hbar = \epsilon_1$, then*

$$\|\partial_s v(1,\cdot)\|^2_{\mathcal{C}^0([0,1]\sqcup[0,\delta])} \leq C_1 \int_{[0,1]\sqcup[0,\delta]} |\nabla_t \partial_s v(1,t)|^2 \, dt.$$

By contradiction we find sequences $\delta^\nu \in (0,1]$ and $v^\nu \in \widehat{\mathcal{M}}_{\delta^\nu}(x^-, x^+)$ that satisfy (60), but there is no uniform constant C_1 with which the estimate holds. Then as in Step 1 we obtain (for a subsequence) $\mathcal{C}^1$-convergence $v^\nu \to x = (x_{02}, \hat{x})$ on $[\frac{1}{2}, 2] \times ([0,1] \sqcup [0,\delta^\nu])$ to constants $x_{02} \in L_0 \times L_2$, $x_1 \in M_1$ with $\hat{x} = (x_{02}, x_1, x_1) \in L_{01} \times L_{12}$. By assumption L_{02} and $(L_0 \times L_2)$ intersect transversely in x_{02}, and hence we have for all $\xi_{02} : [0,1] \to T_{x_{02}} M_{02}$ with $\xi_{02}(1) \in T_{x_{02}}(L_0 \times L_2)$

$$\|\xi_{02}\|_{\mathcal{C}^0([0,1])} \leq C\big(\|\nabla_t \xi_{02}\|_{L^2([0,1])} + \big|\pi^\perp_{02}\xi_{02}(0)\big|\big).$$

Now consider in addition $\hat{\xi} : [0,\delta] \to T_{\hat{x}} M_{0211}$ such that $\hat{\xi}(\delta) \in T_{\hat{x}}(L_{01} \times L_{12})$ and $\xi|_{t=0} = (\xi_{02}, \hat{\xi})|_{t=0} \in T_x(\Delta_{M_0 \times M_2} \times \Delta_1)$. We integrate for all $t \in [0,\delta]$

$$\big|\hat{\xi}(t) - \hat{\xi}(\delta)\big| \leq \int_0^\delta |\nabla_t \hat{\xi}(t)| dt \leq \sqrt{\delta} \bigg(\int_0^\delta |\nabla_t \hat{\xi}(t)|^2 dt \bigg)^{1/2}. \tag{61}$$

Combining this with Lemma 5.1.3 and using the boundary conditions we obtain

$$\big|\pi^\perp_{02}\xi_{02}(0)\big| \leq \big|\pi^\perp_{0211}\hat{\xi}(\delta)\big| + \big|\pi^\perp_{0211}\big(\hat{\xi}(0) - \hat{\xi}(\delta)\big)\big| + \big|\xi'_1(0) - \xi_1(0)\big| \leq C\sqrt{\delta}\bigg(\int_0^\delta |\nabla_t \hat{\xi}(t)|^2 dt\bigg)^{1/2},$$

and thus

$$\|\xi_{02}\|^2_{\mathcal{C}^0([0,1])} \leq C^2 \int_{[0,1]\sqcup[0,\delta]} |\nabla_t \xi|^2 \, dt.$$

We moreover obtain from Lemma 5.1.3 with uniform constants C, C', C''

$$\begin{aligned}\big|\hat{\xi}(\delta)\big| &\leq C\big(\big|\pi_{02}\xi'_{02}(\delta)\big| + \big|(\xi'_1(\delta) - \xi_1(\delta))\big|\big) \\ &\leq C'\big(\big|\xi_{02}(0)\big| + \big|\hat{\xi}(0) - \hat{\xi}(\delta)\big|\big) \;\leq\; C''\bigg(\int_{[0,1]\sqcup[0,\delta]} \big|\nabla_t(\xi_{02}, \hat{\xi})\big|^2 dt\bigg)^{1/2}.\end{aligned}$$

Together with (61) this implies

$$\|\xi\|^2_{\mathcal{C}^0([0,1]\sqcup[0,\delta])} \leq C_1 \int_{[0,1]\sqcup[0,\delta]} |\nabla_t \xi|^2 \, dt$$

with some uniform constant C_1 for all $\delta \in (0,1]$ and all sections ξ over x satisfying the boundary conditions. Due to the $\mathcal{C}^1$-convergence $v^\nu \to x$ this estimate continues to hold with a uniform constant for sufficiently large ν for sections $\xi_{02} \in \mathcal{C}^1([0,1], v_{02}^\nu|_{s=1}{}^* TM_{02})$, $\hat{\xi} \in \mathcal{C}^1([0,\delta^\nu], \hat{v}|_{s=1}^* TM_{0211})$ that satisfy the analogous boundary conditions. (We can write $v^\nu|_{s=1} = e_x(\zeta^\nu)$ with $\|\zeta^\nu\|_{\mathcal{C}^1} \to 0$ and use $de_x(\zeta^\nu)^{-1}$ to map $(\xi_{02}, \hat{\xi})$ to a section over x. This preserves the boundary conditions by construction of e.) In particular, we can apply this new estimate to $\xi = \partial_s v^\nu|_{s=1}$, which provides a uniform estimate and thus finishes the proof by contradiction.

Step 3: *There are uniform constants $\epsilon_2, \Delta > 0$ and C_2 such that the following holds for all for all $\delta \in (0,1]$. If $v \in \widehat{\mathcal{M}}_\delta(x^-, x^+)$ satisfies (58) with $\hbar = \epsilon_2$, then for all $s_0 \geq 2$*

$$\int_{[0,1]\sqcup[0,\delta]} |\partial_s v(s_0,t)|^2 dt \leq C_2 e^{-\Delta s_0} \int_1^2 \int_{[0,1]\sqcup[0,\delta]} |\partial_s v(s,t)|^2 dt ds.$$

Consider the function $f : [1,\infty) \to [0,\infty)$ defined by

$$f(s) := \tfrac{1}{2} \int_{[0,1]\sqcup[0,\delta]} |\partial_s v(s,t)|^2 dt.$$

We can use the equation $\overline{\partial}_J v = \big(\partial_s v_{02} + J_{02}(v_{02})\partial_t v_{02}\, ,\, \partial_s \hat{v} + \hat{J}(\hat{v})\partial_t \hat{v}\big) = 0$ and the bound $\|\partial_s v\|_\infty \leq \kappa$ from Step 1 to calculate for $s \geq 1$

$$\begin{aligned}
f''(s) &= \int_{[0,1]\sqcup[0,\delta]} \Big(|\nabla_s \partial_s v|^2 + \langle \partial_s v\, ,\, \nabla_s^2 \partial_s v\rangle \Big) \\
&= \int_{[0,1]\sqcup[0,\delta]} \Big(|J\nabla_t \partial_s v + (\nabla_{\partial_s v} J)\partial_t v|^2 - \langle \partial_s v\, ,\, J\nabla_t \nabla_s \partial_s v\rangle \Big) \\
&\quad - \int_{[0,1]\sqcup[0,\delta]} \Big(\langle \partial_s v\, ,\, JR(\partial_s v, \partial_t v)\partial_s v + 2(\nabla_{\partial_s v} J)\nabla_s \partial_t v + \nabla_s (\nabla_{\partial_s v} J)\partial_t v\rangle \Big) \\
&\geq \int_{[0,1]\sqcup[0,\delta]} \Big(2|J\nabla_t \partial_s v|^2 + \partial_t \big(\omega(\partial_s v, \nabla_s \partial_s v)\big) - C|\partial_s v|^2 \big(|\partial_s v|^2 + |\nabla_t \partial_s v|\big) \Big) \\
&\geq \big(2 - C\kappa\big) \int_{[0,1]\sqcup[0,\delta]} |J\nabla_t \partial_s v(s,t)|^2 dt \; - C'\big(\kappa + \kappa^2\big) \|\partial_s v(s,\cdot)\|^2_{\mathcal{C}^0([0,1]\sqcup[0,\delta])}.
\end{aligned}$$

The last step uses $2|\partial_s v|^2 |\nabla_t \partial_s v| \leq \kappa |\partial_s v|^2 + \kappa |\nabla_t \partial_s v|^2$ and the claim

$$\left| \int_{[0,1]\sqcup[0,\delta]} \partial_t \big(\omega(\partial_s v, \nabla_s \partial_s v)\big) \right| \leq C\big(|\partial_s v_{02}(1)|^3 + |\partial_s \hat{v}(\delta)|^3\big).$$

To prove the claim we first use the diagonal boundary conditions to obtain

$$\left| \int_{[0,1]\sqcup[0,\delta]} \partial_t \big(\omega(\partial_s v, \nabla_s \partial_s v)\big) \right| = \big|\omega_{02}(\partial_s v_{02}, \nabla_s \partial_s v_{02})|_{t=1} + \omega_{02}(\partial_s \hat{v}, \nabla_s \partial_s \hat{v})|_{t=\delta}\big|.$$

Then we use a smooth family of orthonormal frames $(\gamma_i)_{i=1,\ldots,k} \in \Gamma(T(L_0 \times L_2))$ near $w(s) := v_{02}(s,1)$ (and similarly for $\hat{v}$),

$$\partial_s w(s) = \sum \lambda^i(s)\gamma_i(w(s)), \quad \nabla_s \partial_s w(s) = \sum \Big(\partial_s \lambda^i(s)\gamma_i(w(s)) + \lambda^i(s)\nabla_{\partial_s w(s)}\gamma_i \Big)$$

with $\lambda : \mathbb{R} \to \mathbb{R}^k$. By the orthonormality we have $|\lambda(s)| = |\partial_s w(s)|$, and using the identities $\omega(\gamma_i, \gamma_j) = 0$ one obtains $\big|\omega(\partial_s w, \nabla_s \partial_s w)\big| \leq C|\partial_s w|^3$, where the constant C only depends on $\nabla \gamma_i$. Since L is compact this holds with a uniform constant.

We can now choose $\kappa > 0$ sufficiently small and then fix $\hbar \le \min\{\epsilon_1, \epsilon_\kappa\}$ such that Step 1 and Step 2 (applied to time-shifts of v) together with the above calculation yield for all $s \ge 1$

$$f''(s) \ge \int_{[0,1]\sqcup[0,\delta]} |J\nabla_t \partial_s v(s,t)|^2 dt \ge ((1+\delta)C_1)^{-1} \int_{[0,1]\sqcup[0,\delta]} |\partial_s v(s,t)|^2 dt \ge \Delta^2 f(s)$$

with $\Delta > 0$. Any such nonnegative convex function satisfies for all $s \ge 2$ and $T \ge s$

$$f(s) \le Ce^{-\Delta s}\Big(\int_{[1,2]} f(t)dt + \int_{[2T,2T+1]} f(t)dt \Big)$$

with a constant C that only depends on Δ. A detailed proof can be found in e.g. [35, Lemma 3.7] (use the estimate for $\hat{f}(s-T-1)$, where the function $\hat{f}$ is shifted by $T+1$). If we let $T \to \infty$ then $\int_{[2T,2T+1]} f(t)dt \to 0$ by the finite energy condition $\int_0^\infty f(s)ds < \hbar$, and this proves the claim.

Step 4: *There are constants $\epsilon_3 > 0$ and C_3 such that the following holds for all $\delta \in (0,1]$. If $v \in \widehat{\mathcal{M}}_\delta(x^-,x^+)$ satisfies (58) with $\hbar = \epsilon_3$, then*

$$\|\partial_s v\|_{\mathcal{C}^0_{1,\delta}([1,2])} \le C_3 \, \|\partial_s v\|_{L^2_{1,\delta}([\frac12,\frac52])} .$$

By contradiction we find sequences $\delta^\nu \in (0,1]$ and $v^\nu \in \widehat{\mathcal{M}}_{\delta^\nu}(x^-,x^+)$ that satisfy (60), but the assertion fails, i.e. we cannot find a constant C_3 for which the estimate is satisfied. Then as in Step 1 we obtain (for a subsequence) $\mathcal{C}^1$-convergence $v^\nu \to x = (x_{02}, \hat{x})$ on $[\frac12,\frac52]\times([0,1]\sqcup[0,\delta^\nu])$ to constants $x_{02} \in L_0 \times L_2$, $x_1 \in M_1$ with $\hat{x} = (x_{02}, x_1, x_1) \in L_{01} \times L_{12}$. So we can find sections $\xi^\nu \in \Gamma_{1,\delta^\nu}$ over $u = x$ such that $v^\nu|_{s\in[\frac12,\frac52]} = e_x(\xi^\nu)$. The equation $\overline{\partial}_J v^\nu$ then becomes

$$\nabla_s \xi^\nu + J(\xi^\nu)\nabla_t \xi^\nu = 0$$

and we have the boundary conditions $\nabla_s \xi^\nu_{02}|_{t=1} \in T_{x_{02}}(L_0 \times L_2)$ and $\nabla_s \hat{\xi}^\nu|_{t=\delta^\nu} \in T_{\hat{x}}(L_{01} \times L_{12})$. We fix two cutoff functions $h, \tilde{h} \in \mathcal{C}^\infty(\mathbb{R},[0,1])$ with $h|_{[1,2]} \equiv 1$, $\tilde{h}|_{\operatorname{supp} h} \equiv 1$ and $\operatorname{supp}(h), \operatorname{supp}(\tilde{h}) \subset (\frac12,\frac52)$ and consider the sections $h\xi^\nu, \tilde{h}\xi^\nu \in \Gamma_{1,\delta^\nu}$. Note that $\partial_s v^\nu = de_x(\xi^\nu)\nabla_s\xi^\nu$ with $de_x(\xi^\nu) \approx \mathrm{Id}$. So for sufficiently large ν we have

$$\begin{aligned}
\|\partial_s v^\nu\|_{\mathcal{C}^0_{1,\delta^\nu}([1,2])} &\le 2\|h\nabla_s\xi^\nu\|_{\mathcal{C}^0_{1,\delta^\nu}} \le 2C_S\|h\nabla_s\xi^\nu\|_{H^2_{1,\delta^\nu}}, \\
\|\nabla_s\xi^\nu\|_{L^2_{1,\delta^\nu}([\frac12,\frac52])} &\le 2\|\partial_s v^\nu\|_{L^2_{1,\delta^\nu}([\frac12,\frac52])},
\end{aligned}$$

where we used Lemma 5.1.4. Now we apply Lemma 5.2.1 (b) to the sections $\xi = h\nabla_s\xi^\nu$ and $\xi = \tilde{h}\nabla_s\xi^\nu$ (for which the boundary terms vanish since $\nabla_s\xi^\nu, \nabla_s^2\xi^\nu, \nabla_s^3\xi^\nu$ satisfy the boundary conditions) and $\zeta = \xi^\nu$ (which satisfy $\|\xi^\nu\|_\infty \to 0$ and $\|\nabla\xi^\nu\|_\infty \to 0$) to obtain with uniform constants C, C'

$$\begin{aligned}
\|h\nabla_s\xi^\nu\|_{H^2_{1,\delta^\nu}} &\le C_1\big(\|\big(\nabla_s + J(\xi^\nu)\nabla_t\big)h\nabla_s\xi^\nu\|_{H^1_{1,\delta^\nu}} + \|h\nabla_s\xi^\nu\|_{H^0_{1,\delta^\nu}}\big) \\
&= C_1\big(\|h'\nabla_s\xi^\nu\|_{H^1_{1,\delta^\nu}} + \|h\nabla_s\xi^\nu\|_{H^0_{1,\delta^\nu}}\big) \\
&\le C\|\nabla_s\xi^\nu\|_{H^1_{1,\delta^\nu}(\operatorname{supp} h)} \ \le \ C\|\tilde{h}\nabla_s\xi^\nu\|_{H^1_{1,\delta^\nu}} \\
&\le CC_1\big(\|\big(\nabla_s + J(\xi^\nu)\nabla_t\big)\tilde{h}\nabla_s\xi^\nu\|_{H^0_{1,\delta^\nu}} + \|\tilde{h}\nabla_s\xi^\nu\|_{H^0_{1,\delta^\nu}}\big) \\
&\le C'\|\nabla_s\xi^\nu\|_{H^0_{1,\delta^\nu}([\frac12,\frac52]}.
\end{aligned}$$

Now the contradiction follows,

$$\|\partial_s v^\nu\|_{\mathcal{C}^0_{1,\delta^\nu}([1,2])} \leq 2\|h\nabla_s\xi^\nu\|_{H^2_{1,\delta^\nu}} \leq 2C'\|\nabla_s\xi^\nu\|_{H^0_{1,\delta^\nu}([\frac{1}{2},\frac{5}{2}])} \leq 4C'\|\partial_s v^\nu\|_{L^2_{1,\delta^\nu}([\frac{1}{2},\frac{5}{2}])}.$$

Step 5: *We prove the claim, that is for every $s \geq 3$*

$$d_{\mathcal{C}^0([0,1]\sqcup[0,\delta])}(v(s,\cdot),x^+)^2 + \|\partial_s v(s,\cdot)\|^2_{\mathcal{C}^0([0,1]\sqcup[0,\delta])} \leq Ce^{-\Delta s}E(v)$$

with

$$E(v) := \int_0^2 \int_{[0,1]\sqcup[0,\delta]} |\partial_s v(s,t)|^2 dtds.$$

We choose $\hbar = \min\{\epsilon_2, \epsilon_3\}$, then Step 3 and Step 4 (applied to appropriately shifted solutions) combine as follows for all $s \geq 3$

$$\begin{aligned}\|\partial_s v\|^2_{\mathcal{C}^0_{1,\delta}([s-\frac{1}{2},s+\frac{1}{2}])} &\leq C_3^2 \int_{s-1}^{s+1} \int_{[0,1]\sqcup[0,\delta]} |\partial_s v(s,t)|^2 dt \\ &\leq C_3^2 C_2 \int_{s-1}^{s+1} e^{-\Delta s}E(v)ds \;\leq\; C_3^2 C_2 \Delta^{-1} e^{\Delta} e^{-\Delta s}E(v).\end{aligned}$$

This proves the second part of the claim. The estimate on $d_{\mathcal{C}^0([0,1]\sqcup[0,\delta])}(v(S,\cdot),x^+)$ now simply follows by integration: For all $S \geq 3$ and $t \in [0,1]$

$$\begin{aligned} d_{M_{02}}(v_{02}(S,t),x^+) &\leq \int_S^\infty |\partial_s v_{02}(s,t)|ds \\ &\leq C\int_S^\infty e^{-\Delta s/2}\sqrt{E(v)}ds \\ &= 2C\Delta^{-1}e^{-\Delta S/2}\sqrt{E(v)}, \end{aligned}$$

and similarly for $\hat{v}$. □

5.3. **Compactness.** The surjectivity of the map $\mathcal{T}_\delta : \mathcal{M}^1_0(x^-,x^+) \to \mathcal{M}^1_\delta(x^-,x^+)$, as introduced in the previous section, will be a direct consequence of the following compactness result. Here we choose $\epsilon_0 \in (0,\epsilon]$ with $\epsilon > 0$ from in Theorem 5.1.1. Then $v = e_u(\xi)$ with $\xi \in \Gamma_{1,\delta}(\epsilon_0) \cap K_0$ implies that $[v_u] = \mathcal{T}_\delta([u])$ by the definition of $\mathcal{T}_\delta$ via theorem 5.1.1. We will denote the time-shift by $\tau^\sigma v(s,t) := v(\sigma+s,t)$.

Theorem 5.3.1. *Given $\epsilon_0 > 0$ there exists $\delta_0 > 0$ such that for every $\delta \in (0,\delta_0]$ and $v \in \widehat{\mathcal{M}}^1_\delta(x^-,x^+)$ there exist $u \in \widetilde{\mathcal{M}}^1_0(x^-,x^+)$ and $\sigma \in \mathbb{R}$ such that $\tau^\sigma v = e_u(\xi)$ with $\xi \in \Gamma_{1,\delta}\cap K_0$ and $\|\xi\|_{\Gamma_{1,\delta}} \leq \epsilon_0$. Moreover, the moduli space $\widehat{\mathcal{M}}^1_\delta(x^-,x^+)$ is regular for all $\delta \in (0,\delta_0]$ in the sense that the linearized operator D_v is surjective for every $v \in \widehat{\mathcal{M}}^1_\delta(x^-,x^+)$.*

Proof. We assume by contradiction that there is an $\epsilon_0 > 0$, a sequence $\delta^\nu \to 0$, and solutions $v^\nu = (v^\nu_{02}, \hat{v}^\nu) \in \widehat{\mathcal{M}}^1_{\delta^\nu}(x^-,x^+)$ for which the assertion of the theorem fails. The energy $A(v^\nu) = A < \infty$ is fixed by the energy-index relation in Remark 4.2.3 (applied to the corresponding triple strip solutions in $\widetilde{\mathcal{M}}^1_{\frac{1}{\delta^\nu}}(x^-,x^+)$). We can exclude bubbling by the following argument based on Lemma 5.3.2 below:

If $|dv^\nu_{02}|$ is unbounded near a point $z \in \mathbb{R}\times(0,1]$, then the standard rescaling method gives rise to a nontrivial holomorphic sphere or disk in (M_0,L_0), or in (M_2,L_2), or in both. Thus some fixed amount of energy $\hbar > 0$ would have to concentrate near z. The same energy quantization holds for blowup of $d\hat{v}$ or $dv_{02}|_{t=0}$ by Lemma 5.3.2. So the energy densities $|dv^\nu|$ can only blow up at finitely many points. On the complement the same compactness proof as in the next paragraph provides a $\mathcal{C}^0_{\rm loc}$ convergent subsequence $v_{02}^\nu \to u_{02}$, where the limit corresponds to a solution $u = (u_0, u_2) \in \widetilde{\mathcal{M}}_0(x^-, x^+)$ with finitely many singularities and energy $A(u) < A(v^\nu)$. These can be removed by the standard proofs for pseudoholomorphic curves with Lagrangian boundary condition [26, Theorem 4.1.2], so we would obtain a solution $\tilde{u} \in \widetilde{\mathcal{M}}_0(x^-, x^+)$ of energy $A(\tilde{u}) < A(v^\nu)$. Its index $I(\tilde{u}) < I(v^\nu) = 1$ would be negative due to the energy-index relation and the fact that the index can only change by multiples of 2, see Remark 4.2.3. This poses a contradiction to the assumption of regularity for the moduli space $\widetilde{\mathcal{M}}_0(x^-, x^+)$.

So we excluded bubbling and thus from now on assume that $|dv^\nu| \le C_0$ is uniformly bounded. Then we have $d_{\mathcal{C}^0}\big(v_{02}^\nu|_{t=\delta^\nu}, L_{02}\big) \to 0$ since as in Lemma 5.1.3 it is bounded by $d_{\mathcal{C}^0}\big(v_1'^\nu|_{t=\delta^\nu}, v_1^\nu|_{t=\delta^\nu}\big) \le d_{\mathcal{C}^0}\big(\hat{v}^\nu|_{t=\delta^\nu}, \hat{v}^\nu|_{t=0}\big) \le C_0\delta^\nu$. So we can fix $p > 2$ and find a subsequence and map $u_{02} \in \mathcal{C}^0 \cap W^{1,p}_{\rm loc}(\mathbb{R} \times [0,1], M_0 \times M_2)$ such that $v_{02}^\nu \to u_{02}$ in the $\mathcal{C}^0$-topology and the weak $W^{1,p}$-topology on every compact subset of $\mathbb{R} \times [0,1]$. The limit u_{02} corresponds to a solution $(u_0, u_2) \in \widetilde{\mathcal{M}}^1_0(x^-, x^+)$. We also conclude that $\hat{v} \to \bar{u} = (u_{02}|_{t=0}, \bar{u}_1, \bar{u}_1)$ in $\mathcal{C}^0([-T,T] \times [0,\delta^\nu])$ for all $T > 0$, where $\bar{u}_1$ is determined uniquely by $(u_{02}|_{t=0}, \bar{u}_1, \bar{u}_1) \in L_{01} \times L_{12}$. Indeed, $\hat{v}^\nu|_{t=0} = (v_{02}^\nu, v_1^\nu, v_1^\nu)|_{t=0}$ satisfies $d_{\mathcal{C}^0}(\hat{v}^\nu|_{t=0}, u_{02} \times \Delta_1) \to 0$ as well as $d_{\mathcal{C}^0}(\hat{v}^\nu|_{t=0}, L_{01} \times L_{12}) \le d_{\mathcal{C}^0}(\hat{v}^\nu|_{t=0}, \hat{v}^\nu|_{t=\delta^\nu}) \to 0$, so $v_1|_{t=0}$ must converge to $\bar{u}_1$ on compact sets, and the convergence for $t_0 \in [0,\delta^\nu]$ follows from $d_{\mathcal{C}^0}(\hat{v}^\nu|_{t=0}, \hat{v}^\nu|_{t=t_0}) \le C_0\delta^\nu \to 0$. In summary we have $v^\nu \to u := (u_{02}, \bar{u})$ in the $\mathcal{C}^0$-topology on every set $\{|s| \le T\}$ for fixed T. In the following, we will strengthen this convergence using uniform nonlinear estimates and exponential decay, to find sections $\xi^\nu \in \Gamma_{1,\delta^\nu}(\epsilon_0)$ such that $v^\nu = e_u(\xi^\nu)$ and D_{v^ν} is surjective in contradiction to the assumption.

Let us first note that the limit u here cannot be a broken trajectory between critical points $x^-, y_1, \ldots, y_\ell, x^+$ since each of the trajectories would lie in a regular moduli space of index at least 1. Their total energy would then exceed that of v^ν due to the energy-index relation in Remark 4.2.3. We also cannot have $A(u) < A(v^\nu)$ since, by the same relation, that would lead to a negative index of u. Hence the energies $A(u) = A(v^\nu)$ and thus the indices $I(u) = I(v^\nu) = 1$ agree. In the next step we strengthen the local convergence:

For fixed $T > 0$ and sufficiently large $\nu \ge \nu_0$ we can write $v^\nu|_{\{|s| \le T\}} = e_u(\xi^\nu)$ with a section $\xi^\nu \in \Gamma_{1,\delta^\nu}$ (extended smoothly to $\{|s| > T\}$). The extension of ξ^ν can be chosen such that $\|\xi^\nu\|_\infty \to 0$ and $\sup_\nu \|\nabla\xi^\nu\|_\infty < \infty$ follows from the $\mathcal{C}^0$-convergence and $\mathcal{C}^1$-boundedness of $v^\nu|_{\{|s| \le T\}}$. For the latter note that $\nabla\xi^\nu = de_u(\xi^\nu)^{-1}\nabla v^\nu - \partial_1 e(u, \xi^\nu)\nabla u$, where ∇v^ν is uniformly bounded, and $de_u(\xi^\nu) \to \mathrm{Id}$ as $|\xi^\nu| \to 0$. This puts us into the position where Lemma 5.2.1 applies with $\zeta = \xi^\nu$. We fix a cutoff function $h \in \mathcal{C}^\infty_0([-T,T],[0,1])$ with $h|_{[-T+1,T-1]} \equiv 1$, then

$$\begin{aligned}\|h\xi^\nu\|_{H^1_{1,\delta}} \le C_1\Big(&\|\big(\nabla_s + J(\xi^\nu)\nabla_t\big)h\xi^\nu\|_{H^0_{1,\delta^\nu}} + \|h\xi^\nu\|_{H^0_{1,\delta^\nu}} \\ &+ \|h\hat{\xi}^\nu|_{t=\delta^\nu}\|_{H^0(\mathbb{R})} + \|h\xi_{02}^\nu|_{t=1}\|_{H^0(\mathbb{R})}\Big).\end{aligned}$$

Now we can use (52), $\overline{\partial}_J v^\nu = 0$, $\overline{\partial}_{J_{02}} u_{02} = 0$, and $\partial_t \bar{u} = 0$ to obtain

$$\begin{aligned}\big\|h\big(\nabla_s + \hat{J}(\hat{\xi}^\nu)\nabla_t\big)\hat{\xi}^\nu\big\|_{L^2(\mathbb{R}\times[0,\delta^\nu])} &= \big\|h \cdot de_{\bar{u}}(\hat{\xi}^\nu)^{-1}\big(\partial_1 e(\bar{u}, \hat{\xi}^\nu)\partial_s\bar{u}\big)\big\|_{L^2([-T,T]\times[0,\delta^\nu])} \\ &\le C\|\partial_s\bar{u}\|_{L^2([-T,T]\times[0,\delta^\nu])} \le C\sqrt{\delta^\nu}\|\partial_s\bar{u}\|_{L^2([-T,T])},\end{aligned}$$

and furthermore, using the fact that $\partial_1 e(u_{02}, 0) = \mathrm{Id}$ commutes with $J(u_{02})$,

$$\begin{aligned}&\big\|h\big(\nabla_s + J_{02}(\xi_{02}^\nu)\nabla_t\big)\xi_{02}^\nu\big\|_{L^2(\mathbb{R}\times[0,1])}\\ &= \big\|h\cdot de_{u_{02}}(\xi_{02}^\nu)^{-1}\big(\partial_1 e(u_{02},\xi_{02}^\nu)J(u_{02})\partial_t u_{02} - J_{02}(u_{02})\partial_1 e(u_{02},\xi_{02}^\nu)\partial_t u_{02}\big)\big\|_{L^2(\mathbb{R}\times[0,1])}\\ &\le C\|\xi_{02}^\nu\|_{L^2([-T,T]\times[0,1])}.\end{aligned}$$

Hence we have

$$\|\xi^\nu\|_{H^1_{1,\delta^\nu}(\{|s|\le T-1\})} \le C\Big(\sqrt{\delta^\nu} + \|\xi^\nu\|_{H^0_{1,\delta^\nu}(\{|s|\le T\})} + \|h\hat{\xi}^\nu|_{t=\delta^\nu}\|_{H^0(\mathbb{R})} + \|h\xi_{02}^\nu|_{t=1}\|_{H^0(\mathbb{R})}\Big),$$

which converges to zero, and thus $v_{02}^\nu \to u_{02}$ in the H^1-norm on every compact set. Now we can verify the assumptions of Lemma 5.2.3 (with the constant $\hbar > 0$) and achieve uniform exponential decay: Pick $T > 0$ such that $\int_{[-T,T]\times[0,1]} |\partial_s u_{02}|^2 \ge A(u) - \frac{1}{2}\hbar$ and pick ν_0 such that for all $\nu \ge \nu_0$ we have $\|\partial_s u_{02}\|^2_{L^2([-T,T]\times[0,1])} - \|\partial_s v_{02}^\nu\|^2_{L^2([-T,T]\times[0,1])} \le \frac{1}{2}\hbar$ and thus

$$\int_{\{|s|>T\}}\Big(\int_{[0,1]}|\partial_s v_{02}^\nu|^2 + \int_{[0,\delta^\nu]}|\partial_s \hat{v}|^2\Big) \;\le\; A(v^\nu) + \tfrac{1}{2}\hbar - A(u) + \tfrac{1}{2}\hbar \;=\; \hbar.$$

Now the exponential decay Lemma 5.2.3 combined with the local $\mathcal{C}^0$-convergence implies that

$$d_{\mathcal{C}^0}(v_{02}^\nu, u_{02}) + d_{\mathcal{C}^0}(\hat{v}^\nu, \bar{u}) \to 0$$

uniformly for all s,t. Thus for sufficiently large ν we can write $v^\nu = e_u(\xi^\nu)$ with $\xi^\nu \in H^2_{1,\delta^\nu}$ and $\|\xi^\nu\|_\infty \to 0$. In fact, the uniform exponential decay implies global convergence,

$$\|\xi^\nu\|_\infty \to 0, \qquad \|\xi^\nu\|_{L^p_{1,\delta}} \to 0 \quad \forall p \ge 1, \qquad \|\nabla\xi^\nu\|_\infty \le c_0 < \infty.$$

This puts us into the position where Lemma 5.2.1 and 5.2.2 apply with $\zeta = \xi^\nu$,

$$\begin{aligned}&\|\xi^\nu\|_{H^2_{1,\delta^\nu}} + \|\nabla\xi^\nu\|_{L^4_{1,\delta^\nu}}\\ &\le C_1\Big(\|\nabla_s\xi^\nu + J(\xi^\nu)\nabla_t\xi^\nu\|_{H^1_{1,\delta^\nu}} + \|\nabla_s\xi^\nu + J(\xi^\nu)\nabla_t\xi^\nu\|_{L^4_{1,\delta^\nu}}\\ &\qquad\qquad + \|\xi^\nu\|_{H^0_{1,\delta^\nu}} + \|\hat{\xi}^\nu|_{t=\delta^\nu}\|_{H^1(\mathbb{R})} + \|\xi_{02}^\nu|_{t=1}\|_{H^1(\mathbb{R})}\Big)\\ &\le C_1(1+C_2)\Big(\|\nabla_s\xi^\nu + J(\xi^\nu)\nabla_t\xi^\nu\|_{H^1_{1,\delta^\nu}} + \|\nabla_s\xi^\nu + J(\xi^\nu)\nabla_t\xi^\nu\|_{L^4_{1,\delta^\nu}}\\ &\qquad\qquad\qquad + \|\xi^\nu\|_{H^0_{1,\delta^\nu}} + \sqrt{\delta^\nu}\|\nabla_t\hat{\xi}^\nu\|_{H^1(\mathbb{R}\times[0,\delta^\nu])}\Big).\end{aligned}$$

The terms in the last line converge to zero or can be absorbed into the left hand side for δ^ν sufficiently small. We claim that the penultimate line also converges to zero and we thus obtain the convergence $\|\xi^\nu\|_{\Gamma_{1,\delta}} \to 0$. To check this we recall from (52) that $\overline{\partial}_J v^\nu = 0$ implies

$$\nabla_s\xi^\nu + J(\xi^\nu)\nabla_t\xi^\nu = -de_u(\xi^\nu)^{-1}\big(\partial_1 e(u,\xi^\nu)\partial_s u + J(u)\partial_1 e(u,\xi^\nu)\partial_t u\big). \tag{62}$$

Recall that

$$\partial_1 e(u,0) = \mathrm{Id}_{T_uM}, \qquad \partial_2 e(u,0) = de_u(0) = \mathrm{Id}_{T_uM}. \tag{63}$$

So in zeroth order we have, using the equations $\partial_t \bar u = 0$ and $\partial_s u_{02} = -J_{02}(u_{02})\partial_t u_{02}$,

$$\begin{aligned}\bigl|\nabla_s\hat\xi^\nu + \hat J(\hat\xi^\nu)\nabla_t\hat\xi^\nu\bigr| &\le \bigl|de_{\bar u}(\hat\xi^\nu)^{-1}\bigl(\partial_1 e(\bar u,\hat\xi^\nu)\partial_s\bar u\bigr)\bigr| \le C|\partial_s\bar u|,\\ \bigl|\nabla_s\xi^\nu_{02} + J_{02}(\xi^\nu_{02})\nabla_t\xi^\nu_{02}\bigr| &\le \bigl|de_{u_{02}}(\xi^\nu_{02})^{-1}\bigl(\partial_1 e(u_{02},\xi^\nu_{02})J_{02}(u_{02})\\ &\qquad - J_{02}(u_{02})\partial_1 e(u_{02},\xi^\nu_{02})\bigr)\partial_t u_{02}\bigr| \;\le\; C|\xi^\nu_{02}|,\end{aligned}$$

and thus

$$\begin{aligned}&\|\nabla_s\xi^\nu + J(\xi^\nu)\nabla_t\xi^\nu\|_{L^2_{1,\delta^\nu}} + \|\nabla_s\xi^\nu + J(\xi^\nu)\nabla_t\xi^\nu\|_{L^4_{1,\delta^\nu}}\\ &\le C\bigl(\|\xi^\nu_{02}\|_{L^2(\mathbb{R}\times[0,1])} + \|\xi^\nu_{02}\|_{L^4(\mathbb{R}\times[0,1])} + (\delta^\nu)^{1/2}\|\partial_s\bar u\|_{L^2(\mathbb{R})} + (\delta^\nu)^{1/4}\|\partial_s\bar u\|_{L^4(\mathbb{R})}\bigr) \to 0.\end{aligned}$$

For the first derivative we calculate from (62), denoting all uniform constants by C,

$$\begin{aligned}\bigl|\nabla\bigl(\nabla_s\hat\xi^\nu + \hat J(\hat\xi^\nu)\nabla_t\hat\xi^\nu\bigr)\bigr| &\le C(1+|\nabla\hat\xi^\nu|)\bigl|\partial_1 e(\bar u,\hat\xi^\nu)\partial_s\bar u\bigr| + C\bigl|\nabla\bigl(\partial_1 e(\bar u,\hat\xi^\nu)\partial_s\bar u\bigr)\bigr|\\ &\le C\bigl(1+|\nabla\hat\xi^\nu|\bigr)\bigl(|\partial_s\bar u| + |\nabla_s\partial_s\bar u|\bigr),\end{aligned}$$

and (in between dropping the subscript from ξ^ν_{02})

$$\begin{aligned}\bigl|\nabla\bigl(\nabla_s\xi^\nu_{02} + J_{02}(\xi^\nu_{02})\nabla_t\xi^\nu_{02}\bigr)\bigr| &\le C(1+|\nabla\xi^\nu|)\bigl|\partial_1 e(u,\xi^\nu)J(u)\partial_t u - J(u)\partial_1 e(u,\xi^\nu)\partial_t u\bigr|\\ &\quad + C\bigl|\nabla\bigl(\partial_1 e(u,\xi^\nu)J(u) - J(u)\partial_1 e(u,\xi^\nu)\bigr)\bigr|\cdot|\partial_t u|\\ &\quad + C\bigl|\partial_1 e(u,\xi^\nu)J(u) - J(u)\partial_1 e(u,\xi^\nu)\bigr|\cdot|\nabla\partial_t u|\\ &\le C|\xi^\nu_{02}|\bigl(1+|\nabla\xi^\nu_{02}|\bigr).\end{aligned}$$

Here the estimate for the second summand follows from (63) and the identity

$$\nabla_s(\partial_1 e(u,\xi)X) = \partial_1 e(u,\xi)\nabla_s X + (\nabla_{(\partial_s u,\nabla_s\xi)}\partial_1 e)(u,\xi)X$$

(and similarly for $\nabla_t(\partial_1 e(u,\xi)X)$), where we have $\bigl(\nabla_{(\partial_s u,\nabla_s\xi)}\partial_1 e\bigr)(u,0) = 0$ since

$$\bigl(\nabla_{(Y,0)}\partial_1 e\bigr)(u,0) = \nabla_Y \mathrm{Id}_{T_u M} = 0$$

and, calculating in local normal coordinates with an extension $\tilde Y \in \Gamma(TM)$ of $Y \in T_u M$ that is covariantly constant along $\tau \mapsto \exp_u(\tau X)$,

$$\bigl(\nabla_{(0,Y)}\partial_1 e\bigr)(u,0)X = \partial_\sigma|_{\sigma=0}\partial_\tau|_{\tau=0} e(\exp_u(\tau X),\sigma Y) = \partial_\tau|_{\tau=0}\tilde Y(\exp_u(\tau X)) = 0.$$

Now the uniform estimate $\|\nabla\xi^\nu\|_\infty \le c_0$ and the exponential decay of $\bar u = \bar u(s)$ imply

$$\bigl\|\nabla\bigl(\nabla_s\xi^\nu + J(\xi^\nu)\nabla_t\xi^\nu\bigr)\bigr\|_{L^2_{1,\delta^\nu}} \le C(1+c_0)\bigl(\|\xi^\nu_{02}\|_{L^2(\mathbb{R}\times[0,1])} + (\delta^\nu)^{1/2}\|\partial_s\bar u\|_{H^1(\mathbb{R})}\bigr) \to 0.$$

This proves

$$\|\xi^\nu\|_{\Gamma_{1,\delta^\nu}} \to 0.$$

It remains to find a time-shift such that $\tau^\sigma v^\nu = e_u(\xi^\nu(\sigma))$ with some $\xi^\nu(\sigma) \in K_0$ but still $\|\xi^\nu(\sigma)\|_{\Gamma_{1,\delta^\nu}} \le \epsilon_0$. In order to find this shift we write $\tau^\sigma v^\nu = e_u(\xi^\nu(\sigma))$ with

$$\xi^\nu(\sigma) := \bigl(e_u^{-1}\circ\tau^\sigma\circ e_u\bigr)(\xi^\nu) \in \Gamma_{1,\delta^\nu}. \tag{64}$$

This will satisfy

$$\|\xi^\nu(\sigma)\|_{\Gamma_{1,\delta^\nu}} \le C\bigl(\|\xi^\nu\|_{\Gamma_{1,\delta^\nu}} + |\sigma|\|du\|_{\Gamma_{1,\delta^\nu}}\bigr),$$

so it is well defined whenever $|\sigma| \leq \sigma_0$, where we fixed $\sigma_0 = \frac{1}{2}\epsilon_0 C^{-1}\|du\|^{-1}_{\Gamma_{1,\delta^\nu}}$ such that $\|\xi^\nu(\sigma)\|_{\Gamma_{1,\delta^\nu}} \leq \epsilon_0$ is ensured for sufficiently large $\nu \geq \nu_0$. The L^2-estimate on $\xi^\nu(\sigma)$ can be seen from the pointwise estimate

$$\begin{aligned}
\big|e_u^{-1}\tau^\sigma e_u(\xi)\big| &\leq \big|e_u^{-1}\tau^\sigma e_u(\xi) - e_u^{-1}\tau^\sigma e_u(0)\big| + \big|e_u^{-1}\tau^\sigma u - e_u^{-1}u\big| \\
&\leq C\big(d\big(\tau^\sigma e_u(\xi), \tau^\sigma e_u(0)\big) + d\big(\tau^\sigma u, u\big)\big) \\
&\leq C\big(\big|\tau^\sigma \xi\big| + \sigma|\partial_s u|\big).
\end{aligned}$$

Here C is a continuity constant for e_u^{-1}. The higher derivatives of $\xi(\sigma) = e_u^{-1}\tau^\sigma e_u(\xi)$ are estimated similarly. Now consider the function

$$\Theta^\nu(\sigma) := \langle \xi^\nu_{02}(\sigma), \partial_s u_{02}\rangle_{L^2}.$$

It satisfies

$$|\Theta^\nu(0)| \leq \|\partial_s u\|_{L^2_{1,\delta^\nu}} \|\xi^\nu\|_{L^2_{1,\delta^\nu}} \to 0$$

and (dropping the 02-subscript) we obtain from (64)

$$\begin{aligned}
&\Big|\tfrac{\partial}{\partial\sigma}\Theta^\nu(\sigma) - \|\partial_s u\|^2_{L^2}\Big| \\
&= \Big|\langle\big(de_u(\xi^\nu(\sigma))^{-1}\tau^\sigma\big(\partial_1 e(u,\xi^\nu)\partial_s u + de_u(\xi^\nu)\partial_s\xi^\nu\big) - \tau^\sigma\partial_s u\big), \partial_s u\rangle_{L^2} \\
&\qquad\qquad\qquad + \langle\big(\tau^\sigma\partial_s u - \partial_s u\big), \partial_s u\rangle_{L^2}\Big| \\
&\leq C\big(\|\xi^\nu\|_{H^1}\|\partial_s u\|_{L^2} + \|\xi^\nu\|_\infty\|\partial_s u\|^2_{L^2} + |\sigma|\|\nabla_s\partial_s u\|_{L^2}\|\partial_s u\|_{L^2}\big).
\end{aligned}$$

The latter is an arbitrarily small error for large ν and small σ. Hence we will find solutions $\sigma^\nu \sim -\Theta^\nu(0)/\|\partial_s u_{02}\|^2_{L^2} \in [-\sigma_0, \sigma_0]$ of $\Theta^\nu(\sigma^\nu) = 0$. With these we have $\tau^{\sigma^\nu}v^\nu = e_u(\xi^\nu(\sigma))$, where $\xi^\nu \in K_0 = \big\{\xi \in \Gamma_{1,\delta}\big|\langle\xi_{02}, \partial_s u_{02}\rangle_{L^2} = 0\big\}$ and $\|\xi^\nu(\sigma)\|_{\Gamma_{1,\delta^\nu}} \leq \epsilon_0$. So with this small time-shift on v^ν we obtain a contradiction to the assumption that $\mathcal{T}^\nu_\delta$ is not surjective.

Finally, to prove the transversality we need to check that $D_{v^\nu} = D_{e_u(\xi^\nu)}$ is surjective. (The same then holds for the time shifts $\tau^{\sigma^\nu}v^\nu$.) This follows from the quadratic estimate in Lemma 5.1.5 : Let $Q : \Omega_{1,\delta^\nu} \to \Gamma_{1,\delta^\nu}$ be the right inverse of $D^\delta = d\mathcal{F}_u(0)$, then

$$\begin{aligned}
\|\Phi_u(\xi^\nu)^{-1}D_{e_u(\xi^\nu)}E_u(\xi^\nu)Q - \mathrm{Id}\| &\leq \|\Phi_u(\xi^\nu)^{-1}D_{e_u(\xi^\nu)}E_u(\xi^\nu) - d\mathcal{F}_u(0)\| \cdot \|Q\| \\
&\leq 2C_2\|Q\|\|\xi^\nu\|_{\Gamma_{1,\delta^\nu}},
\end{aligned}$$

where $\|Q\| < \infty$ by (51) and $\|\xi^\nu\|_{\Gamma_{1,\delta^\nu}} \to 0$. This shows that $\Phi_u(\xi^\nu)^{-1}D_{e_u(\xi^\nu)}E_u(\xi^\nu)Q$ and hence also the operator $\Phi_u(\xi^\nu)^{-1}D_{e_u(\xi^\nu)}E_u(\xi^\nu)$ has a right inverse for all sufficiently large $\nu \geq \nu_0$. Here the parallel transport $\Phi_u(\xi^\nu)$ is an isomorphism on the target and $E_u(\xi^\nu)$ identifies $\Gamma_{1,\delta}$ with the domain of $D_{e_u(\xi^\nu)}$. For the latter see the discussion before Lemma 5.1.5 and recall that $E_u(0) = \mathrm{Id}$. So we have established that D_{v^ν} is surjective, and this finishes the proof. ∎

Lemma 5.3.2. *There exists a universal constant $\hbar > 0$ such that the following holds for any sequence of Floer trajectories $v^\nu \in \widehat{\mathcal{M}}_{\delta^\nu}(x^+, x^-)$ with $\delta^\nu \to 0$. If for some $s \in \mathbb{R}$*

$$\liminf_{\nu\to\infty}\big(\|dv^\nu_{02}\|_{L^\infty(B_\epsilon(s,0))} + \|d\hat{v}^\nu\|_{L^\infty(B_\epsilon(s,0))}\big) = \infty \qquad \forall\epsilon > 0,$$

then there exists a sequence $\epsilon^\nu \to 0$ such that

$$\liminf_{\nu\to\infty}\left(\int_{B_{\epsilon^\nu}(s,0)}|dv^\nu_{02}|^2 + \int_{B_{\epsilon^\nu}(s,0)}|d\hat{v}^\nu|^2\right) \geq \hbar.$$

Here $B_\epsilon(s,0)$ *is the* ϵ*-ball in* $\mathbb{R}\times[0,1]$ *or* $\mathbb{R}\times[0,\delta^\nu]$ *respectively.*

Careful analysis of the blowup behavior and appropriate rescaling will lead to convergence to one of the following limit objects:

- a sphere bubble in M_0, M_1, or M_2 ,
- a disk bubble in $(M_0\times M_1, L_{01})$ or $(M_1\times M_2, L_{12})$,
- a disk bubble in $(M_0\times M_2, L_{01}\circ L_{12})$,
- a *figure eight bubble*, that is a triple of J-holomorphic maps

  $$v_0:\mathbb{R}\times(-\infty,-1]\to M_0,\quad v_1:\mathbb{R}\times[-1,1]\to M_1,\quad v_2:\mathbb{R}\times[1,\infty)\to M_2$$

  such that

  $$(v_0(\tau,-1),v_1(\tau,-1))\in L_{01},\quad (v_1(\tau,1),v_2(\tau,1))\in L_{12}.$$

Viewed from $z=\infty$ the lines $\mathrm{Im}(z)=\pm1$ appear as a figure eight, as in Figure 12. We conjecture that the maps (v_0,v_1,v_2) can be extended continuously to S^2 by a point $(v_0(\infty),v_1(\infty),v_2(\infty))$ that lies in both $L_{01}\times M_2$ and $M_0\times L_{12}$.

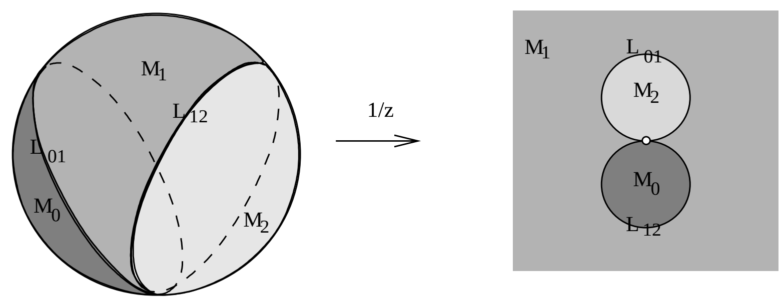


Figure 12. Figure Eight bubble

However, we cannot in general prove this removal of singularities at $z=\infty$ for figure eight bubbles. Instead, as in [47] we prove energy quantization without giving a geometric description of the bubble.

Proof. For notational convenience we introduce the noncontinuous function $|dv|:\mathbb{R}\times[0,1]\to[0,\infty)$ given by $|dv(s,t)|^2=|dv_{02}(s,t)|^2+|d\hat{v}(s,t)|^2$ for $t\in[0,\delta]$ and $|dv(s,t)|=|dv_{02}(s,t)|$ for $t\in(\delta,1]$.

Suppose the lemma is false, that is, for every $k\in\mathbb{N}$ there exists a sequence $v^{k,\nu}\in\widehat{\mathcal{M}}_{\delta^{k,\nu}}(x^+,x^-)$ with $\delta^{k,\nu}\to0$ such that (after time shift to $s=0$) $R_k^\nu:=|dv^{k,\nu}(s_k^\nu,t_k^\nu)|\to\infty$ for some $(s_k^\nu,t_k^\nu)\to(0,0)$, but

$$\liminf_{\nu\to\infty}\int_{B_{\epsilon^\nu}(0)}|dv^{\nu,k}|^2\le\frac{1}{k}.$$

for every sequence $\epsilon^\nu\to0$. In particular, this will hold for a fixed sequence $\epsilon_k^\nu\to0$ that satisfies in addition $\epsilon_k^\nu\ge\delta_k^\nu$, $(s_k^\nu,t_k^\nu)\in B_{\frac14\epsilon_k^\nu}(0)$ and $\epsilon_k^\nu R_k^\nu\to\infty$. We can then find diagonal sequences $v^k\in\widehat{\mathcal{M}}_{\delta_k}(x^+,x^-)$ with $\delta_k\to0$, and $\epsilon_k\to0$, $(s_k,t_k)\in B_{\frac14\epsilon_k}(0)$ such that $\epsilon_kR_k:=\epsilon_k|dv^k(s_k,t_k)|\to\infty$ and

$$\int_{B_{\epsilon_k}(0)}|dv^k|^2\to0. \tag{65}$$

Next, we use Lemma 5.3.3 to refine the choice of the blowup points (s_k,t_k). For that purpose we consider the spaces $X_{02}=\mathbb{R}\times[0,1]$, $\hat{X}=\mathbb{R}\times[0,\delta_k]$, and $X=\mathbb{R}\times[0,1]$, with the obvious inclusion $\pi: X_{02}\cup\hat{X}\to X$. Using the function $f=|dv_{02}^k|$ on X_{02} and $f=|d\hat{v}^k|$ on $\hat{X}$ one can then vary the point $\pi(x)=(s_k,t_k)\in\mathbb{R}\times[0,1]$ by $2\rho=\frac{1}{4}\epsilon_k$ to find $(s_k,t_k)\in B_{\frac{1}{2}\epsilon_k}(0)$ and $\epsilon_k'\le\frac{1}{8}\epsilon_k$, such that $\epsilon_k'R_k:=\epsilon_k'|dv^k(s_k,t_k)|\to\infty$ and $|dv^k|\le 4R_k$ on $B_{\epsilon_k'}(s_k,t_k)$. Here (65) continues to hold on $B_{\epsilon_k(0)}\supset B_{\epsilon_k'(s_k,t_k)}$.

Now in a first step we will prove that figure eight bubbles (arising from rescaling in the case $\delta_kR_k\to\Delta\in(0,\infty)$) have a minimal energy (possibly depending on $\Delta>0$.) More precisely, we claim that (65) implies

$$t_kR_k\to 0,\qquad\text{and}\qquad\delta_kR_k\to 0. \tag{66}$$

In a second step we will then see that this gives rise to a disk bubble in $(M_0\times M_2, L_{01}\circ L_{12})$.

Step 1: *We prove (66).*

First consider the case $|dv_{02}^k(s_k,t_k)|\ge\frac{1}{2}|dv^k(s_k,t_k)|$ and $t_k\ge\frac{1}{2}\delta_k$. Then for all sufficiently large k we can apply the mean value inequality [26, Lemma 4.3.1] to $|dv_{02}^k|$ on the ball $B_{r_k}(s_k,t_k)\subset\mathbb{R}\times(0,1)\cap B_{\epsilon_k}(0)$ with $r_k:=\min\{t_k,\epsilon_k'\}$,

$$\tfrac{1}{4}(r_kR_k)^2\le r_k^2|dv_{02}^k(s_k,t_k)|^2\le c\int_{B_{r_k}(s_k,t_k)}|dv_{02}^k|^2\to 0.$$

Here we cannot have $r_k=\epsilon_k'$ since $\epsilon_k'R_k\to\infty$, so we have $r_k=t_k$ and thus $\frac{1}{2}\delta_kR_k\le t_kR_k\to 0$ as claimed.

In the case $|d\hat{v}^k(s_k,t_k)|\ge\frac{1}{2}|dv^k(s_k,t_k)|$ and $\delta_k\ge t_k\ge\frac{1}{2}\delta_k$ we can apply the mean value inequality [47, Theorem 1.3, Lemma A.1] to $|d\hat{v}^k|$ with boundary condition $\hat{v}^k|_{t=\delta_k}\in L_{01}\times L_{12}$ on the partial ball $B_{r_k(s_k,t_k)}\subset\mathbb{R}\times(0,\delta_k]\cap B_{\epsilon_k}(0)$ for $r_k:=\min\{\frac{1}{2}\delta_k,\epsilon_k'\}$,

$$\tfrac{1}{4}(r_kR_k)^2\le r_k^2|d\hat{v}^k(s_k,t_k)|^2\le c\int_{B_{r_k}(s_k,t_k)}|d\hat{v}^k|^2\to 0.$$

As before we cannot have $r_k=\epsilon_k'$ since $\epsilon_k'R_k\to\infty$, so we have $r_k=\frac{1}{2}\delta_k$ and thus $t_kR_k\le\delta_kR_k\to 0$ as claimed.

In the remaining case $t_k\le\frac{1}{2}\delta_k$ we consider the holomorphic curve

$$w^k:=(v_{02}^k,\hat{v}^k):\mathbb{R}\times[0,\delta_k]\to M_0\times M_2\times M_0\times M_2\times M_1\times M_1,$$

which satisfies the Lagrangian boundary condition $w^k|_{t=0}\in\Delta_0\times\Delta_2\times\Delta_1$. By the above we have $|dw^k(s_k,t_k)|\ge R_k\to\infty$ and $\int_{B_{\epsilon_k}(0)}|dw^k|^2\to 0$. So for all sufficiently large k we can apply the mean value inequality [47, Theorem 1.3, Lemma A.1] on the partial ball $B_{r_k(s_k,t_k)}\subset\mathbb{R}\times[0,\delta_k)\cap B_{\epsilon_k}(0)$ for $r_k:=\min\{\frac{1}{2}\delta_k,\epsilon_k'\}$,

$$(r_kR_k)^2\le r_k^2|dw^k(s_k,t_k)|^2\le c\int_{B_{r_k}(s_k,t_k)}|dw^k|^2\to 0.$$

Again we cannot have $r_k=\epsilon_k'$ since $\epsilon_k'R_k\to\infty$, so we have $r_k=\frac{1}{2}\delta_k$ and thus $2t_kR_k\le\delta_kR_k\to 0$ as claimed.

Step 2: *We prove the lemma.*

We consider the rescaled maps $w^k=(w_{02}^k,\hat{w}^k)$, where $w_{02}^k: B_{\epsilon_kR_k}(0)\cap\mathbb{H}^2\to M_0\times M_2$ is defined on half balls of radius $\epsilon_kR_k\to\infty$ in the half space $\mathbb{H}^2:=\mathbb{R}\times[0,\infty)$ by $w_{02}^k(s,t):=v_{02}^k(s_k+s/R^k,t/R^k)$, and $\hat{w}^k: B_{\epsilon_kR_k}(0)\cap(\mathbb{R}\times[0,\delta_kR_k])\to M_0\times M_2\times M_1\times M_1$ is defined by $\hat{w}^k(s,t) := \hat{v}^k(s_k + s/R^k, t/R^k)$ on balls of radius $\epsilon_k R_k$ intersected with the strip of width $\delta_k R_k \to 0$.

This rescaling preserves the nontriviality $|dw^k(0, t_k R_k)| \geq 1$, but on both domains $|dw^k|$ is uniformly bounded. Hence we can find a subsequence of the w_{02}^k that converges in the $\mathcal{C}^0$-topology on the unit half ball $D_1 := B_1(0) \cap \mathbb{H}^2$. The (scaling invariant) energy $\int_{B_{\epsilon_k R_k}(0)} |dw_{02}^k|^2$ converges to zero by (65), so the limit has to be constant. In fact, we have $w_{02}^k \to x_{02} \in L_{02}$ since the boundary values $w_{02}^k|_{t=0}$ converge to $L_{01} \circ L_{12} = L_{02}$ in $\mathcal{C}^0([-1,1])$. To see the latter use the transversality of the Lagrangians as in Lemma 5.1.3 and integrate the bound on $|\partial_t \hat{w}^k|$ to obtain

$$d\big(\hat{w}^k(s,0), \hat{w}^k(s,\delta_k)\big) \leq \int_0^{\delta_k} |\partial_t \hat{w}^k(s,t)| dt \leq \delta_k 2R_k \to 0.$$

This also proves that $\hat{w}^k \to x_1$ in $\mathcal{C}^0([-1,1] \times [0, \delta_k R_k])$, where $x_1 \in M_1$ is uniquely determined by $\bar{x} := (x_{02}, x_1, x_1) \in L_{01} \times L_{12}$. The maps w_{02}^k are $\bar{J}_{02}$-holomorphic, so by elliptic regularity the convergence $w_{02}^k \to x_{02}$ is in the $\mathcal{C}^\infty$-topology on every compact subset of $\mathbb{H}^2 \setminus \partial\mathbb{H}^2$. However, in order to obtain a contradiction to the fact that $|dw^k(0, t_k R_k)| \geq 1$ with $t_k R_k \to 0$ we need to establish $\mathcal{C}^1$-convergence on D_1 up to the boundary.

We begin by noting that due to the $\mathcal{C}^0$-convergence we can express $w^k = e_x(\xi^k)$ in terms of sections $\xi^k = (\xi_{02}^k, \hat{\xi}^k) \in H^2(D_1, x_{02}^* T(M_0 \times M_2)) \times H^2([0,1] \times [0, \delta_k R_k], \bar{x}^* T(M_0 \times M_2 \times M_1 \times M_1))$ using the exponential map centered at $x = (x_{02}, \bar{x})$. These sections satisfy the diagonal and Lagrangian boundary conditions $\xi^k|_{t=0} \in T_x(\Delta_0 \times \Delta_2 \times \Delta_1)$ and $\hat{\xi}^k|_{t=\delta_k R_k} \in T_{\bar{x}}(L_{01} \times L_{12})$, the $\mathcal{C}^0$-convergence $\|\xi^k\|_\infty \to 0$, and a uniform bound $\|\nabla \xi^k\|_\infty \leq c_0$. Since $\overline{\partial}_J w^k = 0$ and $\nabla x = 0$ we obtain from (52)

$$\nabla_s \xi^k + J(\xi^k) \nabla_t \xi^k = 0.$$

Now $dw^k = de_x(\xi^k) \nabla_s \xi^k ds + de_x(\xi^k) J(\xi^k) \nabla_s \xi^k dt$, so it suffices to prove the $\mathcal{C}^0$-convergence of $\nabla_s \xi^k$ near 0. For that purpose we multiply the sections by cutoff functions $h = (h_{02}, \hat{h})$ with $h_{02} : \mathbb{R} \times [0,1] \to [0,1]$ supported in D_1, $\hat{h} : \mathbb{R} \to [0,1]$ supported in $[-1,1]$, and both equal to 1 near 0. Then we obtain sections on the multistrip $h\xi^k := (h_{02}\xi_{02}^k, \hat{h}\hat{\xi}^k) \in \Gamma_{1,\delta_k R_k}$ that also satisfy the boundary condition $h_{02}\xi_{02}^k|_{t=1} = 0$. These satisfy a uniform bound

$$\sup_k \Big(\|\nabla_s(h\xi^k) + J(\xi^k)\nabla_t(h\xi^k)\|_{H^1_{1,\delta_k R_k}} + \|h\xi^k\|_{H^0_{1,\delta_k R_k}} \Big) \leq \sup_k C \|\xi^k\|_{H^1_{1,\delta_k R_k}(\mathrm{supp}(h))} < \infty$$

due to the bounds on $\|\xi^k\|_\infty$ and $\|\nabla \xi^k\|_\infty$ and the compact support of h. From this Lemma 5.2.1 (b) provides a uniform bound

$$\sup_k \|h\xi^k\|_{H^2_{1,\delta_k R_k}} \leq C_\Gamma < \infty.$$

Indeed, the boundary terms vanish since the constant boundary conditions directly transfer to the derivatives, $\nabla_s \xi_{02}^k|_{t=1}, \nabla_s^2 \xi_{02}^k|_{t=1} \in T_{x_{02}}(L_0 \times L_2)$ and $\nabla_s \hat{\xi}^k|_{t=\delta_k R_k}, \nabla_s^2 \hat{\xi}^k|_{t=\delta_k R_k} \in T_{\hat{x}}(L_{01} \times L_{12})$.

We now fix a pair of cutoff functions h' with support in $h^{-1}(1)$ and still equal to 1 near 0. Then we apply Lemma 5.2.1 (b) to $h'\nabla_s \xi^k$, again with vanishing boundary terms, to obtain

$$\begin{aligned}\sup_k \|h'\nabla_s\xi^k\|_{H^2_{1,\delta_k R_k}} &\leq \sup_k C_1\Big(\big\|\big(\nabla_s + J(\xi^k)\nabla_t\big)h'\nabla_s\xi^k\big\|_{H^1_{1,\delta_k R_k}} + \|h'\nabla_s\xi^k\|_{H^0_{1,\delta_k R_k}}\Big)\\ &\leq \sup_k C(1+c_0)\|h\xi^k\|_{H^2_{1,\delta_k R_k}} < \infty.\end{aligned}$$

We can pick the cutoff functions such that $h'_{02}|_{D_{1/2}} \equiv 1$ on the half ball $D_{1/2} \subset \mathbb{H}^2$ and $\hat{h}|_{[-\frac{1}{2},\frac{1}{2}]} \equiv 1$. Then the compact Sobolev embedding $H^2(D_{1/2}) \hookrightarrow \mathcal{C}^0(D_{1/2})$ provides $\mathcal{C}^0$-convergence of a subsequence $\nabla_s\xi^k_{02}$. We already know that the limit is 0, so we obtain $\nabla_s\xi^k_{02} \to 0$ and $\partial_s w^k_{02} \to 0$ in $\mathcal{C}^0(D_{1/2})$. It remains to establish $\|\nabla_s\hat{\xi}^k\|_{\mathcal{C}^0([-\frac{1}{2},\frac{1}{2}]\times[0,\delta_k R_k])} \to 0$ and thus $\|\partial_s\hat{w}^k\|_{\mathcal{C}^0([-\frac{1}{2},\frac{1}{2}]\times[0,\delta_k R_k])} \to 0$ in contradiction to $|dw^k(0,t_k R_k)| \geq 1$ with $t_k R_k \to 0$. To see this we follow the argument in Lemma 5.1.4. Using the standard Sobolev embedding $H^1([-\frac{1}{2},\frac{1}{2}]) \hookrightarrow \mathcal{C}^0([-\frac{1}{2},\frac{1}{2}])$ we obtain for all $t_0 \in [0,\delta_k R_k]$

$$\begin{aligned}\tfrac{1}{C}\|\nabla_s\hat{\xi}^k|_{t=t_0} - \nabla_s\hat{\xi}^k|_{t=\delta_k R_k}\|^2_{\mathcal{C}^0([-\frac{1}{2},\frac{1}{2}])} &\leq \|\nabla_s\hat{\xi}^k|_{t=t_0} - \nabla_s\hat{\xi}^k|_{t=\delta_k R_k}\|^2_{H^1([-\frac{1}{2},\frac{1}{2}])}\\ &\leq \delta_k R_k \int_0^{\delta_k R_k} \|\nabla_t\nabla_s\hat{\xi}^k\|^2_{H^1([-\frac{1}{2},\frac{1}{2}])}\\ &\leq \delta_k R_k \|\nabla_s\hat{\xi}^k\|^2_{H^2([-\frac{1}{2},\frac{1}{2}]\times[0,\delta_k R_k])} \to 0.\end{aligned} \tag{67}$$

From the above we moreover have $\|\nabla_s\xi'^k_{02}|_{t=0}\|_{\mathcal{C}^0([-\frac{1}{2},\frac{1}{2}])} = \|\nabla_s\xi^k_{02}|_{t=0}\|_{\mathcal{C}^0([-\frac{1}{2},\frac{1}{2}])} \to 0$. Now using Lemma 5.1.3 and the boundary conditions, in particular $(\xi^k_1 - \xi'^k_1)|_{t=0} = 0$, we obtain

$$\begin{aligned}&\|\nabla_s\hat{\xi}^k|_{t=\delta_k R_k}\|_{\mathcal{C}^0([-\frac{1}{2},\frac{1}{2}])}\\ &\leq C\big(\|\pi_{02}(\nabla_s\hat{\xi}^k)|_{t=\delta_k R_k}\|_{\mathcal{C}^0([-\frac{1}{2},\frac{1}{2}])} + \|\nabla_s(\xi^k_1 - \xi'^k_1)|_{t=\delta_k R_k}\|_{\mathcal{C}^0([-\frac{1}{2},\frac{1}{2}])}\big)\\ &\leq C\big(\|\nabla_s\xi'^k_{02}|_{t=0}\|_{\mathcal{C}^0([-\frac{1}{2},\frac{1}{2}])} + 3\|\nabla_s\hat{\xi}^k|_{t=\delta_k R_k} - \nabla_s\hat{\xi}^k|_{t=0}\|_{\mathcal{C}^0([-\frac{1}{2},\frac{1}{2}])}\big) \to 0.\end{aligned}$$

Combining $\|\nabla_s\hat{\xi}^k|_{t=\delta_k R_k}\|_{\mathcal{C}^0([-\frac{1}{2},\frac{1}{2}])} \to 0$ with (67) then proves $\|\nabla_s\hat{\xi}^k\|_{\mathcal{C}^0([-\frac{1}{2},\frac{1}{2}]\times[0,\delta_k R_k])} \to 0$ and thus $|dw^k(0,t_k R_k)| \to 0$ in contradiction to the assumption. □

Lemma 5.3.3. *Let (X,d) be a metric space, $X_1,\ldots,X_n$ topological spaces, $\pi : X_1 \cup \ldots \cup X_n \to X$ a continuous map, and $f : X_1 \cup \ldots X_n \to \mathbb{R}$ a non-negative continuous function. Fix $x \in X_i$ for some $i = 1,\ldots,n$ and $\rho > 0$. Suppose that $\pi^{-1}(B_{2\rho}(\pi(x))) \cap X_i$ is complete for each $i = 1,\ldots,n$. Then there exists an $x' \in X_1 \cup \ldots X_n$ and a positive number $\rho' \leq \rho$ such that*

$$d(\pi(x'),\pi(x)) < 2\rho, \qquad \sup_{\pi^{-1}B_{\rho'}(\pi(x'))} f \leq 2f(x'), \quad \rho' f(x') \geq \rho f(x).$$

Proof. Otherwise, the same argument as in the proof of Hofer's lemma [26, p.93] shows that there exists a sequence $x_\alpha \in X_1 \cup \ldots \cup X_n$ such that

$$x_0 = x, \quad d(\pi(x_\alpha),\pi(x_{\alpha+1})) \leq \rho/2^\alpha, \quad f(x_{\alpha+1}) > 2f(x_\alpha).$$

After passing to a subsequence, we obtain a Cauchy sequence x_α in some X_i with $f(x_\alpha) \to \infty$, which contradicts completeness and continuity of f. □

Remark 5.3.4. To see that the assumption that L_{02} is embedded is necessary, consider the case that M_0, M_2 are points. In this case, if $v : \mathbb{R} \times [0,1] \to M_1$ is a Floer trajectory of index one with limits $x^+ \neq x^-$, we can consider the rescaled maps $v_\delta : \mathbb{R} \times [0,\delta] \to M_1$.

In this case a figure eight bubble always develops in the limit $\delta \to 0$. This shows that the bijection between trajectories fails in this case.

5.4. **Shrinking strips in quilted surfaces.** Consider a quilted surface $\underline{S}$ containing a component S_k that is diffeomorphic to $\mathbb{R} \times [0,1]$ and attached via seams $\{(\ell_-, f_-), (k, e_-)\}$ and $\{(k, e_+), (\ell_+, f_+)\}$. (We can allow one of these seams to be replaced by a boundary component $(k, e_\pm) \in \mathcal{B}$. In that case we set $M_{\ell_\pm} = \{\mathrm{pt}\}$.) Let $\underline{L}$ be Lagrangian boundary conditions for $\underline{S}$ and suppose that the Lagrangians $L_- := L_{(\ell_-,f_-),(k,e_-)} \subset M_{\ell_-}^- \times M_k$, $L_+ := L_{(k,e_+),(\ell_+,f_+)} \subset M_k^- \times M_{\ell_+}$ associated to the boundary components of S_k are such that $L_- \circ L_+$ is smooth and embedded by projection into $M_{\ell_-}^- \times M_{\ell_+}$. Let $\underline{S}'$ denote the quilted surface obtained by removing the component S_k and replacing it with a new seam $\{(\ell_-, f_-), (\ell_+, f_+)\}$. We define Lagrangian boundary conditions $\underline{L}'$ for $\underline{S}'$ by $L'_{(\ell_-,f_-),(\ell_+,f_+)} := L_- \circ L_+$. In this setting we have a canonical identification of Floer chain groups attached to the ends

$$CF(\underline{L}_{\underline{e}}) \xrightarrow{\sim} CF(\underline{L}'_{\underline{e}}) \tag{68}$$

as in Remark 2.3.3 for every $\underline{e} \in \mathcal{E}(\underline{S}) \cong \mathcal{E}(\underline{S}')$. Consider the relative invariants $\Phi_{\underline{S}}$ and $\Phi_{\underline{S}'}$ defined in Section 4.2.

Theorem 5.4.1. *Suppose that all symplectic manifolds in $\underline{M}$ satisfy (M1-2) with the same monotonicity constant, all Lagrangians in $\underline{L}$ satisfy (L1-3), and $\underline{L}$ is monotone and relative spin. Assume moreover that $L_- \circ L_+$ is embedded in the sense of Definition 2.0.5, satisfies (L1),(L3), and $\underline{L}'$ is monotone. Then* (68) *induces isomorphisms in Floer cohomology*

$$\Psi_{\underline{e}} : HF(\underline{L}_{\underline{e}}) \to HF(\underline{L}'_{\underline{e}})$$

and furthermore these maps intertwine with the relative invariants:

$$\Phi_{\underline{S}'} \circ \left(\bigotimes_{\underline{e} \in \mathcal{E}_-} \Psi_{\underline{e}} \right) = \left(\bigotimes_{\underline{e} \in \mathcal{E}_+} \Psi_{\underline{e}} \right) \circ \Phi_{\underline{S}} \, [d_k n_k]$$

where in the shift of degree $[d_k n_k]$, d_k is the number of incoming ends meeting the removed strip minus the number of outgoing ends meeting the removed strip, and $2n_k$ is the dimension of M_k.

Example 5.4.2. To see the necessity of the degree shift in a simple example, suppose that $\underline{S} = (S)$ is the disk with two incoming ends, and Φ_S the corresponding relative invariant described in (26). Suppose that L^0, L^1 intersect in a single point x. Then the theorem above applies, $\underline{S}'$ is empty, $\Phi_{\underline{S}'}$ is the trivial invariant, and $\Psi_{\underline{e}_-}$ maps $\langle x \rangle \otimes \langle x \rangle \mapsto 1$. On the other hand, $\Phi_{\underline{S}}$ is the duality pairing, which has degree $-n$.

On the level of chain complexes the maps $\Psi_{\underline{e}}$ are simply the identity. Therefore it suffices to show that the maps $C\Phi_{\underline{S}}$ and $C\Phi_{\underline{S}'}[d_k n_k]$ are equal for sufficiently small width. See (33) for an explanation of the grading shift. The bijection between pseudoholomorphic quilts follows from essentially the same degeneration argument as was used to prove Theorem 5.0.3. However, in this case the adjoint of the linearized operator is honestly surjective. (There is no translational symmetry, so elements of the zero-dimensional moduli space have linearized operators of index zero, not one.) The surfaces to the left and right of the shrinking strip are arbitrary quilted surfaces, but this is of no relevance in the proof.

As a first application of Theorem 5.4.1 we prove the claim of Remark 5.0.4. For that purpose we denote by

$$\Psi : HF(L_0, L_{01}, L_{12}, L_2) \to HF(L_0, L_{02}, L_2),$$
$$\tilde{\Psi} : HF(L_{02}, (L_{01}, L_{12})) \to HF(L_{02}, L_{02})$$

the isomorphisms given by Theorem 5.0.3. Then we have the following alternative description of Ψ (which a priori depends on L_0 and L_2) in terms of $\tilde{\Psi}$ and the identity morphism $1_{L_{02}} \in HF(L_{02}, L_{02})$, defined in Section 6.7.

Corollary 5.4.3. *Let*

$$\Upsilon : HF(L_{02}, (L_{01}, L_{12})) \otimes HF(L_0, L_{01}, L_{12}, L_2) \to HF(L_0, L_{02}, L_2)$$

denote the relative invariant associated to the quilted surface on the right in Figure 10. Then we have for all $f \in HF(L_0, L_{01}, L_{12}, L_2)$

$$\Psi(f) = \Upsilon(\tilde{\Psi}^{-1}(1_{L_{02}}) \otimes f).$$

Moreover, in the notation of Section 6.8, we have

$$\Upsilon(T \otimes f) = \Phi_T(L_0) \circ f.$$

Proof. We apply Theorem 5.4.1 to $\Upsilon = \Phi_{\underline{S}}$, where the quilted surface $\underline{S}$ contains one simple strip in M_1. (The other surfaces are triangles.) This implies $\Upsilon(T \otimes f) = \Phi_{\underline{S}'}(\tilde{\Psi}(T) \otimes \Psi(f))$, where the quilted surface $\underline{S}'$ is obtained by replacing this strip with a seam condition in $L_{01} \circ L_{12} = L_{02}$. To calculate this for $\tilde{\Psi}(T) = 1_{L_{02}}$ we use the gluing formula (34) to obtain $\Upsilon(\tilde{\Psi}^{-1}(1_{L_{02}}) \otimes f) = \Phi_{\underline{S}''}(\Psi(f))$, where $\underline{S}''$ is the surface that is obtained by gluing the quilted cap of Figure 33 into $\underline{S}'$. Since $\underline{S}''$ is a simple double strip (with seam condition L_{02} and boundary conditions L_0, L_2), and we do not quotient out by translation, the relative invariant $\Phi_{\underline{S}''}$ is the identity, as in Example 4.1.6. This proves the first claim.

The second claim follows from a deformation of the quilt $\underline{S}$ to the glued quilt that corresponds, by (34), to the composition of the natural transformation $T \mapsto \Phi_T(L_0) \in HF((L_0, L_{02}), (L_0, L_{01}, L_{12}))$ (given by the quilt in Figure 38) with the pair of pants product from $HF((L_0, L_{02}), (L_0, L_{01}, L_{12})) \otimes HF((L_0, L_{01}, L_{12}), L_2)$ to $HF((L_0, L_{02}), L_2)$ (given by the quilt in Figure 20). Figure 13 gives a picture summary of these arguments. □

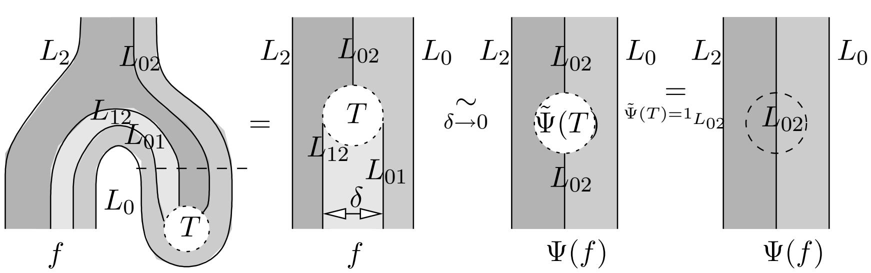


FIGURE 13. Proof by pretty picture

6. Functors associated to Lagrangian correspondences

6.1. **Donaldson-Fukaya category of Lagrangians.** Let (M,ω) be a symplectic manifold satisfying (M1-2). We fix a Maslov cover $\mathrm{Lag}^N(M) \to M$ as in Section 2.2, which will be used to grade Floer cohomology groups. In addition, we fix a *background class* $b \in H^2(M,\mathbb{Z}_2)$, which will be used to fix orientations of moduli spaces and thus define Floer cohomology groups with $\mathbb{Z}$ coefficients. In our examples, b will be either 0 or the second Stiefel-Whitney class $w_2(M)$ of M.

Definition 6.1.1. We say that a compact Lagrangian submanifold $L \subset M$ is *admissible* if the image of $\pi_1(L)$ in $\pi_1(M)$ is torsion and if L has minimal Maslov number $N_L \geq 3$.

The first assumption guarantees that any sequence of admissible Lagrangian submanifolds is monotone. (Alternatively, one could work with the framework of Bohr-Sommerfeld monotone Lagrangians described in Remark 3.1.4.) The last assumption implies that the Floer cohomology of any sequence is well-defined, and can be relaxed to $N_L \geq 2$ by working with matrix factorizations as explained in Section 7.

Definition 6.1.2. A *brane structure* on an admissible L consists of an orientation, a grading, and a relative spin structure with background class b, see Section 2.2 and [46]. An admissible Lagrangian equipped with a brane structure will be called a *Lagrangian brane*.

Remark 6.1.3.
(a) We have not included in the definition of Lagrangian branes the data of a flat vector bundle, in order to save space. The extension of the constructions below to this case is left to the reader.
(b) If one wants only $\mathbb{Z}_2$-gradings on the morphism spaces of the Donaldson-Fukaya category, then the N-fold Maslov cover and gradings of the Lagrangians may be ignored.
(c) If one wants only $\mathbb{Z}_2$ coefficients, then the background class and relative spin structures may be ignored.

Definition 6.1.4. The *Donaldson-Fukaya category*
$$\mathrm{Don}(M) := \mathrm{Don}(M, \mathrm{Lag}^N(M), \omega, b)$$
is defined as follows:
(a) The objects of $\mathrm{Don}(M)$ are Lagrangian branes in M.
(b) The morphism spaces of $\mathrm{Don}(M)$ are the $\mathbb{Z}_N$-graded Floer cohomology groups with $\mathbb{Z}$ coefficients
$$\mathrm{Hom}(L,L') := HF(L,L')$$
constructed using a choice of perturbation datum consisting of a pair (J,H) of a time-dependent almost complex structure J and a Hamiltonian H, as in Section 2.2.
(c) The composition law in the category $\mathrm{Don}(M)$ is defined by
$$\begin{aligned} \mathrm{Hom}(L,L') \times \mathrm{Hom}(L',L'') &\longrightarrow \mathrm{Hom}(L,L'') \\ (f,g) &\longmapsto f \circ g := \Phi_P(f \otimes g), \end{aligned}$$
where Φ_P is the relative invariant associated to the "half-pair of pants" surface P, that is, the disk with three markings on the boundary (two incoming ends, one outgoing end) as in Figure 14.
(d) The identity $1_L \in \mathrm{Hom}(L,L)$ is the relative invariant $1_L := \Phi_S \in HF(L,L)$ associated to a disk S with a single marking (an outgoing end), see Figure 14

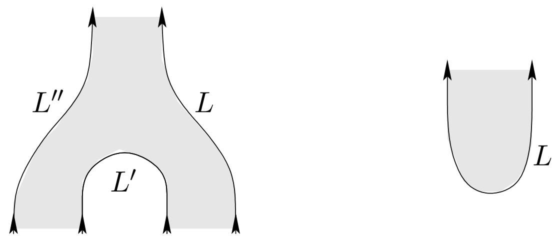


FIGURE 14. Composition and identity in the Donaldson-Fukaya category

The "closed" analog of the category $\mathrm{Don}(M)$, whose morphisms are symplectomorphisms of M, was apparently introduced by Donaldson in a seminar talk, see [26, 12.6]. Subsequently Fukaya introduced an A_∞ category version involving Lagrangian submanifolds.

Associativity of the composition follows from the gluing Theorem 4.1.8 applied to the surfaces in Figure 15: The two ways of composing correspond to two ways of gluing the pair of pants. The resulting surfaces are the same (up to a deformation of the complex structure), hence the resulting compositions are the same. The identity axiom $1_{L_0} \circ f = f = f \circ 1_{L_1}$

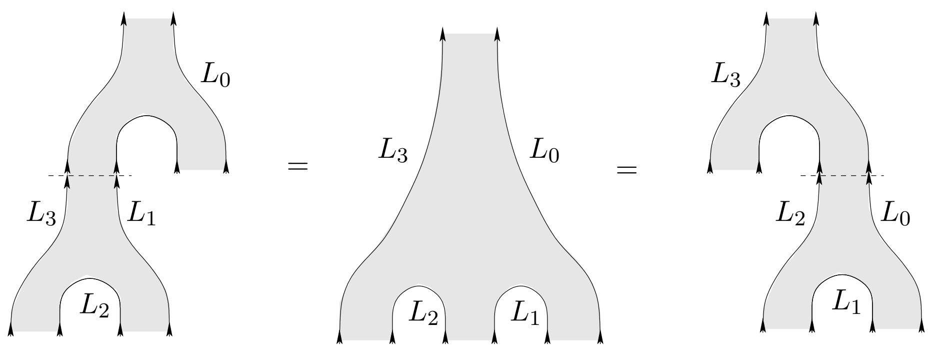


FIGURE 15. Associativity of composition

follows from the same gluing argument applied to the surfaces on the left and right in Figure 16. Here – in contrast to the Floer trajectories – the solutions on the strip are counted without quotienting by $\mathbb{R}$, hence as in Example 4.1.6 this relative invariant is the identity.

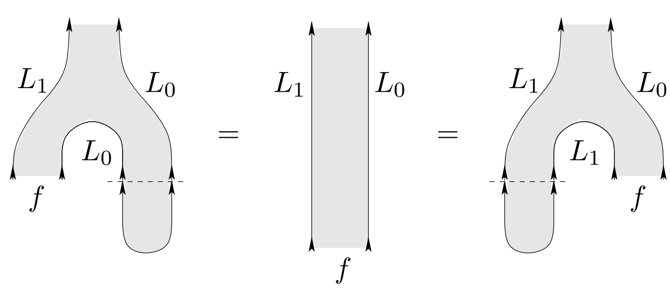


FIGURE 16. Identity axiom

Remark 6.1.5. The category $\mathrm{Don}(M)$ is independent of the choices of perturbation data, up to isomorphism of categories: The relative invariants for the infinite strip with perturbation data interpolating between two different choices gives an isomorphism of the morphism spaces, and the gluing law in Theorem 4.1.8 implies compatibility of these morphisms with compositions and identities.

6.2. **Functors associated to symplectomorphisms.** Let $\psi : M_0 \to M_1$ be a graded symplectomorphism (see Definition 2.2.2 (a)).

Definition 6.2.1. One can define a functor

$$\Phi(\psi) : \mathrm{Don}(M_0) \to \mathrm{Don}(M_1)$$

(a) on the level of objects by $L \mapsto \psi(L)$,
(b) on the level of morphisms

$$HF(L_0, L_1) \to HF(\psi(L_0), \psi(L_1))$$

is induced by the obvious map of chain complexes

$$CF(L_0, L_1) \to CF(\psi(L_0), \psi(L_1)), \quad \langle x\rangle \mapsto \langle \psi(x)\rangle$$

for all $x \in \mathcal{I}(L_0, L_1)$. (Here we use the Hamiltonians $H \in \mathrm{Ham}(L_0, L_1)$ and $H \circ \psi^{-1} \in \mathrm{Ham}(\psi(L_0), \psi(L_1))$.)

Note that $\Phi(\psi)$ satisfies the functor axioms

$$\Phi(\psi)(f \circ g) = \Phi(\psi)(f) \circ \Phi(\psi)(g), \qquad \Phi(\psi)(1_L) = 1_{\psi(L)}.$$

Furthermore if $\psi_{01} : M_0 \to M_1$ and $\psi_{12} : M_1 \to M_2$ are symplectomorphisms then

$$\Phi(\psi_{12} \circ \psi_{01}) = \Phi(\psi_{01}) \circ \Phi(\psi_{12}).$$

In terms of Lagrangian correspondences this functor is $L \mapsto L \circ \mathrm{graph}\, \psi$ on objects. This suggests that one should extend the functor to more general Lagrangian correspondences $L_{01} \subset M_0^- \times M_1$ by $L \mapsto L \circ L_{01}$ on objects. However, these compositions are generically only immersed, so one would have to allow for singular Lagrangians as objects in $\mathrm{Don}(M_1)$. Moreover, it is not clear how to extend the functor on the level of morphisms, that is Floer cohomology groups. In the following sections we propose some alternative definitions of functors associated to general Lagrangian correspondences.

6.3. **First functor for Lagrangian correspondences.** Fix an integer $N > 0$ and let Ab_N be the category of $\mathbb{Z}_N$-graded abelian groups. Let $\mathrm{Don}(M)^\vee$ be the category whose objects are functors from $\mathrm{Don}(M)$ to Ab_N, and whose morphisms are natural transformations.

Let $(M_j, \omega_j), j = 0, 1$ be compact monotone symplectic manifolds equipped with N-fold Maslov coverings $\mathrm{Lag}^N(M_j)$ and background classes b_j, and let $L_{01} \subset M_0^- \times M_1$ be an admissible Lagrangian correspondence equipped with a grading and a relative spin structure with background class $-\pi_0^* b_0 + \pi_1^* b_1$.

Definition 6.3.1. We can define a contravariant functor associated to L_{01},

$$\Phi_{L_{01}} : \mathrm{Don}(M_0) \to \mathrm{Don}(M_1)^\vee.$$

(a) On the level of objects, for every Lagrangian $L_0 \subset M_0$ we define a functor $\Phi_{L_{01}}(L_0) : \mathrm{Don}(M_1) \to \mathrm{Ab}_N$ by

$$L_1 \mapsto HF(L_0, L_{01}, L_1) = HF(L_0 \times L_1, L_{01})$$

on objects $L_1 \subset M_1$, and on morphisms

$$\begin{array}{ccc} HF(L_1, L_1') & \to & \mathrm{Hom}(HF(L_0, L_{01}, L_1), HF(L_0, L_{01}, L_1')) \\ f & \mapsto & \Big\{ g \mapsto \Phi_{\underline{S}_1}(g \otimes f) \Big\} \end{array}$$

is defined by the relative invariant associated to the quilted surface $\underline{S}_1$ shown in Figure 17,

$$\Phi_{\underline{S}_1} : HF(L_0, L_{01}, L_1) \otimes HF(L_1, L_1') \to HF(L_0, L_{01}, L_1').$$

(b) The functor on the level of morphisms associates to every $f \in HF(L_0, L_0')$ a natural transformation

$$\Phi_{L_{01}}(f) : \Phi_{L_{01}}(L_0') \to \Phi_{L_{01}}(L_0),$$

which maps objects $L_1 \subset M_1$ to the Ab_N-morphism

$$\Phi_{L_{01}}(f)(L_1) : \quad \begin{array}{ccc} HF(L_0', L_{01}, L_1) & \to & HF(L_0, L_{01}, L_1) \\ g & \mapsto & \Phi_{\underline{S}_0}(f \otimes g) \end{array}$$

defined by the relative invariant associated to the quilted surface $\underline{S}_0$ shown in Figure 17,

$$\Phi_{\underline{S}_0} : HF(L_0, L_0') \otimes HF(L_0', L_{01}, L_1) \to HF(L_0, L_{01}, L_1).$$

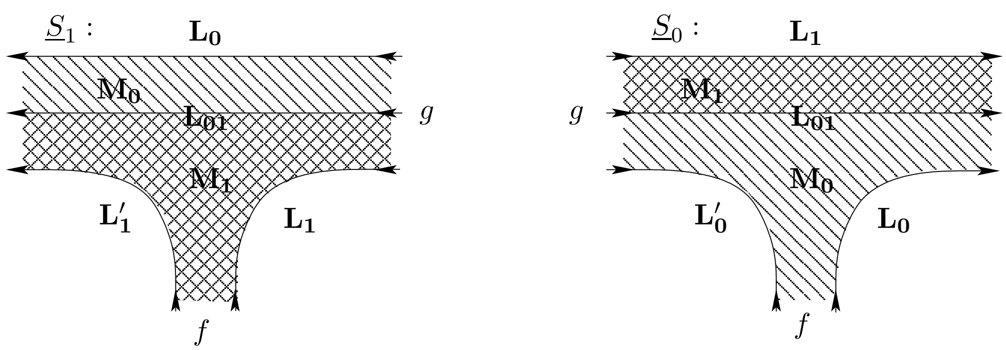


FIGURE 17. Lagrangian functor for morphisms

The composition axiom for the functors $\Phi_{L_{01}}(L_0)$ and the commutation axiom for the natural transformations $\Phi_{L_{01}}(f)$ follow from the gluing formula (34) applied to Figures 18 and 19. Clearly the functor $\Phi_{L_{01}}$ is unsatisfactory, since given two Lagrangian correspondences

$$L_{01} \subset M_0^- \times M_1, \quad L_{12} \subset M_1^- \times M_2$$

it is not clear how to define the composition of the associated functors

$$\Phi_{L_{01}} : \mathrm{Don}(M_0) \to \mathrm{Don}(M_1)^\vee, \quad \Phi_{L_{12}} : \mathrm{Don}(M_1) \to \mathrm{Don}(M_2)^\vee.$$

As a solution (perhaps not the only one) we will define in Section 6.4 a category sitting in between $\mathrm{Don}(M)$ and $\mathrm{Don}(M)^\vee$, whose image in $\mathrm{Don}(M)^\vee$ is roughly speaking the category of functors $\mathrm{Don}(M) \to \mathrm{Ab}_N$ of geometric origin. This will allow for the definition of composable functors for general Lagrangian correspondences in Section 6.5.

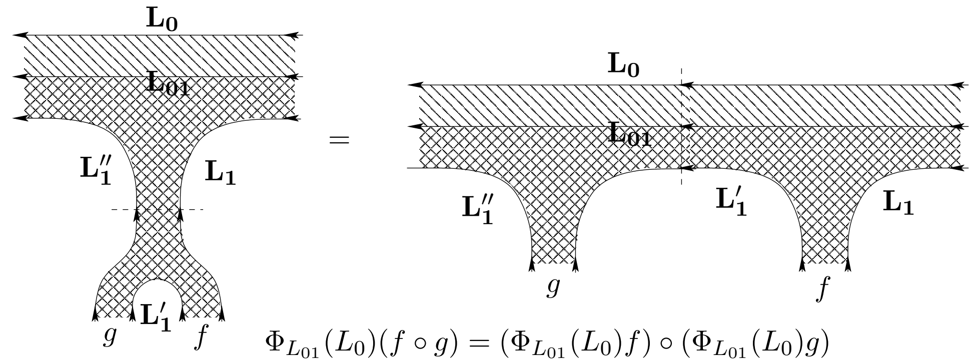


FIGURE 18. Composition axiom for Lagrangian functors

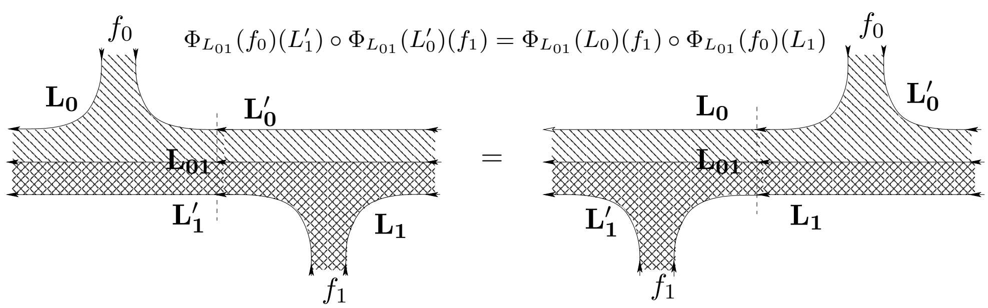


FIGURE 19. Commutation axiom for Lagrangian functors

6.4. **Generalized Donaldson-Fukaya category.** In this section we extend the Donaldson-Fukaya category $\mathrm{Don}(M)$ to a category $\mathrm{Don}^{\#}(M)$ which has generalized Lagrangian submanifolds as objects. Hence $\mathrm{Don}^{\#}(M)$ sits in between $\mathrm{Don}(M)$ and $\mathrm{Don}(M)^{\vee}$. One might draw an analogy here with the way square-integrable functions sit between smooth functions and distributions. $\mathrm{Don}^{\#}(M)$ admits a functor to $\mathrm{Don}(M)^{\vee}$, whose image is roughly speaking the subcategory of $\mathrm{Don}(M)^{\vee}$ generated by objects of geometric origin.

Remark 6.4.1. For readers familiar with the A_∞ set-up, we remark that it seems to be an open question whether the derived Fukaya category is self-dual. If it is, one could do without these constructions by working directly in the derived category of the dual of the Fukaya category. But then we would have to work directly with A_∞ categories from the beginning, which would substantially complicate the exposition.

Let (M, ω) be a symplectic manifold satisfying (M1-2) with monotonicity constant $\tau \geq 0$. We fix a Maslov cover $\mathrm{Lag}^N(M) \to M$ and a background class $b \in H^2(M, \mathbb{Z}_2)$.

Let $\underline{L}$ be a generalized Lagrangian submanifold of M, i.e. $\underline{L} = (L_{(-r)(-r+1)}, \ldots, L_{(-1)0})$ is a sequence of compact Lagrangian correspondences $L_{(i-1)i} \subset N_{i-1}^- \times N_i$ for a sequence $N_{-r}, \ldots, N_0$ of any length $r \geq 0$ of symplectic manifolds with $N_{-r} = \{\mathrm{pt}\}$ a point and $N_0 = M$. We call $\underline{L}$ *admissible* if each N_i satisfies (M1-2) with the monotonicity constant $\tau \geq 0$, the image of each $\pi_1(L_{(i-1)i})$ in $\pi_1(N_{i-1}^- \times N_i)$ is torsion, and each $L_{(i-1)i}$ has minimal Maslov number $N_{L_{(i-1)i}} \geq 3$. (Alternatively, we could work in the framework of Bohr-Sommerfeld monotone Lagrangians described in Remark 3.1.4.) An (admissible) Lagrangian submanifold $L \subset M$ is an (admissible) generalized Lagrangian with $r = 0$. Recall moreover that a generalized Lagrangian $\underline{L}$ is a generalized Lagrangian correspondence from $\{\mathrm{pt}\}$ to M, in the sense of Definition 2.1.1. We picture $\underline{L}$ as a sequence

$$\{\mathrm{pt}\} \xrightarrow{L_{(-r)(-r+1)}} N_{-r} \xrightarrow{L_{(-r+1)(-r+2)}} \cdots \xrightarrow{L_{(-2)(-1)}} N_{-1} \xrightarrow{L_{(-1)0}} N_0 = M \ .$$

Given two generalized Lagrangians $\underline{L}, \underline{L}'$ of M we can transpose one and concatenate them to a sequence of Lagrangian correspondences from $\{\mathrm{pt}\}$ to $\{\mathrm{pt}\}$,

$$\{\mathrm{pt}\} \xrightarrow{L_{(-r)(-r+1)}} \cdots \xrightarrow{L_{(-1)0}} N_0 = M = N_0' \xrightarrow{(L'_{(-1)0})^t} \cdots \xrightarrow{(L'_{(-r')(-r'+1)})^t} \{\mathrm{pt}\} \ .$$

The Floer cohomology of this sequence (as defined in Section 3.3 is the natural generalization of the Floer cohomology for pairs of Lagrangian submanifolds. Hence we define

$$HF(\underline{L}, \underline{L}') := HF(L_{(-r)(-r+1)}, \ldots, L_{(-1)0}, (L'_{(-1)0})^t, \ldots, (L'_{(-r')(-r'+1)})^t). \tag{69}$$

Note here that every such sequence arising from a pair of admissible generalized Lagrangians is automatically monotone in the sense of Section 3.3, by Lemma 3.1.3.

Definition 6.4.2. We define the *generalized Donaldson-Fukaya category*

$$\mathrm{Don}^{\#}(M) := \mathrm{Don}^{\#}(M, \mathrm{Lag}^N(M), \omega, b).$$

(a) The objects of $\mathrm{Don}^{\#}(M)$ are admissible generalized Lagrangians of M, equipped with orientations, a grading, and a relative spin structure (see Definitions 2.3.1, 3.3.1).
(b) The morphism spaces of $\mathrm{Don}^{\#}(M)$ are the $\mathbb{Z}_N$-graded Floer cohomology groups (see (69))
$$\mathrm{Hom}(\underline{L}, \underline{L}') := HF(\underline{L}, \underline{L}')[d]$$
given by choices of a perturbation datum and widths as described in Section 4.3; the second group is shifted by degree
$$d = \frac{1}{2}\Big(\sum_k \dim(N_k) + \sum_{k'} \dim(N'_{k'})\Big),$$
and for $\mathbb{Z}$-coefficients the Floer homology groups are modified by the inclusion of additional determinant lines as below in (70).
(c) The composition of morphisms in $\mathrm{Don}^{\#}(M)$,
$$\begin{aligned} \mathrm{Hom}(\underline{L}, \underline{L}') \times \mathrm{Hom}(\underline{L}', \underline{L}'') &\longrightarrow \quad \mathrm{Hom}(\underline{L}, \underline{L}'') \\ (f, g) \qquad\qquad &\longmapsto f \circ g := \Phi_{\underline{P}}(f \otimes g) \end{aligned}$$
is defined by the relative invariant $\Phi_{\underline{P}}$ associated to the quilted half-pair of pants surface $\underline{P}$ in Figure 20, with orderings given as follows: The relative invariant is independent of the orderings of the patches with one outgoing end by Remark 4.2.6. The remaining patches have two incoming and zero incoming edges, and these are ordered from the top down, that is, starting with those furthest from the boundary.
(d) Identities $1_{\underline{L}} \in \mathrm{Hom}(\underline{L}, \underline{L})$ are furnished by relative invariants $1_{\underline{L}} := \Phi_{\underline{S}} \in \mathrm{Hom}(\underline{L}, \underline{L})$ associated to the quilted disk $\underline{S}$ with a single outgoing end, as in Figure 21, with components ordered from the bottom up, that is, with the outer-most patches ordered first.

Note that both the identity and composition are degree 0 by (33). The identity and associativity axioms are satisfied with $\mathbb{Z}_2$ coefficients by Theorem 4.2.8 applied to the quilted versions of Figures 15, 16.

Remark 6.4.3. To obtain the axioms with $\mathbb{Z}$ coefficients requires a modification of the Floer cohomology groups, incorporating the determinant lines in a more canonical way. This will be treated in detail in [46], so we only give a sketch here: For each intersection point $\underline{x} \in \mathcal{I}(\underline{L}, \underline{L}')$ we say that an *orientation* for $\underline{x}$ consists of the following data: A partially quilted surface[13] $\underline{S}$ with a single end, complex vector bundles $\underline{E}$ over $\underline{S}$, and totally real subbundles $\underline{F}$ over the boundaries and seams, such that near infinity on the strip-like ends $\underline{E}$ and $\underline{F}$ are given by $(T_{x_i}M_i)$ and $T_{\underline{x}}\underline{L}, T_{\underline{x}}\underline{L}'$; a real Cauchy-Riemann operator $D_{\underline{E},\underline{F}}$; an orientation on the determinant line $\det(D_{\underline{E},\underline{F}})$. We say that two orientations for $\underline{x}$ are *isomorphic* if the two problems have the same bundles $\underline{E}$, and the surfaces, boundary and seam conditions are deformation equivalent after a possible re-ordering of boundary components etc., and the orientations are related by the isomorphism of determinant lines arising from re-ordering. Let $O(\underline{x})$ denote the space of isomorphism classes of orientations for $\underline{x}$. Define

$$\widetilde{CF}(\underline{L}, \underline{L}') = \bigoplus_{\underline{x} \in \mathcal{I}(\underline{L}, \underline{L}')} O(\underline{x}) \otimes_{\mathbb{Z}_2} \mathbb{Z}. \tag{70}$$

The Floer coboundary operator extends canonically to an operator of degree 1 on $\widetilde{CF}(\underline{L}, \underline{L}')$, and let $\widetilde{HF}(\underline{L}, \underline{L}')$ denote its cohomology. This is similar to the definition given in e.g. Seidel [37, 12.19], except that we allow more general surfaces. The group $\widetilde{HF}(\underline{L}, \underline{L}')$ is of infinite rank over $\mathbb{Z}$, but it has finite rank over a suitable graded-commutative Novikov ring generated by determinant lines.

The relative invariants extend to operators $\widetilde{\Phi}_{\underline{S}}$ operating on the tensor product of (extended) Floer cohomologies. In particular, the quilted pair of pants defines an operator

$$\widetilde{\Phi}_{\underline{P}} : \widetilde{HF}(\underline{L}, \underline{L}') \otimes \widetilde{HF}(\underline{L}', \underline{L}'') \to \widetilde{HF}(\underline{L}, \underline{L}'').$$

If we fix orientations for each generator $\langle \underline{x} \rangle$, as in the definition of HF, then the gluing sign for the first gluing (to the second incoming end) in the proof of associativity, Figure 15, is $+1$. For the second gluing (to the first incoming end) when applied to $\langle \underline{x}_1 \rangle \otimes \langle \underline{x}_2 \rangle \otimes \langle \underline{x}_3 \rangle$ the sign is $(-1)^{|\underline{x}_3| \frac{1}{2} \sum_i \dim(N_i^{\underline{L}_1})}$. In addition, the two gluings induce different orderings of patches in the glued quilted surface, which are related by the additional sign $(-1)^{\left(\frac{1}{2}\sum_i \dim(N_i^{\underline{L}_1})\right)\left(\frac{1}{2}\sum_i \dim(N_i^{\underline{L}_2})\right)}$. Combined together, these factors cancel the sign arising from the re-ordering of determinants in the definitions of $\widetilde{\Phi}_{\underline{P}}(\widetilde{\Phi}_{\underline{P}}(\langle \underline{x}_1 \rangle \otimes \langle \underline{x}_2 \rangle) \otimes \langle \underline{x}_3 \rangle)$ and $\widetilde{\Phi}_{\underline{P}}(\langle \underline{x}_1 \rangle \otimes \widetilde{\Phi}_{\underline{P}}(\langle \underline{x}_2 \rangle \otimes \langle \underline{x}_3 \rangle))$.

The identity axiom involves gluing a quilted cup with a quilted pair of pants; the orderings of the patches for the quilted cup and quilted pants above are chosen so that the gluing sign for gluing the quilted cup with quilted pants to obtain a quilted strip is $+1$ for gluing into the second argument, and $(-1)^{|\underline{x}| \frac{1}{2} \sum_i \dim(N_i)}$ for gluing into the first argument. Again, the additional sign is absorbed into the isomorphism of determinant lines induced by gluing.

Remark 6.4.4. To simplify pictures of quilts we will use the following conventions indicated in Figure 22 : A generalized Lagrangian submanifold $\underline{L}$ of M can be used as "boundary condition" for a surface mapping to M in the sense that the boundary arc that is labeled by the sequence $\underline{L} = (L_{(-r)(-r+1)}, \ldots, L_{(-1)0})$ of Lagrangian correspondences from $\{\mathrm{pt}\}$ to

[13] See [46] for the definition of partial quilts. For example, the standard cup orientation for $\underline{x} = (x_1, \ldots, x_N)$ will use unquilted cups S_i associated to each $T_{x_i}M_i$, and identified via seams on the strip-like ends.

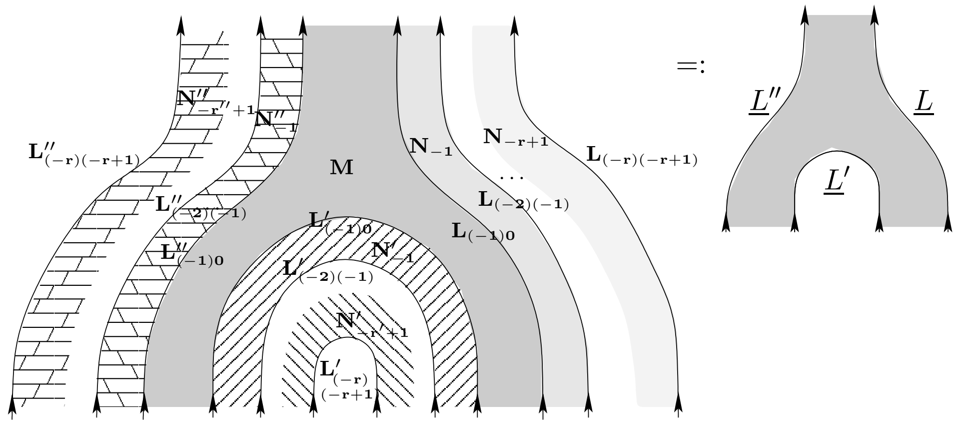


FIGURE 20. Quilted pair of pants

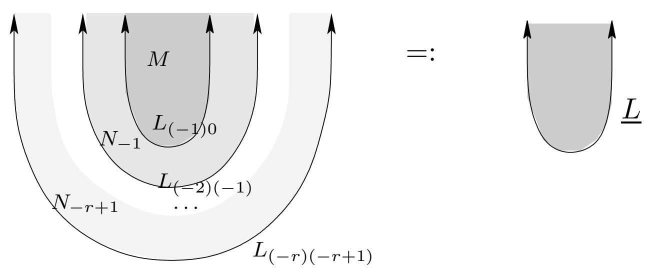


FIGURE 21. Quilted identity

M is replaced by a sequence of strips mapping to $N_{-1}, \ldots, N_{-r+1}$, with seam conditions in $L_{(-1)0}, \ldots, L_{(-r+2)(-r+1)}$ and a final boundary condition in $L_{(-r)(-r+1)}$.

Similarly, a generalized Lagrangian correspondence (see Section 6.5) $\underline{L}$ between M_- and M_+ can be used as "seam condition" between surfaces mapping to $M_\pm$ in the sense that the seam that is labeled by the sequence $\underline{L} = (L_{01}, \ldots, L_{(r-1)r})$ of Lagrangian correspondences from M_- to M_+ is replaced by a sequence of strips mapping to $M_1, \ldots, M_{r-1}$ with seam conditions in $L_{01}, \ldots, L_{(r-1)r}$.

Remark 6.4.5. As for $\mathrm{Don}(M)$, the category $\mathrm{Don}^\#(M)$ is independent of the choices of perturbation data and widths up to isomorphism of categories, see Remark 6.1.5 and Proposition 4.3.1.

Proposition 6.4.6. *The map $\underline{L} \mapsto \underline{L}^\vee$, which for all general Lagrangians $\underline{L}$ in M is given by*

$$\underline{L}^\vee(L_0) := \mathrm{Hom}(\underline{L}, L_0) = HF(L_{-r(-r+1)}, \ldots, L_{(-1)0}, L_0)[d]$$

for all Lagrangian submanifolds $L_0 \subset M$ and with degree shift $d = \frac{1}{2}\sum_k \dim(N_k)$, extends to a contravariant functor $\mathrm{Don}^\#(M) \to \mathrm{Don}(M)^\vee$.

Proof. The functor $\underline{L}^\vee : \mathrm{Don}(M) \to \mathrm{Ab}_N$ can be defined on morphisms by

$$\underline{L}^\vee : \begin{array}{ccc} \mathrm{Hom}(L_1, L_1') & \to & \mathrm{Hom}(\mathrm{Hom}(\underline{L}, L_1), \mathrm{Hom}(\underline{L}, L_1')) \\ f & \mapsto & \left\{ g \mapsto g \circ f = \Phi_{\underline{P}}(g \otimes f) \right\} \end{array}$$

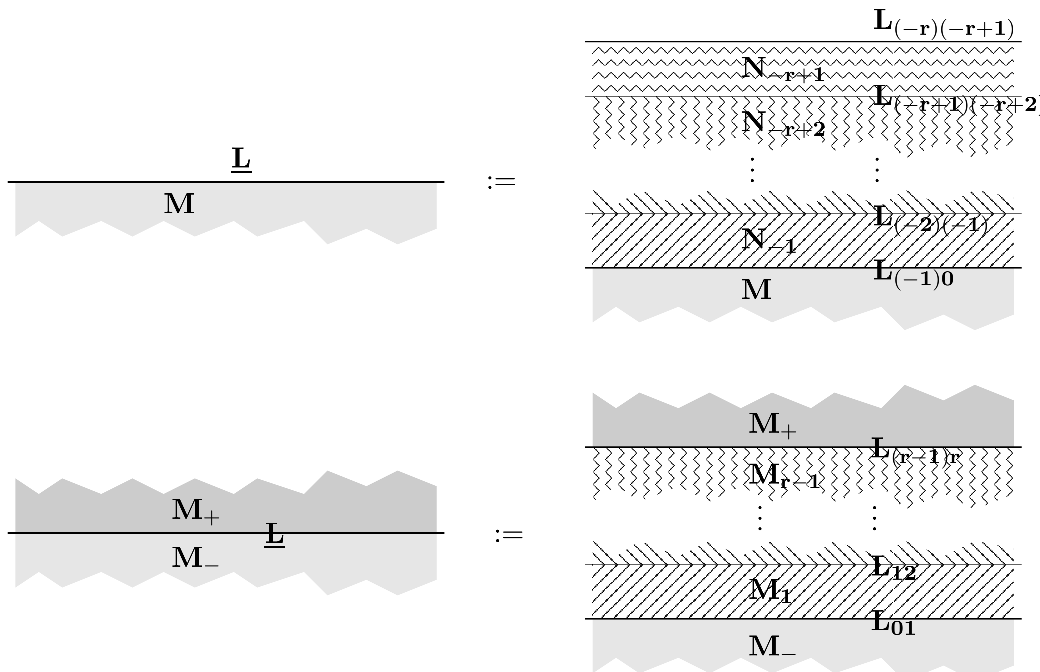


FIGURE 22. Conventions on using generalized Lagrangians and Lagrangian correspondences as boundary and seam conditions

using the composition on $\mathrm{Don}^\#(M)$. To morphisms $f \in \mathrm{Hom}(\underline{L}, \underline{L}')$ of $\mathrm{Don}^\#(M)$ we can then associate the natural transformation $f^\vee : \underline{L}'^\vee \to \underline{L}^\vee$, which maps every object $L_1 \subset M$ of $\mathrm{Don}(M)$ to the following Ab_N-morphism $f^\vee(L_1)$:

$$\begin{aligned} \mathrm{Hom}(\underline{L}', L_1) \to &\quad \mathrm{Hom}(\underline{L}, L_1) \\ g \qquad &\mapsto f \circ g = \Phi_{\underline{P}}(f \otimes g), \end{aligned}$$

again given by composition on $\mathrm{Don}^\#(M)$. The axioms follow from the gluing formula (34) applied to jazzed-up versions of Figures 18 and 19 (which show the example $\underline{L} = (L_0, L_{01})$, $\underline{L}' = (L_0', L_{01})$). In this case the orientations are independent of the ordering of patches since all have one boundary component and one outgoing end. □

6.5. **Composable functors for Lagrangian correspondences.** Let M_0 and M_1 be two symplectic manifolds satisfying (M1-2) with the same monotonicity constant $\tau \geq 0$. We fix Maslov covers $\mathrm{Lag}^N(M_i) \to M_i$ and background classes $b_i \in H^2(M_i, \mathbb{Z}_2)$. Given a smooth, compact Lagrangian correspondence $L_{01} \subset M_0^- \times M_1$ we can now define a functor $\Phi(L_{01}) : \mathrm{Don}^\#(M_0) \to \mathrm{Don}^\#(M_1)$. For this purpose we need admissibility assumptions and an additional brane structure on L_{01}. We will give the precise definitions in Section 6.7. For now let us use the preliminary definition that for any $\underline{L} \in \mathrm{Obj}(\mathrm{Don}^\#(M_0))$ we have $(\underline{L}, L_{01}) \in \mathrm{Obj}(\mathrm{Don}^\#(M_1))$.

Definition 6.5.1. We define a functor

$$\Phi(L_{01}) : \mathrm{Don}^\#(M_0) \to \mathrm{Don}^\#(M_1).$$

(a) On the level of objects, $\Phi(L_{01})$ is defined by appending the Lagrangian correspondence to the sequence of Lagrangian correspondences: For a generalized Lagrangian

$\underline{L} = (L_{-r(-r+1)}, \ldots, L_{(-1)0})$ of M_0 with corresponding sequence of symplectic manifolds $(\{\mathrm{pt}\}, N_{-r+1}, \ldots, N_{-1}, M_0)$ we put

$$\Phi(L_{01})(\underline{L}) := (\underline{L}, L_{01}) := (L_{(-r)(-r+1)}, \ldots, L_{(-1)0}, L_{01})$$

with the corresponding sequence $(\{\mathrm{pt}\}, N_{-r+1}, \ldots, N_{-1}, M_0, M_1)$ of symplectic manifolds.

(b) On the level of morphisms, for any pair $\underline{L}, \underline{L}'$ of generalized Lagrangians in M_0, we define

$$\Phi(L_{01}) := \Phi_{\underline{S}} : \mathrm{Hom}(\underline{L}, \underline{L}') \to \mathrm{Hom}(\Phi(L_{01})(\underline{L}), \Phi(L_{01})(\underline{L}'))$$

to be the relative invariant associated to the quilted surface $\underline{S}$ with two punctures and one interior circle, as in Figure 23.

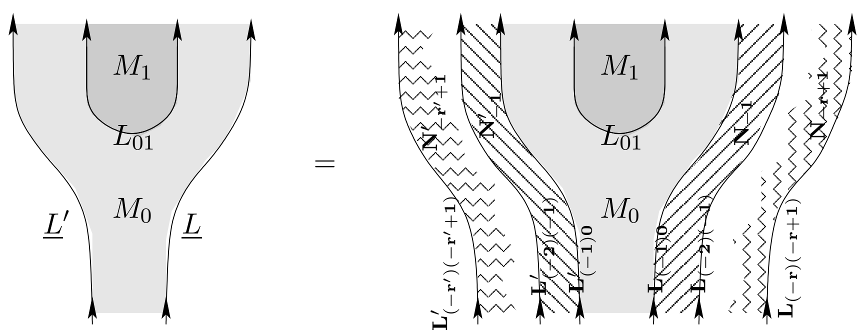


FIGURE 23. The Lagrangian correspondence functor $\Phi(L_{01})$ on morphisms

Remark 6.5.2. In the case that M_1 is a point, the map for morphisms is the dual of the pair of pants product.

For composable morphisms $f \in \mathrm{Hom}(\underline{L}, \underline{L}')$, $g \in \mathrm{Hom}(\underline{L}', \underline{L}'')$ one shows $\Phi_{L_{01}}(f \circ g) = \Phi_{L_{01}}(f) \circ \Phi_{L_{01}}(g)$ by applying (34) to the gluings shown in Figure 24 (simplifying the picture by the convention of Figure 22), which yield homotopic quilted surfaces. The gluing signs for both gluings are positive. Similarly, the second gluing shows that $\Phi(L_{01})(1_{\underline{L}}) = 1_{\Phi(L_{01})(\underline{L})}$, since we have ordered the patches of the quilted cup from the outside in.

Remark 6.5.3. The surfaces of the first gluing in Figure 24 can equivalently be represented as degenerations of one quilted disk. The corresponding one-parameter family in Figure 25 is the one-dimensional *multiplihedron* of Stasheff, see [42], [24, p. 113], to which we will return in [25].

With this new definition, any two functors associated to smooth, compact, admissible Lagrangian correspondences, $\Phi(L_{01}) : \mathrm{Don}^{\#}(M_0) \to \mathrm{Don}^{\#}(M_1)$ and $\Phi(L_{12}) : \mathrm{Don}^{\#}(M_1) \to \mathrm{Don}^{\#}(M_2)$, are clearly composable. More generally, consider a sequence

$$\underline{L}_{0r} = (L_{01}, \ldots, L_{(r-1)r})$$

of Lagrangian correspondences $L_{(j-1)j} \subset M_{j-1}^- \times M_j$. (That is, $\underline{L}_{0r}$ is a generalized Lagrangian correspondence from M_0 to M_r in the sense of Definition 2.1.1.) Assume that $\underline{L}_{0r}$ is admissible in the (preliminary) sense that, for any $\underline{L} \in \mathrm{Obj}(\mathrm{Don}^{\#}(M_0))$ and $k = 1, \ldots, r$,

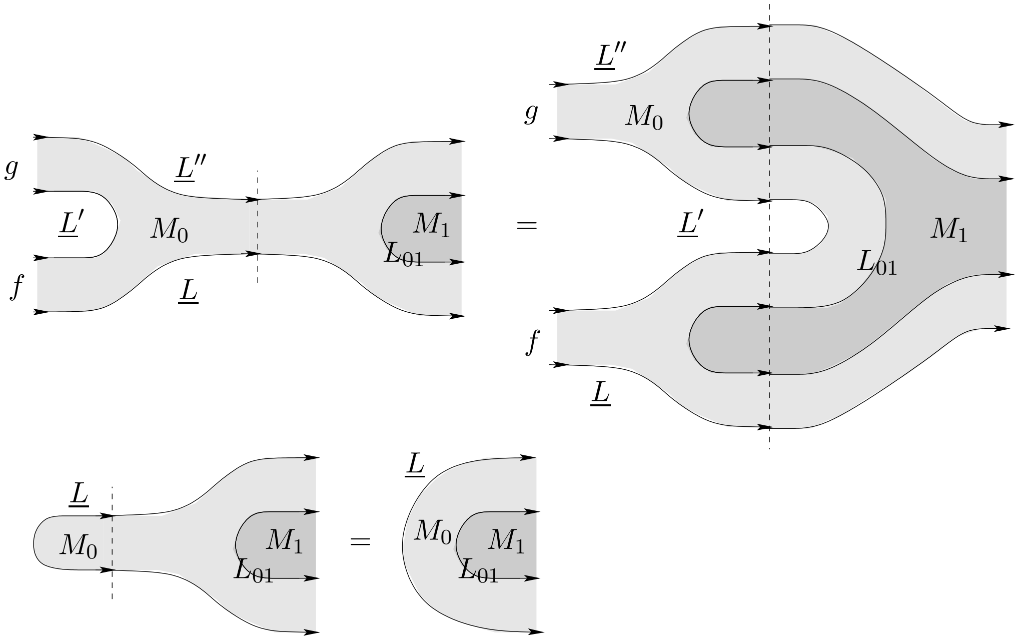


FIGURE 24. The functor axioms for $\Phi_{L_{01}}$

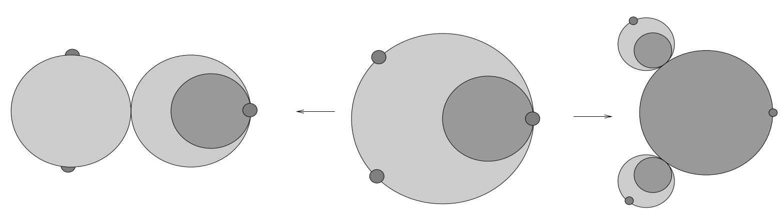


FIGURE 25. Degeneration view of the first functor axiom

we have $(\underline{L}, L_{01}, \ldots, L_{(k-1)k}) \in \mathrm{Obj}(\mathrm{Don}^{\#}(M_k))$. We can then define a functor by concatenation

$$\Phi(\underline{L}_{0r}) := \Phi(L_{01}) \circ \ldots \circ \Phi(L_{(r-1)r}) : \mathrm{Don}^{\#}(M_0) \to \mathrm{Don}^{\#}(M_r). \tag{71}$$

Remark 6.5.4. On the level of morphisms, the functor $\Phi(\underline{L}_{0r})$ is given by the relative invariant associated to the quilted surface $\underline{S}$ in Figure 26,

$$\Phi(\underline{L}_{0r}) = \Phi_{\underline{S}} : \mathrm{Hom}(\underline{L}, \underline{L}') \to \mathrm{Hom}(\Phi(\underline{L}_{0r})(\underline{L}), \Phi(\underline{L}_{0r})(\underline{L}'))$$

for all generalized Lagrangian submanifolds $\underline{L}, \underline{L}' \in \mathrm{Obj}(\mathrm{Don}^{\#}(M_0))$, with patches with two outgoing ends ordered from bottom up. This follows from (34) applied to the gluing shown in Figure 26.

6.6. **Functors for composed Lagrangian correspondences and graphs.** As first application of our main Theorem 1.0.1 we will show that the composed functor $\Phi(L_{01}) \circ \Phi(L_{12}) : \mathrm{Don}^{\#}(M_0) \to \mathrm{Don}^{\#}(M_2)$ is isomorphic to the functor $\Phi(L_{01} \circ L_{12})$ of the geometric composition $L_{01} \circ L_{12} \subset M_0^- \times M_2$, if the latter is embedded. More precisely and more generally, we have the following result.

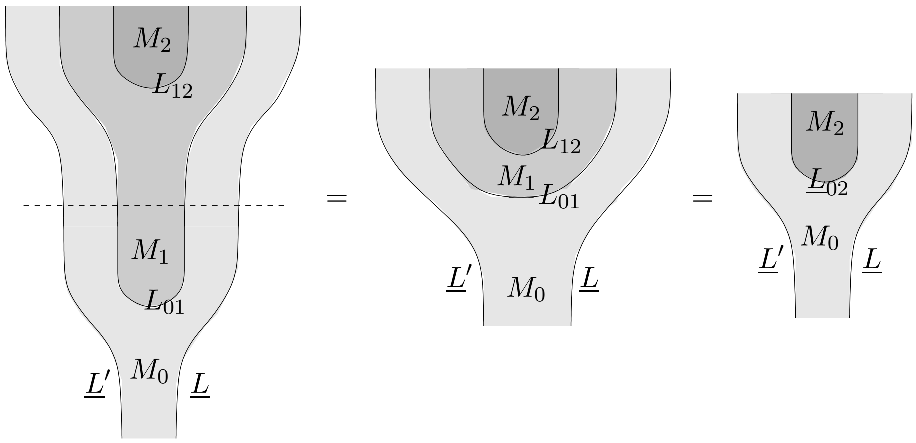


FIGURE 26. The composition $\Phi(L_{01}) \circ \ldots \circ \Phi(L_{(r-1)r})$ is given by a relative invariant for the sequence $\underline{L}_{0r} = (L_{01}, \ldots, L_{(r-1)r})$. (Here $r = 2$.)

Theorem 6.6.1. *Let $\underline{L}_{0r} = (L_{01}, \ldots, L_{(r-1)r})$ and $\underline{L}'_{0r'} = (L'_{01}, \ldots, L'_{(r'-1)r'})$ be two admissible generalized Lagrangian correspondence from M_0 to $M_r = M_{r'}$. Suppose that they are equivalent in the sense of Section 2.1 through a series of embedded compositions of consecutive Lagrangian correspondences, and such that each intermediate generalized Lagrangian correspondence is admissible (see Section 6.7).*

Then for any two generalized Lagrangian submanifolds $\underline{L}, \underline{L}' \in \mathrm{Obj}(\mathrm{Don}^{\#}(M_0))$ there is an isomorphism

$$\Psi : \mathrm{Hom}(\Phi(\underline{L}_{0r})(\underline{L}), \Phi(\underline{L}_{0r})(\underline{L}')) \to \mathrm{Hom}(\Phi(\underline{L}'_{0r'})(\underline{L}), \Phi(\underline{L}'_{0r'})(\underline{L}'))$$

which intertwines the functors on the morphism level,

$$\Psi \circ \Phi(\underline{L}_{0r}) = \Phi(\underline{L}'_{0r'}) \;:\; \mathrm{Hom}(\underline{L}, \underline{L}') \to \mathrm{Hom}(\Phi(\underline{L}'_{0r'})(\underline{L}), \Phi(\underline{L}'_{0r'})(\underline{L}')).$$

Proof. By assumption there exists a sequence of admissible generalized Lagrangian correspondences $\underline{L}^j$ connecting $\underline{L}^0 = \underline{L}_{0r}$ to $\underline{L}^N = \underline{L}'_{0r'}$. In each step two consecutive Lagrangian correspondences L_-, L_+ in the sequence $\underline{L}^j = (\ldots, L_-, L_+, \ldots)$ are replaced by their embedded, monotone composition $L_- \circ L_+$ in $\underline{L}^{j\pm1} = (\ldots, L_- \circ L_+, \ldots)$. To each $\underline{L}^j$ we associate seam conditions for the quilted surface $\underline{S}^j$ on the right of Figure 26. Replacing the consecutive correspondences by their composition corresponds to shrinking a strip in this surface. So Theorem 5.4.1 provides an isomorphism $\Psi_{\underline{e}^j_+}$ associated to the outgoing end $\underline{e}^j_+$ of each surface $\underline{S}^j$ such that $\Psi_{\underline{e}^j_+} \circ \Phi_{\underline{S}^j} = \Phi_{\underline{S}^{j\pm1}}$. Figure 27 shows an example of this degeneration. The isomorphism Ψ is given by concatenation of the isomorphisms $\Psi_{\underline{e}^j_+}$ (and their inverses in case the composition is between $\underline{L}^j$ and $\underline{L}^{j-1}$). It intertwines $\Phi_{\underline{S}^0} = \Phi(\underline{L}_{0r})$ and $\Phi_{\underline{S}^N} = \Phi(\underline{L}'_{0r'})$ as claimed. $\square$

Next, let $\psi : M_0 \to M_1$ be a symplectomorphism and $\mathrm{graph}\,\psi \subset M_0^- \times M_1$ its graph. The functor $\Phi(\psi)$ defined in Section 6.2 extends to a functor

$$\Phi(\psi) : \mathrm{Don}^{\#}(M_0) \to \mathrm{Don}^{\#}(M_1)$$

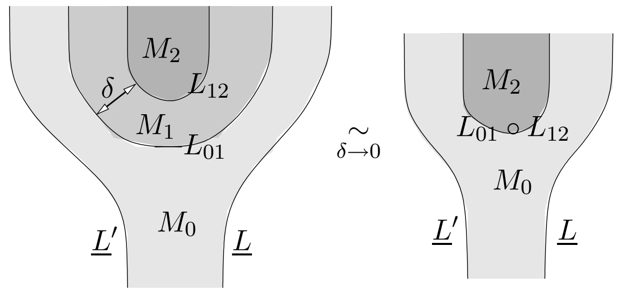


FIGURE 27. Isomorphism between the functors $\Phi(L_{01})\circ\Phi(L_{12})$ and $\Phi(L_{01}\circ L_{12})$

on the level of objects by

$$\underline{L}=(L_{-r(-r+1)},\ldots,L_{-10})\mapsto(L_{-r(-r+1)},\ldots,(1_{N_{-1}}\times\psi)(L_{-10}))=:\psi(\underline{L}).$$

On the level of morphisms, the functor $\Phi(\psi):\mathrm{Hom}(\underline{L},\underline{L}')\to\mathrm{Hom}(\Phi(\psi)(\underline{L}),\Phi(\psi)(\underline{L}'))$ is defined by $\langle(x_{-r},\ldots,x_{-1},x_0)\rangle\mapsto\langle(x_{-r},\ldots,x_{-1},\psi(x_0)\rangle$ on the generators $\mathcal{I}(\underline{L},\underline{L}')$ of the chain complex. As another application of our main Theorem we will show that this functor is in fact isomorphic to the functor $\Phi(\operatorname{graph}\psi):\mathrm{Don}^{\#}(M_0)\to\mathrm{Don}^{\#}(M_1)$ that we defined for the Lagrangian correspondence $\operatorname{graph}\psi$.

Proposition 6.6.2. *$\Phi(\psi)$ and $\Phi(\operatorname{graph}\psi)$ are canonically isomorphic as functors from $\mathrm{Don}^{\#}(M_0)$ to $\mathrm{Don}^{\#}(M_1)$. More precisely, there exists a canonical natural transformation $\alpha:\Phi(\psi)\to\Phi(\operatorname{graph}\psi)$, that is $\alpha(\underline{L})\in\mathrm{Hom}(\Phi(\psi)(\underline{L}),\Phi(\operatorname{graph}\psi)(\underline{L}))$ for every $\underline{L}\in\mathrm{Obj}(\mathrm{Don}^{\#}(M_0))$ such that $\alpha(\underline{L})\circ\Phi(\operatorname{graph}\psi)(f)=\Phi(\psi)(f)\circ\alpha(\underline{L}')$ for all $f\in\mathrm{Hom}(\underline{L},\underline{L}')$, and all $\alpha(\underline{L})$ are isomorphisms in $\mathrm{Don}^{\#}(M_1)$.*

Proof. Consider a generalized Lagrangian submanifold $\underline{L}=(L_{(-r)(-r+1)},\ldots,L_{(-1)0})\in\mathrm{Obj}(\mathrm{Don}^{\#}(M_0))$. By Theorem 1.0.1 we have canonical isomorphisms from

$$\begin{aligned}\mathrm{Hom}(\Phi(\psi)\underline{L},\Phi(\operatorname{graph}\psi)\underline{L})&=\mathrm{Hom}(\psi(\underline{L}),(\underline{L},\operatorname{graph}\psi))\\&=\mathrm{Hom}(\ldots(1\times\psi)(L_{(-1)0}),(\operatorname{graph}\psi)^t,(L_{(-1)0})^t\ldots)\end{aligned}$$

to all three of

$$\begin{aligned}&\mathrm{Hom}(\ldots(1\times\psi)(L_{(-1)0}),(L_{(-1)0}\circ(\operatorname{graph}\psi))^t\ldots)=\mathrm{Hom}(\psi(\underline{L}),\psi(\underline{L})),\\&\mathrm{Hom}(\ldots L_{(-1)0},\operatorname{graph}\psi,(\operatorname{graph}\psi)^t,(L_{(-1)0})^t\ldots)=\mathrm{Hom}((\underline{L},\operatorname{graph}\psi)(\underline{L},\operatorname{graph}\psi)),\\&\mathrm{Hom}(\ldots(1\times\psi)(L_{(-1)0})\circ\operatorname{graph}(\psi^{-1}),(L_{(-1)0})^t\ldots)=\mathrm{Hom}(\underline{L},\underline{L}),\end{aligned}$$

see Figure 28.[14] The isomorphisms are by $(\psi(\underline{x}),\underline{x})\mapsto\psi(\underline{x})$, $(\underline{x},\psi(x_0),\underline{x})$, or $\underline{x}$, respectively, on the level of perturbed intersection points $\underline{x}=(x_{-r},\ldots,x_0)\in\mathcal{I}(\underline{L},\underline{L})$. The first two isomorphisms also intertwine the identity morphisms $1_{\psi(\underline{L})}\cong 1_{(\underline{L},\operatorname{graph}\psi)}$ by Theorem 5.0.3 and the degeneration of the quilted identity indicated in Figure 28; this is the identity axiom for the functor $\Phi(\operatorname{graph}\psi)$. The identity axiom for $\Phi(\psi)$ implies that the above isomorphisms (their composition which coincides with $\Phi(\psi):\mathrm{Hom}(\underline{L},\underline{L})\to\mathrm{Hom}(\psi(\underline{L}),\psi(\underline{L}))$) also intertwine $1_{\underline{L}}$ with $1_{\psi(\underline{L})}$. We define $\alpha(\underline{L})\in\mathrm{Hom}(\Phi(\psi)\underline{L},\Phi(\operatorname{graph}(\psi))\underline{L})$ to be the element corresponding to the identities $1_{\Phi(\psi)(\underline{L})}\cong 1_{\Phi(\operatorname{graph}\psi)(\underline{L})}\cong 1_{\underline{L}}$ under these isomorphisms.

[14]Strictly speaking, one has to apply the shift functor Ψ_{M_0} of Definition 6.6.4 to adjust the relative spin structure on $\underline{L}$. However, $HF(\Psi_{M_0}(\underline{L}),\Psi_{M_0}(\underline{L}))$ is canonically isomorphic to $HF(\underline{L},\underline{L})$.

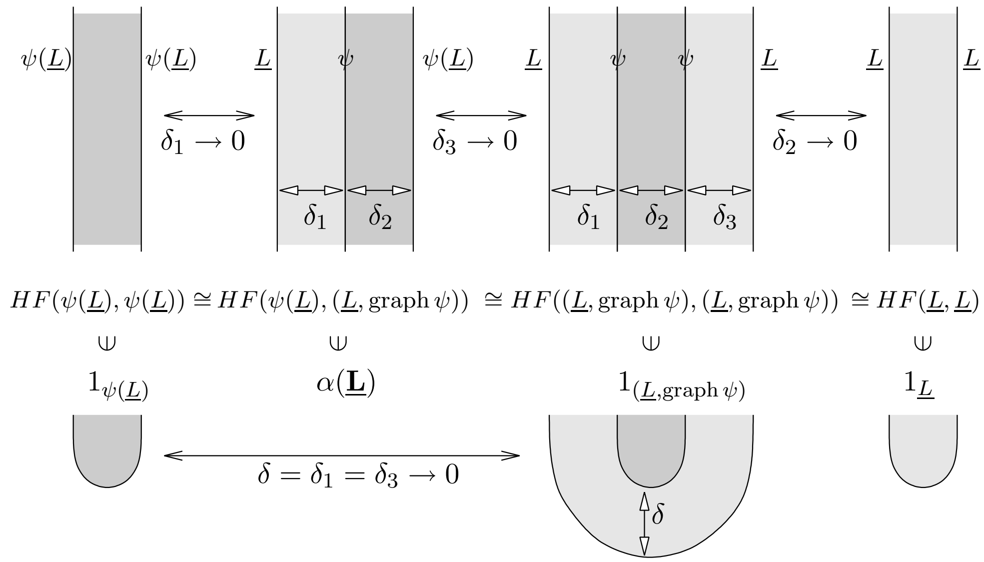


FIGURE 28. Natural isomorphisms of Floer cohomology groups and definition of the natural transformation α : The light and dark shaded surfaces are mapped to M_0 and M_1 respectively and we abbreviate $\operatorname{graph}\psi$ by ψ and $\Phi(\psi)(\underline{L})$ by $\psi(\underline{L})$.

Each $\alpha(\underline{L})$ is an isomorphism since $\alpha(\underline{L}) \circ f = I_1(f)$ for all $f \in HF(\Phi(\operatorname{graph}\psi)\underline{L}, \underline{L}'')$ and $f \circ \alpha(\underline{L}) = I_2(f)$ for all $f \in HF(\underline{L}'', \Phi(\psi)\underline{L})$, with the isomorphisms (again from Theorem 5.0.3)

$$I_1 : HF((\underline{L}, \operatorname{graph}\psi), \underline{L}'') \to HF(\psi(\underline{L}), \underline{L}''),$$
$$I_2 : HF(\underline{L}'', \psi(\underline{L})) \to HF(\underline{L}'', (\underline{L}, \operatorname{graph}\psi)).$$

These identities can be seen from the gluing formula (34) and Theorem 5.0.3, applied to the gluings and degenerations indicated in Figure 29. The quilted surfaces can be deformed to a strip resp. a quilted strip (which corresponds to a strip in $M_0^- \times M_1$). These relative invariants both are the identity since the solutions are counted without quotienting by $\mathbb{R}$, see Example 4.1.6. For $f \in \operatorname{Hom}(\underline{L}, \underline{L}')$ this already shows the first equality in

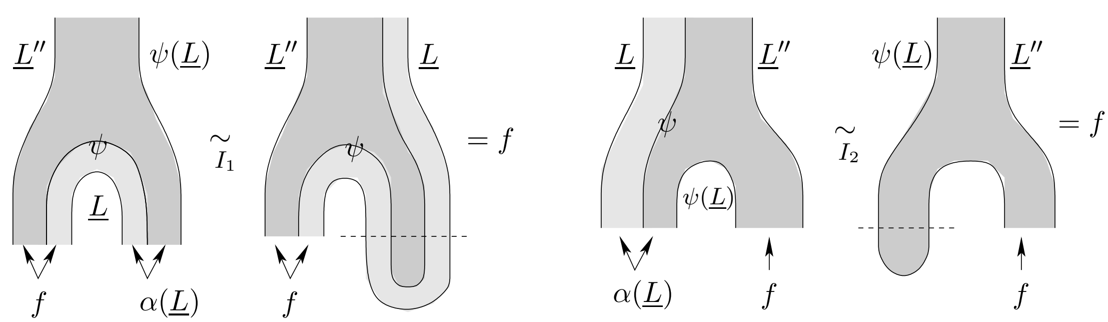


FIGURE 29. $\alpha(\underline{L})$ is an isomorphism in $\operatorname{Don}^{\#}(M_1)$

$\Phi(\psi)(f) \circ \alpha(\underline{L}') = I(f) = \alpha(\underline{L}) \circ \Phi(\operatorname{graph}\psi)(f)$ with the isomorphism $I : HF(\underline{L}, \underline{L}') \to$

$HF(\psi(\underline{L}), (\underline{L}', \operatorname{graph}\psi))$. More precisely, on the chain level for $x \in \mathcal{I}(\underline{L}, \underline{L}')$

$$\Phi(\psi)(x) \circ \alpha(\underline{L}') = (\psi(x), x) = \alpha(\underline{L}) \circ \Phi(\operatorname{graph}\psi)(x).$$

The second identity is proven by repeatedly using Theorem 5.4.1 and the gluing formula (34), see Figure 30. □

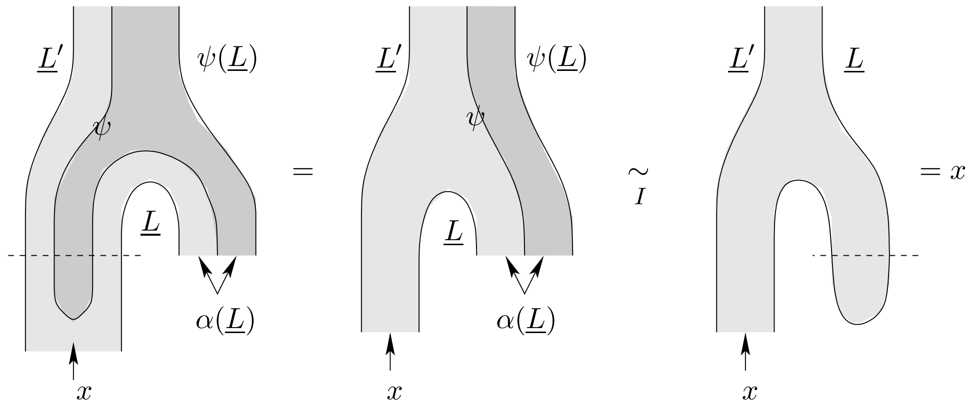


FIGURE 30. Isomorphism of functors for a symplectomorphism and its graph, using shrinking strips

Remark 6.6.3. There is an analytically easier proof of the previous Proposition 6.6.2 in the special case when one of the Lagrangian correspondences is the graph of a symplectomorphism: Instead of shrinking a strip as in Theorems 5.0.3 and Theorem 5.4.1 one can apply the symplectomorphism to the whole strip; for a suitable choice of perturbation data it then attaches smoothly to the other surface in the quilt, and the seam can be removed.

The functor $\Phi(\mathrm{Id}_{\mathrm{M}_0})$ associated to the identity map on M_0 clearly is the identity functor on $\mathrm{Don}^\#(M_0)$. So Proposition 6.6.2 gives a (rather indirect) isomorphism between the functor for the diagonal and the identity functor. To be more precise, taking into account the relative spin structure of the diagonal, we need to introduce the following shift functor.

Definition 6.6.4. We define a shift functor

$$\Psi_{M_0}: \ \mathrm{Don}^\#(M_0, \mathrm{Lag}^N(M_0), \omega_0, b_0) \to \mathrm{Don}^\#(M_0, \mathrm{Lag}^N(M_0), \omega_0, b_0 - w_2(M_0)).$$

(a) On the level of objects, Ψ_{M_0} maps every generalized Lagrangian $\underline{L} \in \mathrm{Don}^\#(M_0)$ to itself but shifts the relative spin structure to one with background class $b_0 - w_2(M_0)$, as explained in [46].
(b) On the level of morphisms, $\Psi_{M_0} : \mathrm{Hom}(\underline{L}, \underline{L}') \to \mathrm{Hom}(\Psi_{M_0}(\underline{L}), \Psi_{M_0}(\underline{L}'))$ is the canonical isomorphism for shifted spin structures from [46].

Remark 6.6.5. Let $\Delta \subset M_0^- \times M_0$ denote the diagonal. Throughout, we will equip Δ with the orientation and relative spin structure that are induced by the projection to the second factor (see [46]). Then Δ is an admissible Lagrangian correspondence from M_0 to M_1, where $M_1 = M_0$ with the same symplectic structure $\omega_1 = \omega_0$ and Maslov cover $\mathrm{Lag}^N(M_1) = \mathrm{Lag}^N(M_0)$, but with a shifted background class $b_1 = b_0 - w_2(M_0)$. In other words, Δ is an object in the category $\mathrm{Don}^\#\big(M_0, M_1\big)$ that is introduced in Section 6.7 below.

In the following, we will drop the Maslov cover and symplectic form from the notation.

Corollary 6.6.6. *The functor* $\Phi(\Delta) : \mathrm{Don}^{\#}(M_0, b_0) \to \mathrm{Don}^{\#}(M_0, b_0 - w_2(M_0))$ *associated to the diagonal is canonically isomorphic to the shift functor* Ψ_{M_0}.

6.7. **Composition in the Donaldson-Fukaya category of correspondences.** The set of sequences of Lagrangian correspondences forms a category in its own right, which we define in close analogy to the generalized Donaldson category in Section 6.4. We will then be able to define a composition functor for these categories.

Let M_a and M_b be symplectic manifolds satisfying (M1-2) with the same monotonicity constant $\tau \geq 0$. We fix an integer $N > 0$, N-fold Maslov covers $\mathrm{Lag}^N(M_{(\cdot)}) \to M_{(\cdot)}$, and background classes $b_{(\cdot)} \in H^2(M_{(\cdot)}, \mathbb{Z}_2)$. Let $\underline{L} = (L_{01}, L_{12}, \ldots, L_{(r-1)r})$ be a generalized Lagrangian correspondences from M_a to M_b, as defined in Section 2.1. We picture $\underline{L}$ as sequence

$$M_a = N_0 \xrightarrow{L_{01}} N_1 \xrightarrow{L_{12}} \cdots \xrightarrow{L_{(r-1)r}} N_r = M_b \ .$$

We call a generalized Lagrangian correspondence $\underline{L}$ from M_a to M_b *admissible* if each N_j satisfies (M1-2) with the monotonicity constant $\tau \geq 0$, the image of each $\pi_1(L_{(j-1)j})$ in $\pi_1(N_{j-1}^- \times N_j)$ is torsion, and each $L_{(j-1)j}$ has minimal Maslov number $N_{L_{(j-1)j}} \geq 3$. (Alternatively, we could work in the framework of Bohr-Sommerfeld monotone Lagrangians as described in Remark 3.1.4.)

Definition 6.7.1. We define the *Donaldson-Fukaya category of correspondences*

$$\mathrm{Don}^{\#}(M_a, M_b) := \mathrm{Don}^{\#}(M_a, M_b, \mathrm{Lag}^N(M_a), \mathrm{Lag}^N(M_b), \omega_a, \omega_b, b_a, b_b).$$

(a) The objects of $\mathrm{Don}^{\#}(M_a, M_b)$ are admissible generalized Lagrangian correspondences from M_a to M_b, equipped with orientations, gradings, and relative spin structures (see Definitions 2.3.1, 3.3.1).[15]

(b) The morphism spaces of $\mathrm{Don}^{\#}(M_a, M_b)$ are the $\mathbb{Z}_N$-graded Floer cohomology groups (defined in Sections 3.3 and 4.3)

$$\mathrm{Hom}(\underline{L}, \underline{L}') := HF(\underline{L}, \underline{L}')[d],$$

where the second group is shifted by $d = \frac{1}{2}(\sum_k \dim(N_k) + \sum_{k'} \dim(N'_{k'}))$. For $\mathbb{Z}$-coefficients one has to introduce determinant lines as in Remark 6.4.3. See Figure 31 for alternative views of the quilted holomorphic cylinders of Figure 7, which are counted (modulo $\mathbb{R}$-shift) as Floer trajectories.

(c) The composition of morphisms in $\mathrm{Don}^{\#}(M_a, M_b)$,

$$\begin{aligned} \mathrm{Hom}(\underline{L}, \underline{L}') \times \mathrm{Hom}(\underline{L}', \underline{L}'') &\longrightarrow \ \mathrm{Hom}(\underline{L}, \underline{L}'') \\ (f, g) \qquad\qquad &\longmapsto f \circ g := \Phi_{\underline{P}}(f \otimes g) \end{aligned}$$

is defined by the relative invariant $\Phi_{\underline{P}}$ associated to the quilted pair of pants surface $\underline{P}$ (this time the pair of pants is an honest one, not just the front) in Figure 32, where the patches without outgoing ends are ordered from M_a to M_b.

(d) The identity $1_{\underline{L}} \in \mathrm{Hom}(\underline{L}, \underline{L})$ for a generalized Lagrangian correspondence $\underline{L}$ is given by the relative invariant $1_{\underline{L}} := \Phi_{\underline{S}}$ associated to the quilted cap in Figure 33, where the patches without outgoing ends are ordered from M_b to M_a.

[15] In the previous notation, a grading on $\underline{L}$ is a collection of N-fold Maslov covers $\mathrm{Lag}^N(N_j) \to N_j$ for $j = 0, \ldots, r$ and gradings of the Lagrangian correspondences $L_{(j-1)j}$. Here the gradings on $N_0 = M_a$ and $N_r = M_b$ are the fixed ones. A relative spin structure on $\underline{L}$ is a collection of background classes $b_j \in H^2(N_j, \mathbb{Z}_2)$ for $j = 0, \ldots, r$ and relative spin structures on $L_{(j-1)j}$ with background classes $-\pi_{j-1}^* b_{j-1} + \pi_j^* b_j$. Here $b_0 = b_a$ and $b_r = b_b$ are the fixed background classes in M_a and M_b.

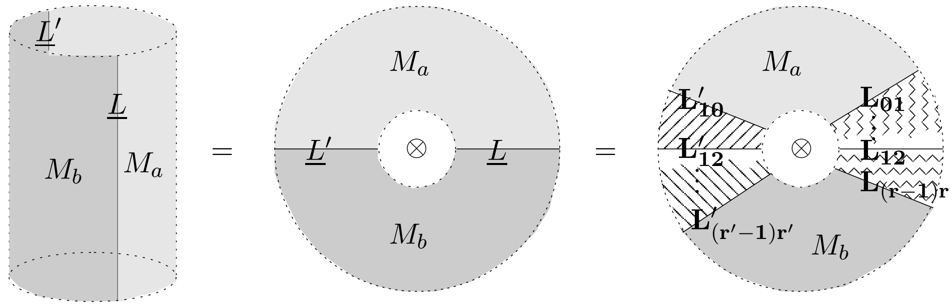


FIGURE 31. Floer trajectories for pairs of generalized Lagrangian correspondences

Remark 6.7.2. In Figure 31 and the following pictures, the outer circles will always be outgoing ends. The inner circles are usually incoming ends, indicated by a $\otimes$ or marked with the incoming morphism. Ends at the top resp. bottom of pictures will always be outgoing resp. incoming, unless otherwise indicated by arrows.

Note that all objects $\underline{L} \in \mathrm{Obj}(\mathrm{Don}^{\#}(M_a, M_b))$ satisfy the preliminary admissibility assumption of Section 6.5 : For any $\underline{L}_a \in \mathrm{Obj}(\mathrm{Don}^{\#}(M_a))$ we have $(\underline{L}_a, \underline{L}) \in \mathrm{Obj}(\mathrm{Don}^{\#}(M_b))$. The associativity and identity axiom for this category follow from the gluing formula (34) applied to the gluings (indicated by dashed lines) in Figure 34. Note that – in contrast to Figure 31 – the solutions on the quilted annulus (i.e. cylinder) are counted without quotienting by $\mathbb{R}$, hence as in Example 4.1.6 this relative invariant is the identity.

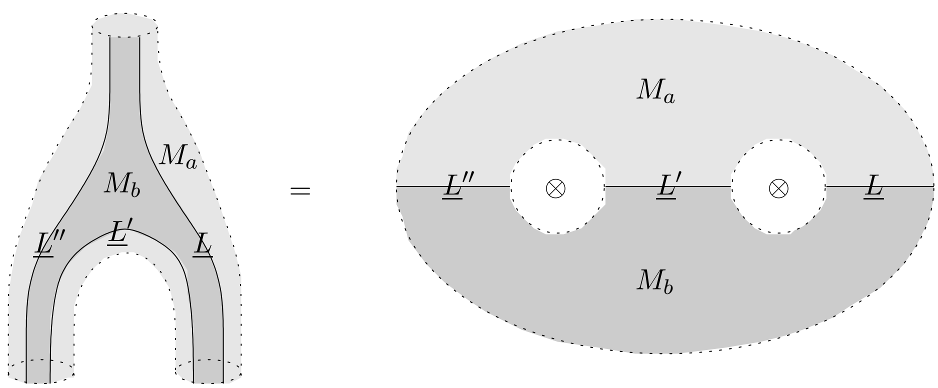


FIGURE 32. Quilted pair of pants: Composition of morphisms for Lagrangian correspondences

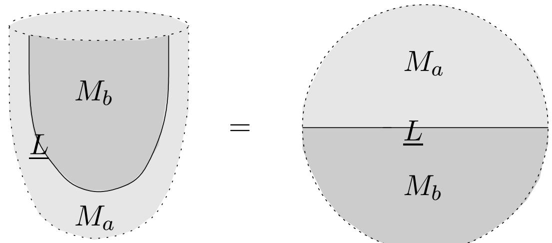


FIGURE 33. Quilted cap: Identity for Lagrangian correspondences

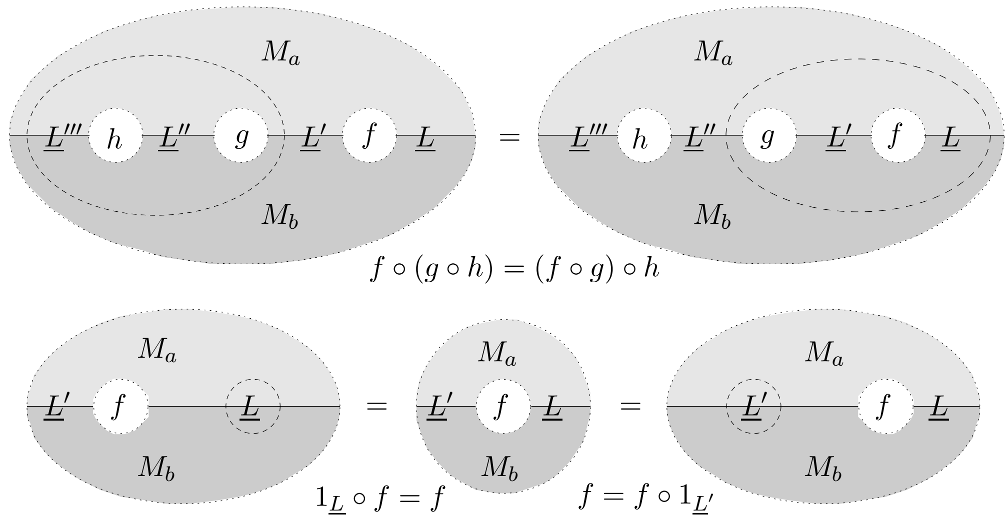


FIGURE 34. Axioms for Donaldson-Fukaya category of correspondences

Remark 6.7.3. Consider the case where the symplectic manifolds $M_a = M_b = M$ agree (including Maslov cover and background class). Then for any admissible generalized Lagrangian correspondence $\underline{L} \in \mathrm{Obj}(\mathrm{Don}^\#(M,M))$ the composition of morphisms in (c) defines a ring structure on $\mathrm{Hom}(\underline{L},\underline{L})$, and (d) provides an identity element. Another application of our main theorem shows that this ring structure is isomorphic under embedded compositions of correspondences: Let $\underline{L}$ and $\underline{L}'$ be two admissible generalized Lagrangian correspondences from M to itself. Suppose that they are equivalent in the sense of Section 2.1 through a series of embedded compositions of consecutive Lagrangian correspondences, and such that each intermediate generalized Lagrangian correspondence is admissible. Then there is a canonical ring isomorphism

$$\big(\mathrm{Hom}(\underline{L},\underline{L}),\circ\big) \simeq \big(\mathrm{Hom}(\underline{L}',\underline{L}'),\circ\big)$$

which intertwines the identity elements $1_{\underline{L}}$ and $1_{\underline{L}'}$.

Indeed, by assumption there exists a sequence of admissible generalized Lagrangian correspondences $\underline{L}^j$ connecting $\underline{L}^0 = \underline{L}$ to $\underline{L}^N = \underline{L}'$ as in the proof of Theorem 6.6.1. In each step two consecutive Lagrangian correspondences in the sequence $\underline{L}^j = (\ldots, L_-, L_+, \ldots)$ are replaced by their embedded, monotone composition in $\underline{L}^{j\pm1} = (\ldots, L_- \circ L_+, \ldots)$. Theorem 5.0.3 provides isomorphisms $\Psi^j : HF(\underline{L}^j,\underline{L}^j) \to HF(\underline{L}^{j\pm1},\underline{L}^{j\pm1})$ by shrinking the strip between L_- and L_+. Theorem 5.4.1 applies to the corresponding strips in the pair of pants surface and the quilted cap surface of Definition 6.7.1 (c) and (d) and shows that the isomorphisms Ψ^j intertwine the ring structures and identity morphisms. The full ring isomorphism is given by a composition of these isomorphisms or their inverses.

Next, consider a triple of symplectic manifolds M_a, M_b, M_c satisfying (M1-2) with the same monotonicity constant τ, equipped with Maslov covers $\mathrm{Lag}^N(M_{(\cdot)}) \to M_{(\cdot)}$ (with the same N) and background classes $b_{(\cdot)} \in H^2(M_{(\cdot)},\mathbb{Z}_2)$. We denote by $\mathrm{Don}^\#(M_a,M_b) \times \mathrm{Don}^\#(M_b,M_c)$ the product category. That is, objects are pairs $(\underline{L}_{ab},\underline{L}_{bc})$ of objects of $\mathrm{Don}^\#(M_a,M_b)$ and $\mathrm{Don}^\#(M_b,M_c)$. Morphisms are pairs (f,g) with $f \in \mathrm{Hom}(\underline{L}_{ab},\underline{L}'_{ab})$, $g \in$

$\mathrm{Hom}(\underline{L}_{bc}, \underline{L}'_{bc})$. Composition is given by

$$(f,g)\circ(f',g') := (-1)^{|f'||g|}(f\circ f', g\circ g')$$

for $f \in \mathrm{Hom}(\underline{L}_{ab}, \underline{L}'_{ab})$, $f' \in \mathrm{Hom}(\underline{L}'_{ab}, \underline{L}''_{ab})$, $g \in \mathrm{Hom}(\underline{L}_{bc}, \underline{L}'_{bc})$, $g' \in \mathrm{Hom}(\underline{L}'_{bc}, \underline{L}''_{bc})$.

Definition 6.7.4. We define a *composition functor*

$$\# : \mathrm{Don}^{\#}(M_a, M_b) \times \mathrm{Don}^{\#}(M_b, M_c) \to \mathrm{Don}^{\#}(M_a, M_c). \tag{72}$$

(a) On the level of objects $\#$ is defined by concatenation:

$$\begin{gathered}\mathrm{Obj}(\mathrm{Don}^{\#}(M_a, M_b)) \times \mathrm{Obj}(\mathrm{Don}^{\#}(M_b, M_c)) \to \mathrm{Obj}(\mathrm{Don}^{\#}(M_a, M_c)) \\ (\underline{L}_{ab}, \underline{L}_{bc}) \mapsto \underline{L}_{ab}\#\underline{L}_{bc},\end{gathered}$$

where

$$(L^{ab}_{01}, \ldots, L^{ab}_{(r-1)r})\#(L^{bc}_{01}, \ldots, L^{bc}_{(r'-1)r'}) := (L^{ab}_{01}, \ldots, L^{ab}_{(r-1)r}, L^{bc}_{01}, \ldots, L^{bc}_{(r'-1)r'}).$$

(b) On the level of morphisms, $\#$ is defined for $\underline{L}_{ab}, \underline{L}'_{ab} \in \mathrm{Obj}(\mathrm{Don}^{\#}(M_a, M_b))$ and $\underline{L}_{bc}, \underline{L}'_{bc} \in \mathrm{Obj}(\mathrm{Don}^{\#}(M_b, M_c))$ by

$$\begin{array}{ccc}\mathrm{Hom}(\underline{L}_{ab}, \underline{L}'_{ab}) \times \mathrm{Hom}(\underline{L}_{bc}, \underline{L}'_{bc}) & \to & \mathrm{Hom}(\underline{L}_{ab}\#\underline{L}_{bc}, \underline{L}'_{ab}\#\underline{L}'_{bc}) \\ (f,g) & \mapsto & f\#g := \Phi_{\underline{P}}(f\otimes g),\end{array}$$

where $\Phi_{\underline{P}}$ is the relative invariant associated to the quilted pair of pants $\underline{P}$, where now every seam connects one of the incoming cylindrical ends to the outgoing cylindrical end, as in Figure 35.

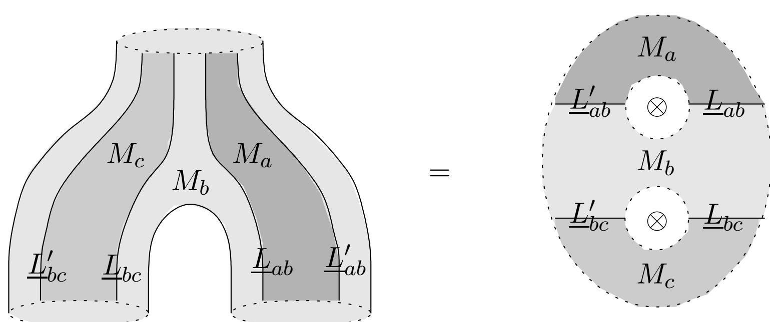


FIGURE 35. Composition functor on Donaldson categories of correspondences

The composition axiom for the functor $\#$ follows from the gluing formula (34) applied to the two degenerations of the five-holed sphere shown in Figure 36: For all $f \in \mathrm{Hom}(\underline{L}_{ab}, \underline{L}'_{ab})$, $f' \in \mathrm{Hom}(\underline{L}'_{ab}, \underline{L}''_{ab})$, $g \in \mathrm{Hom}(\underline{L}_{bc}, \underline{L}'_{bc})$, $g' \in \mathrm{Hom}(\underline{L}'_{bc}, \underline{L}''_{bc})$ we obtain

$$\#\big((f,g)\circ(f',g')\big) = (-1)^{|f'||g|}(f\circ f')\#(g\circ g') = (f\#g)\circ(f'\#g').$$

The identity axiom for the concatenation functor, $1_{\underline{L}_{ab}}\#1_{\underline{L}_{bc}} = 1_{\underline{L}_{ab}\#\underline{L}_{bc}}$, follows similarly from the gluing formula (34) applied to the degenerations shown in Figure 37.

Remark 6.7.5. The construction of functors associated to Lagrangian correspondences in Section 6.5 has an obvious extension (71) for generalized Lagrangian correspondences. For $\underline{L}_{ab} \in \mathrm{Don}^{\#}(M_a, M_b)$ the functor $\Phi(\underline{L}_{ab}) : \mathrm{Don}^{\#}(M_a) \to \mathrm{Don}^{\#}(M_b)$ acts on objects $\underline{L} \in \mathrm{Obj}(\mathrm{Don}^{\#}(M_a))$ by concatenation $\Phi(\underline{L}_{ab}) = \underline{L}\#\underline{L}_{ab}$, and on morphisms $\Phi(\underline{L}_{ab})$:

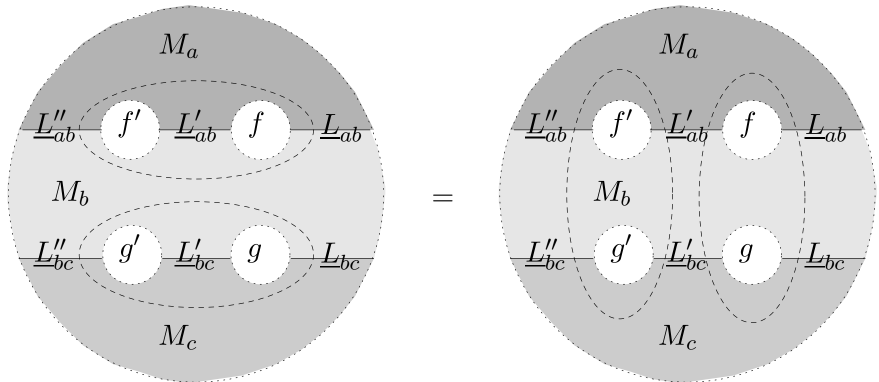


FIGURE 36. Composition axiom for the concatenation functor

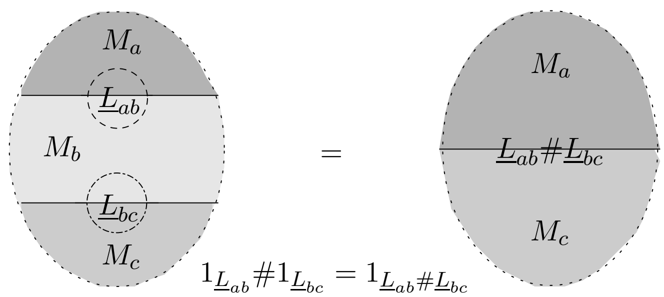


FIGURE 37. Identity axiom for the concatenation functor

$HF(\underline{L},\underline{L}') \to HF(\underline{L}\#\underline{L}_{ab},\underline{L}'\#\underline{L}_{ab})$ is defined by composition $\Phi(L_{01}) \circ \ldots \circ \Phi(L_{(r-1)r})$ of the functors associated to the simple Lagrangian correspondences $(L_{01},\ldots,L_{(r-1)r}) = \underline{L}_{ab}$. Alternatively, the map $\Phi(\underline{L}_{ab})$ on morphisms can be defined directly by the relative invariant in Figure 26, see Remark 6.5.4. Using the first definition, we have a tautological equality of functors

$$\Phi(\underline{L}_{ab}) \circ \Phi(\underline{L}_{bc}) = \Phi(\underline{L}_{ab}\#\underline{L}_{bc}) \tag{73}$$

for any two objects $\underline{L}_{ab} \in \mathrm{Don}^{\#}(M_a,M_b)$ and $\underline{L}_{bc} \in \mathrm{Don}^{\#}(M_b,M_c)$.

6.8. **Natural transformations.** Let M_a and M_b be as in the previous section and let $\underline{L}_{ab},\underline{L}'_{ab}$ be objects in $\mathrm{Don}^{\#}(M_a,M_b)$.

Definition 6.8.1. Given a morphism $T \in \mathrm{Hom}(\underline{L}_{ab},\underline{L}'_{ab})$ we define a natural transformation

$$\Phi_T :\ \Phi(\underline{L}_{ab}) \to \Phi(\underline{L}'_{ab})$$

as follows: To any object $\underline{L}$ in $\mathrm{Don}^{\#}(M_a)$ we assign the morphism

$$\Phi_T(\underline{L}) \in \mathrm{Hom}(\Phi(\underline{L}_{ab})(\underline{L}),\Phi(\underline{L}'_{ab})(\underline{L}))$$

given by the relative invariant associated to the surface in Figure 38, which is independent of the ordering of the patches. (Note that the end where T is inserted is cylindrical in the sense that the strip-like ends glue together to a cylindrical end.)

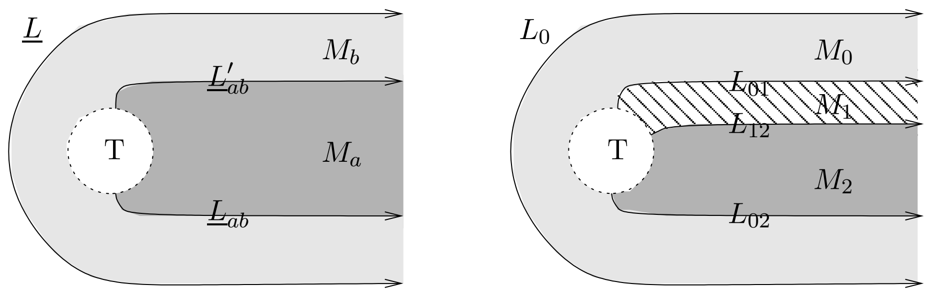


FIGURE 38. Natural transformation associated to a Floer cohomology class: General case and an example, where $\underline{L}$ consists of a single Lagrangian L_0, $\underline{L}_{ab}$ consists of a single Lagrangian L_{02}, and $\underline{L}'_{ab}$ consists of a pair (L_{01}, L_{12}).

To see that Φ_T is a natural transformation of functors $\Phi(\underline{L}_{ab}) \to \Phi(\underline{L}'_{ab})$ we must show that for any two objects $\underline{L}, \underline{L}'$ in $\mathrm{Don}^{\#}(M_a)$ and any morphism $f \in \mathrm{Hom}(\underline{L}, \underline{L}')$ we have

$$\Phi(\underline{L}_{ab})(f) \circ \Phi_T(\underline{L}') = (-1)^{|T||f|} \Phi_T(\underline{L}) \circ \Phi(\underline{L}'_{ab})(f). \tag{74}$$

This identity follows from (34) applied the gluing shown in Figure 39.

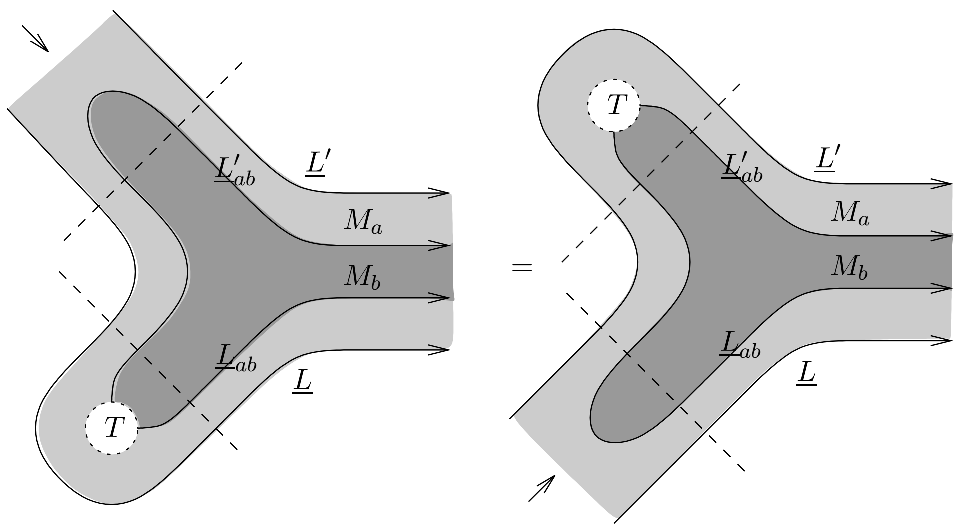


FIGURE 39. Natural transformation axiom

Proposition 6.8.2. *The maps* $\underline{L}_{ab} \mapsto \Phi(\underline{L}_{ab})$ *and* $T \mapsto \Phi_T$ *define a functor*

$$\mathrm{Don}^{\#}(M_a, M_b) \to \mathrm{Fun}(\mathrm{Don}^{\#}(M_a), \mathrm{Don}^{\#}(M_b)).$$

Proof. We apply (34) to the gluings in Figure 40 to deduce the composition axiom

$$\Phi_T(\underline{L}) \circ \Phi_{T'}(\underline{L}) = \Phi_{T \circ T'}(\underline{L})$$

for all $T \in \mathrm{Hom}(\underline{L}_{ab}, \underline{L}'_{ab})$, $T' \in \mathrm{Hom}(\underline{L}'_{ab}, \underline{L}''_{ab})$, and $\underline{L} \in \mathrm{Obj}(\mathrm{Don}^{\#}(M_a))$. The identity axiom

$$\Phi_{1_{\underline{L}_{ab}}}(\underline{L}) = 1_{\Phi(\underline{L}_{ab})(\underline{L})}$$

for $T = 1_{\underline{L}_{ab}} \in \mathrm{Hom}(\underline{L}_{ab}, \underline{L}_{ab})$ and $\underline{L} \in \mathrm{Obj}(\mathrm{Don}^{\#}(M_a))$ follows from (34) applied to the gluing in Figure 41. $\square$

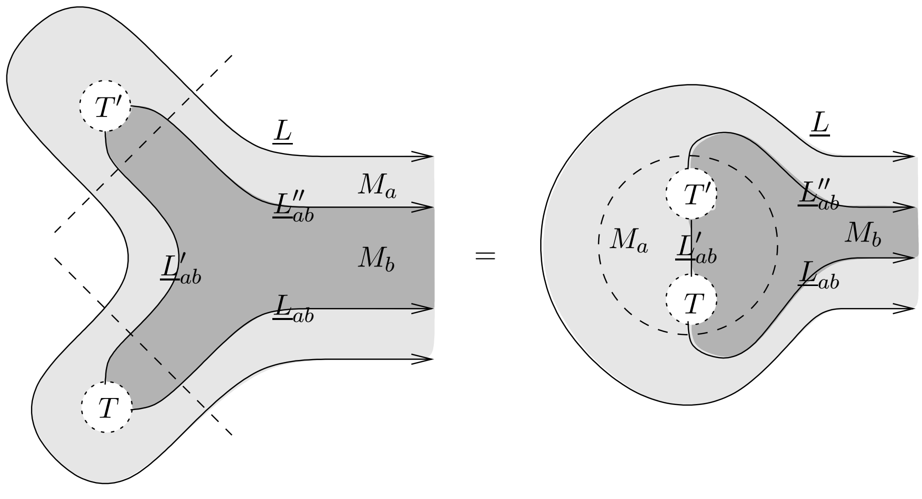


Figure 40. Composition axiom for natural transformations

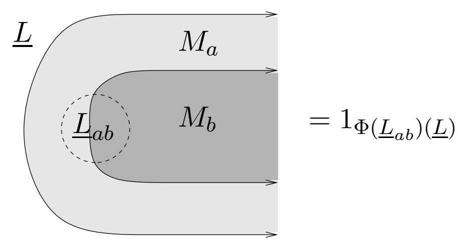


Figure 41. Identity axiom for natural transformations

Remark 6.8.3. In this remark we discuss the special case of the diagonal $\Delta \subset M^- \times M$, which gives rise to the so-called open-closed maps in $2D$ TQFT. By [32] there is a ring isomorphism between the Floer cohomology of the diagonal $HF(\Delta, \Delta)$ and the quantum cohomology $HF(\mathrm{Id})$. Our construction gives for any element $\alpha \in HF(\Delta, \Delta) \simeq HF(\mathrm{Id})$ an automorphism of the identity functor $\Phi(\Delta)$ (more precisely, of the shift functor $\Phi(\Delta) \simeq \Psi_M$ in case $w_2(M) \neq 0$). In particular, we obtain elements $\Phi_\alpha(L) \in HF((L, \Delta), (L, \Delta)) \simeq HF(L, L)$ for each admissible Lagrangian submanifold $L \subset M$. (Here $HF((L, \Delta), (L, \Delta)) \simeq HF(L, L)$ is a ring isomorphism by Remark 6.7.3 .) Proposition 6.8.2 gives

$$\Phi_{\alpha \circ \beta}(L) = \Phi_\alpha(L) \circ \Phi_\beta(L).$$

That is, the closed-open map $HF(\mathrm{Id}) \to HF(L, L)$ is a ring homomorphism. The closed-open maps in Floer theory are discussed in more detail in Albers [2, Theorem 3.1]. More relations of this type, in the general setting of open-closed TQFT, are discussed in [27].

For any pair of Lagrangians $L^0, L^1 \subset M$, combining the ring homomorphism $HF(\mathrm{Id}) \to HF(L^k, L^k)$ with the composition $HF(L^k, L^k) \times HF(L^0, L^1) \to HF(L^0, L^1)$ gives a module structure on $HF(L^0, L^1)$ over $HF(\mathrm{Id})$. The module structure is independent of $k = 0, 1$, by the natural transformation axiom (74) with $\underline{L} = \underline{L}' = \Delta$. It is equal to the module structure induced by the isomorphism $HF(L^0 \times L^1, \Delta) \to HF(L^0, L^1)$ of [46].

Note that if $HF(\mathrm{Id}) \to HF(L, L)$ is a surjection and $HF(\mathrm{Id})$ is semisimple then $HF(L, L)$ is again semisimple, and in particular nilpotent free.

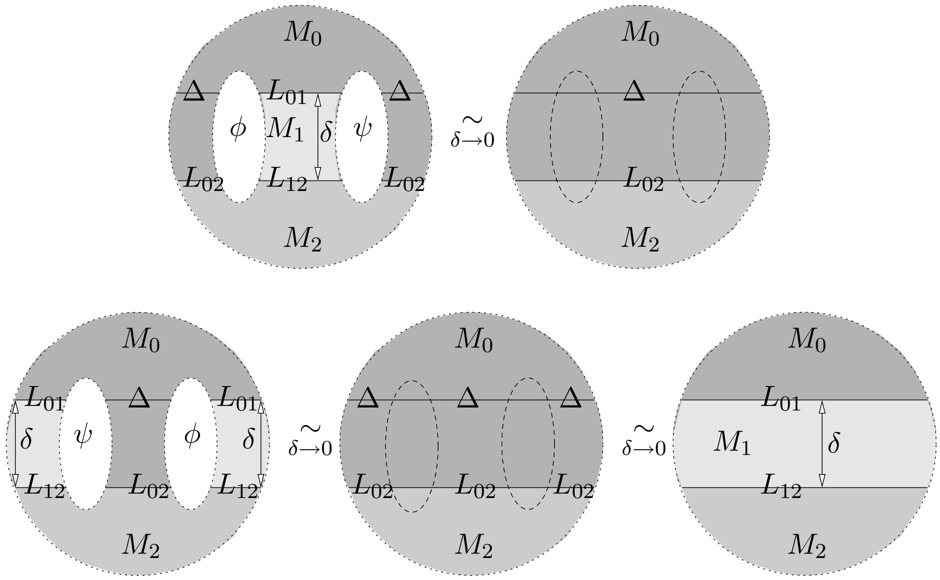


FIGURE 42. Isomorphism of composition and concatenation

Next, we show that embedded composition of Lagrangian correspondences gives rise to isomorphic objects in the Donaldson-Fukaya category. For simplicity we restrict to the case of simple Lagrangian correspondences, i.e. sequences of length 1. The statement and argument for the general case is analogous.

Theorem 6.8.4. *Let* $L_{01} \in \mathrm{Obj}(\mathrm{Don}^{\#}(M_0, M_1))$ *and* $L_{12} \in \mathrm{Obj}(\mathrm{Don}^{\#}(M_1, M_2))$ *be admissible Lagrangian correspondences. Suppose that* $L_{01} \times_{M_1} L_{12} \to M_0^- \times M_2$ *embeds to a smooth, admissible Lagrangian correspondence* $L_{02} := L_{01} \circ L_{12} \in \mathrm{Obj}(\mathrm{Don}^{\#}(M_0, M_2))$. *Then* $\Delta_{M_0} \# L_{02}$, $L_{02} \# \Delta_{M_2}$, *and* $L_{01} \# L_{12}$ *are all isomorphic in* $\mathrm{Don}^{\#}(M_0, M_2)$.

Remark 6.8.5. If in Theorem 6.8.4 we moreover assume $w_2(M_0) = 0$ or $w_2(M_2) = 0$, then we in fact have an isomorphism between $L_{01} \# L_{12}$ and $L_{01} \circ L_{12}$, by Proposition 6.8.6 below.

Proof. By Theorem 5.0.3, $\mathrm{Hom}(L_{01} \# L_{12}, \Delta_{M_0} \# L_{02})$ resp. $\mathrm{Hom}(\Delta_{M_0} \# L_{02}, L_{01} \# L_{12})$ is isomorphic to $\mathrm{Hom}(\Delta_{M_0} \# L_{02}, \Delta_{M_0} \# L_{02})$; let ϕ resp. ψ denote the inverse image of the identity $1_{\Delta_{M_0} \# L_{02}}$. To establish the isomorphism $L_{01} \# L_{12} \simeq \Delta_{M_0} \# L_{02}$ we show that

$$\psi \circ \phi = 1_{\Delta_{M_0} \# L_{02}}, \qquad \phi \circ \psi = 1_{L_{01} \# L_{12}}$$

for the composition by the pair of pants products. These are special cases of Theorem 5.4.1 applied to the degenerations shown in Figure 42. The isomorphism $L_{01} \# L_{12} \simeq L_{02} \# \Delta_{M_2}$ is proven in the same way. □

Proposition 6.8.6. *Suppose that* M_0 *satisfies* $w_2(M_0) = 0$. *Then the diagonal* $\Delta_{M_0} \in \mathrm{Don}^{\#}(M_0, M_0)$ *is an identity of the composition* $\#$ *up to isomorphism. That is, for every generalized Lagrangian* $\underline{L} \in \mathrm{Obj}(\mathrm{Don}^{\#}(M_0, M_1))$ *the objects* $\Delta_{M_0} \# \underline{L}$ *and* $\underline{L}$ *are isomorphic in* $\mathrm{Don}^{\#}(M_0, M_1)$, *and for every generalized Lagrangian* $\underline{L} \in \mathrm{Obj}(\mathrm{Don}^{\#}(M_1, M_0))$ *the objects* $\underline{L} \# \Delta_{M_0}$ *and* $\underline{L}$ *are isomorphic in* $\mathrm{Don}^{\#}(M_1, M_0)$.

Proof. By Theorem 5.0.3, both $\mathrm{Hom}(\Delta_{M_0} \# \underline{L}, \underline{L})$ and $\mathrm{Hom}(\underline{L}, \Delta_{M_0} \# \underline{L})$ are isomorphic to $\mathrm{Hom}(\underline{L}, \underline{L})$; let ϕ resp. ψ denote the inverse image of the identity $1_{\underline{L}}$. Then the identities

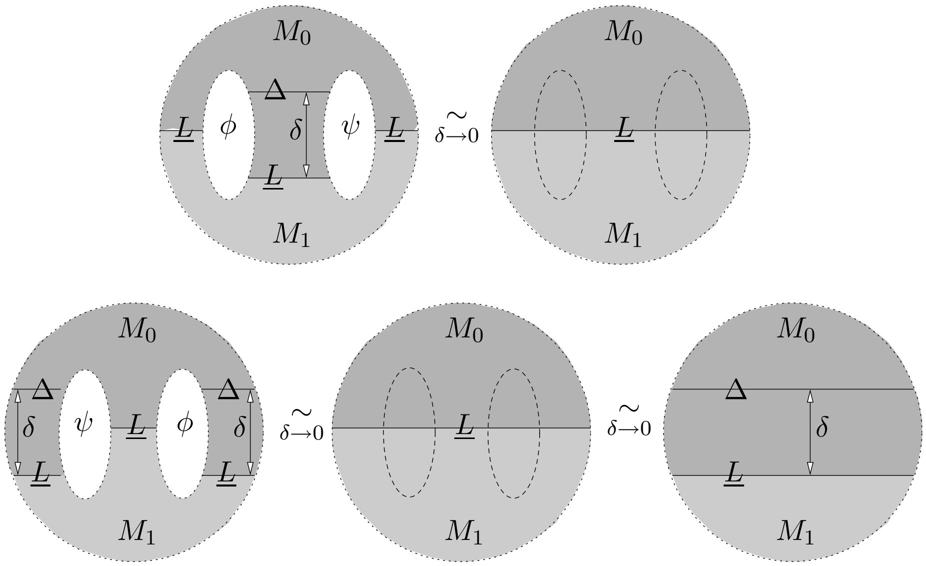


FIGURE 43. Isomorphism of $\Delta_{M_0}\#\underline{L}$ and $\underline{L}$

$\phi \circ \psi = 1_{\underline{L}}$ and $\phi \circ \psi = 1_{\Delta_{M_0}\#\underline{L}}$ follow from Theorem 5.4.1 applied to the degenerations shown in Figure 43. (Alternatively, as mentioned in Section 6.6.2, one could glue the strips instead of shrinking them.) This proves $\Delta_{M_0}\#\underline{L} \simeq \underline{L}$. The isomorphism $\underline{L}\#\Delta_{M_0} \simeq \underline{L}$ is proven in the same way. $\square$

Corollary 6.8.7. *Under the assumptions of Theorem 6.8.4 the functors* $\Psi_{M_0} \circ \Phi(L_{01} \circ L_{12})$*,* $\Phi(L_{01} \circ L_{12}) \circ \Psi_{M_2}$*, and* $\Phi(L_{01}) \circ \Phi(L_{12})$ *are all isomorphic in the category of functors from* $\mathrm{Don}^{\#}(M_0)$ *to* $\mathrm{Don}^{\#}(M_2)$*.*

Proof. From Theorem 6.8.4 and (73) we obtain isomorphisms between $\Phi(\Delta_{M_0}\#L_{02}) = \Phi(\Delta_{M_0}) \circ \Phi(L_{02})$, $\Phi(L_{02}\#\Delta_{M_2}) = \Phi(L_{02}) \circ \Phi(\Delta_{M_2})$, and $\Phi(L_{01}\#L_{12}) = \Phi(L_{01}) \circ \Phi(L_{12})$. By Proposition 6.6.6 the functors $\Phi(\Delta_{M_k})$ are isomorphic to the shift functors Ψ_{M_k}. Since isomorphisms commute with composition of functors, this proves the corollary. $\square$

6.9. **Weinstein-Floer 2-category.** We can rephrase Theorem 5.0.3 and summarize the constructions of this chapter, using the language of 2-categories.

Definition 6.9.1. A 2-*category* $\mathcal{C}$ consists of the following data:

(a) A class of objects $\mathrm{Obj}(\mathcal{C})$.
(b) For each pair of objects $X, Y \in \mathrm{Obj}(\mathcal{C})$, a small category $\mathrm{Hom}(X, Y)$.
(c) For each triple of objects $X, Y, Z \in \mathrm{Obj}(\mathcal{C})$, a composition functor

$$\circ: \ \mathrm{Hom}(X, Y) \times \mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z).$$

(d) For every $X \in \mathrm{Obj}(\mathcal{C})$ an identity functor $1_X \in \mathrm{Hom}(X, X)$.

These data should satisfy the following axioms:

(Identity): For all $X, Y \in \mathrm{Obj}(\mathcal{C})$ and $f \in \mathrm{Hom}(X, Y)$

$$1_X \circ f = f, \qquad f \circ 1_Y = f.$$

(Associativity): For all composable morphisms f, g, h

$$f \circ (g \circ h) = (f \circ g) \circ h.$$

Objects resp. morphisms in $\mathrm{Hom}(X, Y)$ are called 1-morphisms resp. 2-morphisms. We say that $\mathcal{C}$ has *weak identities* if equality in the identity axiom is replaced by 2-isomorphism.

The basic example of a 2-category is Cat, whose objects are categories, 1-morphisms are functors, and 2-morphisms are natural transformations.

Definition 6.9.2. A 2-*functor* $\mathcal{F} : \mathcal{C}_1 \to \mathcal{C}_2$ between 2-categories $\mathcal{C}_1$ and $\mathcal{C}_2$ consists of

(a) a map $\mathcal{F} : \mathrm{Obj}(\mathcal{C}_1) \to \mathrm{Obj}(\mathcal{C}_2)$,
(b) for each pair $X, Y \in \mathrm{Obj}(\mathcal{C}_1)$, a functor

$$\mathcal{F}(X, Y) : \mathrm{Hom}(X, Y) \to \mathrm{Hom}(\mathcal{F}(X), \mathcal{F}(Y)),$$

respecting composition and identities.

In the following we restrict ourselves to symplectic manifolds that are spin, i.e. $w_2(M) = 0$. Their advantage is that the shift functor $\Psi_M : \mathrm{Don}^\#(M, b) \to \mathrm{Don}^\#(M, b)$ of Definition 6.6.4 is trivial and the diagonal $\Delta_M \subset M^- \times M$ is an object of the category of correspondences $\mathrm{Don}^\#(M, M)$ from (M, b) to itself. We moreover drop the Maslov cover from the data, thus working with ungraded Floer cohomology groups.

Definition 6.9.3. Fix a constant $\tau \geq 0$. We define the *Weinstein-Floer* 2-*category* $\mathrm{Floer}^\#_\tau$ as follows:

(a) Objects are symplectic manifolds (M, ω) that satisfy (M1-2) with monotonicity constant τ and $w_2(M) = 0$, and that are equipped with a background class $b \in H^2(M, \mathbb{Z}_2)$.
(b) The morphism categories of $\mathrm{Floer}^\#$ are the Donaldson categories of Lagrangian correspondences, $\mathrm{Hom}(M_0, M_1) := \mathrm{Don}^\#(M_0, M_1)$; without grading.
(c) Composition is defined by the functor (72),

$$\# : \mathrm{Don}^\#(M_0, M_1) \times \mathrm{Don}^\#(M_1, M_2) \to \mathrm{Don}^\#(M_0, M_2).$$

(d) The diagonal defines a weak identity $\Delta_M \in \mathrm{Don}^\#(M, M)$.

The associativity axiom is immediate on the level of objects: For any triple $\underline{L}_{01} \in \mathrm{Obj}(\mathrm{Don}^\#(M_0, M_1))$, $\underline{L}_{12} \in \mathrm{Obj}(\mathrm{Don}^\#(M_1, M_2))$, $\underline{L}_{23} \in \mathrm{Obj}(\mathrm{Don}^\#(M_2, M_3))$ we have $(\underline{L}_{01} \# \underline{L}_{12}) \# \underline{L}_{23} = \underline{L}_{01} \# (\underline{L}_{12} \# \underline{L}_{23})$. On the level of morphisms we apply (34) to the gluings indicated by dashed lines in Figure 44 to prove that $(f \# g) \# h = f \# (g \# h)$ for all $f \in \mathrm{Hom}(\underline{L}_{01}, \underline{L}'_{01})$, $g \in \mathrm{Hom}(\underline{L}_{12}, \underline{L}'_{12})$, $h \in \mathrm{Hom}(\underline{L}_{23}, \underline{L}'_{23})$. The weak identity axiom follows from Proposition 6.8.6. Hence $\mathrm{Floer}^\#$ is a 2-category with weak identities.

Remark 6.9.4. $\mathrm{Floer}^\#_\tau$ is independent, up to 2-isomorphism of 2-categories, of the choices of perturbation data and strip widths, as in Remarks 6.1.5, 6.4.5, and Proposition 4.3.1.

Theorem 6.8.4 implies that the definition of composition in the Weinstein-Floer 2-category $\mathrm{Floer}^\#_\tau$ agrees with the geometric definition, in the case that geometric composition is smooth, embedded, and monotone.

Theorem 6.9.5. *The map* $M_0 \mapsto \mathrm{Don}^\#(M_0)$ *and the functors*

$$\mathrm{Don}^\#(M_0, M_1) \to \mathrm{Fun}(\mathrm{Don}^\#(M_0), \mathrm{Don}^\#(M_1))$$

as in Proposition 6.8.2 define a categorification 2-functor $\mathrm{Floer}^\#_\tau \to \mathrm{Cat}$ *for every* $\tau \geq 0$.

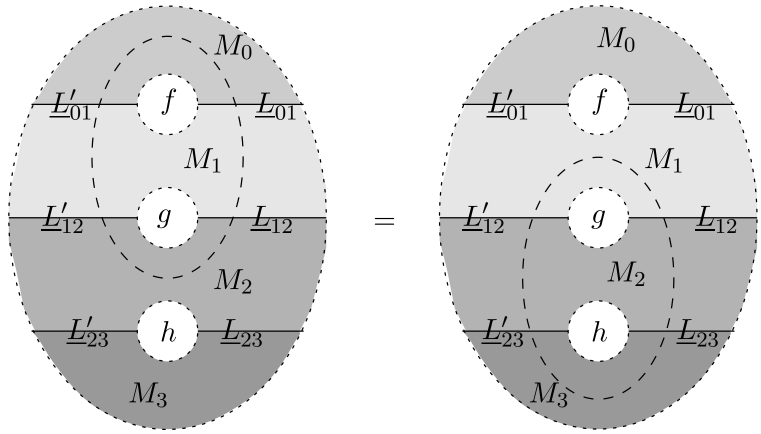


FIGURE 44. Associativity of the concatenation functor

Proof. Compatibility with the composition follows from the identity (73). The weak identities $\Delta_M \in \mathrm{Hom}(M, M)$ are mapped to weak identities $\Phi(\Delta) \simeq 1_{\mathrm{Don}^\#(M)}$ by Corollary 6.6.6. Here the shift functor Ψ_M is the identity since $w_2(M) = 0$. $\square$

Remark 6.9.6. (a) For any genuinely monotone symplectic manifold (i.e. with $\tau > 0$) we can achieve $\tau = 1$ by rescaling. It thus suffices to consider the *exact Weinstein-Floer 2-category* $\mathrm{Floer}^\#_0$ and the *monotone Weinstein-Floer 2-category* $\mathrm{Floer}^\#_1$. Note however that we cannot incorporate Lagrangian correspondences between monotone symplectic manifolds with different monotonicity constants. This is due to bubbling effects which are true obstructions in our present setup. We expect that the A_∞-setup, incorporating all bubbling effects, has better behavior.

(b) One can define an analogous *graded Weinstein-Floer 2-category* $\mathrm{Floer}^\#_{N,\tau}$ for any $\tau \geq 0$ and integer N, whose objects are monotone symplectic manifolds with the additional structure of a Maslov cover $\mathrm{Lag}^N(M) \to M$. Its 1-morphisms are graded generalized Lagrangian correspondences, and its 2-morphism spaces are the graded Floer cohomology groups.

Remark 6.9.7. (a) One can define a strong identity $1_M \in \mathrm{Hom}(M, M)$ by allowing the empty sequence $1_M := \emptyset$ as a generalized Lagrangian correspondence. The various constructions in this Section extend to the case of empty sequences by allowing cylindrical ends as in Remark 4.3.2.

(b) In the case $w_2(M) \neq 0$, the diagonal is not an automorphism but a morphism $\Delta_M \in \mathrm{Hom}((M, b), (M, b - w_2(M)))$, see Remark 6.6.5. Hence

$$\underline{L}\#\Delta_M \in \mathrm{Hom}((M_1, b_1), (M, b - w_2(M))), \quad \underline{L} \in \mathrm{Hom}((M_1, b_1), (M, b))$$

lie in different morphism spaces that are not related by a simple shift in the background class. However, the categorification functor in Theorem 6.9.5 generalizes directly to this setup as follows. The functor maps the special $\mathrm{Floer}^\#_\tau$ 1-morphisms $\Delta_M \in \mathrm{Don}^\#((M, b), (M, b - w_2(M))$ to Cat 1-morphisms that are isomorphic to the shift functors $\Psi_M \in \mathrm{Fun}(\mathrm{Don}^\#(M, b), \mathrm{Don}^\#(M, b - w_2(M)))$.

(c) One can make the diagonal a strong identity by modding out by the equivalence relation discussed in Section 2.1. Let $\mathrm{Brane}^\#_\tau$ denote the 2-category whose objects and 1-morphisms are those of $\mathrm{Floer}^\#_\tau$, modulo the equivalence relation $L_{01}\#L_{12} \sim L_{01} \circ L_{12}$

for embedded compositions, as in Section 2.1, and whose 2-morphisms are defined as follows. Given a pair $[\underline{L}_{01}], [\underline{L}'_{01}]$ of 1-morphisms from M_0 to M_1, define the space of 2-morphisms $\mathrm{Hom}([\underline{L}_{01}], [\underline{L}'_{01}])$ by $\mathrm{Hom}([\underline{L}_{01}], [\underline{L}'_{01}]) = HF(\underline{L}_{01}, \underline{L}'_{01})$ for some choice of representatives $\underline{L}_{01}, \underline{L}'_{01}$. Define composition by concatenation $\#$, as in (72). The equivalence classes of the diagonal $[\Delta_M]$ define true identities in case $w_2(M) = 0$. Our main result, Theorem 5.0.3, implies that $\mathrm{Brane}^{\#}_{\tau}$ is independent of the choice of representatives up to 2-isomorphism of 2-categories. Theorem 6.8.4 implies that the categorification 2-functor of Theorem 6.9.5 induces a 2-functor $\mathrm{Brane}^{\#}_{\tau} \to \mathrm{Cat}$ to the 2-category of categories Cat.

Remark 6.9.8. (a) We continue the comparison with quantization in Remark 2.1.5. The category $\mathrm{Hörm}^{\#}$ of closed manifolds and sequences of Fourier integral operators admits a *quantization functor* to $\mathbb{R}$-families of Hilbert spaces. On the level of objects, the functor is given by mapping a manifold Q to the family $H^s(Q)$ of distributions of Sobolev class $s \in \mathbb{R}$ on Q. On the level of morphisms, the functor is given by mapping a sequence of Fourier integral operators from Q to Q' of total degree d to the operator $H^s(Q) \to H^{s-d}(Q')$ given by composition. It would be interesting to know whether some of the other properties of quantization extend to Fukaya-Floer categorification. For instance, in most quantization schemes, quantization commutes with products, that is, the quantum Hilbert space for a product of two phase spaces is the tensor product of the quantum Hilbert spaces for the factors. The corresponding property for categorifications

$$\mathrm{Don}^{\#}(M_1 \times M_2) \stackrel{?}{=} \mathrm{Don}^{\#}(M_1) \times \mathrm{Don}^{\#}(M_2)$$

is certainly false, except in trivial cases. However, it seems possible that an appropriate version of this axiom holds for Fukaya categories in good situations. Similarly, for a free G-action on a manifold Q one has *quantization commutes with reduction*, which says that the quantum Hilbert space for a quotient is the space of invariant vectors in the quantum Hilbert space with G-action, see [10],[9]. The results of this paper construct a functor

$$\Phi(\mu^{-1}(0)) \;:\; \mathrm{Don}^{\#}(M) \to \mathrm{Don}^{\#}(M/\!\!/G)$$

for a monotone or exact Hamiltonian G-space M, arising from the Lagrangian correspondence $\mu^{-1}(0)) \hookrightarrow M^- \times M/\!\!/G$, see Section 2. Can one characterize its image in terms of invariants, i.e., does *categorification commute with reduction*

$$\mathrm{Don}^{\#}(M/\!\!/G)) \stackrel{?}{\cong} \mathrm{Don}^{\#}(M)^G$$

hold? Even better would be a categorification scheme with some relationship to quantization, as in Khovanov's work [16].

(b) The 2-category $\mathrm{Floer}^{\#}$ has certain similarities with 2-categories of motives in algebraic geometry. It would be interesting to know if homological mirror symmetry extends to the level of correspondences, as an equivalence of "derived 2-categories".

(c) Donaldson-Fukaya categories of cotangent bundles are related to the derived categories of constructible sheaves in Nadler-Zaslow [28]. It is natural to conjecture that the relationship described in that paper can be extended to correspondences and their conormal bundles to give an isomorphism of 2-categories.

7. Derived Floer theory

In this section we describe a framework in which the assumption (L3) of minimal Maslov number at least three can be removed. Namely, even if the Floer differential ∂ does not square to zero, the image $DF(L^0, L^1)$ of $(CF(L^0, L^1), \partial)$ in the *derived category of matrix factorizations* is independent of all choices up to isomorphism. In our application to $SU(r)$ knot Floer cohomology in [45], the invariant associated to a trivalent graph will be an object in such a derived category, and the language is chosen to make it match up with that in Khovanov-Rozansky [18]. Even in the case that the differentials have vanishing square, working in the derived category has certain advantages. For example, it makes duals and tensor products work the way they should. We emphasize that the version of the derived category needed here is not something deep, but essentially only a question of language. We also emphasize that the derived category construction discussed here is separate from the derived category construction applied by Kontsevich to Fukaya's A_∞ category.

7.1. **Matrix factorizations.**

We define categories of matrix factorizations as follows, see e.g. [31, p.17].[16]

Definition 7.1.1. For any $w \in \mathbb{Z}$, let $\mathrm{Fact}(w)$ denote the category of factorizations of $w\,\mathrm{Id}$.

- (a) The objects of $\mathrm{Fact}(w)$ consist of pairs (C, ∂), where
  - (i) C is a $\mathbb{Z}_2$-graded free abelian group $C = C^0 \oplus C^1$;
  - (ii) ∂ is a group homomorphism $\partial : C^\bullet \to C^{\bullet+1}$, satisfying $\partial^2 = w\,\mathrm{Id}$.
- (b) For any pair of objects C, C', the space of morphisms $\mathrm{Hom}_{\mathrm{Fact}}(C, C')$ is the space of grading preserving maps $f : C^\bullet \to (C')^\bullet$ such that $f\partial = \partial' f$.

Given an object $(C, \partial) \in \mathrm{Obj}(\mathrm{Fact}(w))$, there exists a dual object $(C, \partial)^\vee = (C^\vee, \partial^\vee)$, where $C^\vee = \mathrm{Hom}(C^0, \mathbb{Z}) \oplus \mathrm{Hom}(C^1, \mathbb{Z})$ and $\partial^\vee$ is the dual of ∂. Similarly for a morphism $f : (C^0, \partial^0) \to (C^1, \partial^1)$ we obtain a dual morphism $f^\vee : (C^1, \partial^1)^\vee \to (C^0, \partial^0)^\vee$. Thus we obtain a contravariant *dualization* functor

$$\mathrm{Fact}(w) \to \mathrm{Fact}(w), \quad (C, \partial) \mapsto (C, \partial)^\vee.$$

Similarly, the *graded tensor product* defines a covariant functor

$$\mathrm{Fact}(w_0) \times \mathrm{Fact}(w_1) \to \mathrm{Fact}(w_0 + w_1), \quad \big((C^0, \partial^0), (C^1, \partial^1)\big) \mapsto (C^0, \partial^0) \otimes (C^1, \partial^1).$$

For any matrix factorization (C, ∂) let $H((C, \partial) \otimes_{\mathbb{Z}} \mathbb{Z}_w)$ denote the cohomology of the differential obtained from ∂ by tensoring with $\mathbb{Z}_w$: $\partial \otimes_{\mathbb{Z}} \mathbb{Z}_w : C \otimes_{\mathbb{Z}} \mathbb{Z}_w \to C \otimes_{\mathbb{Z}} \mathbb{Z}_w$. Any morphism in $\mathrm{Fact}(w)$ defines a homomorphism of the corresponding cohomology groups, and so we have a *cohomology with coefficients* functor to the category Ab of $\mathbb{Z}_2$-graded abelian groups,

$$\mathrm{Fact}(w) \to \mathrm{Ab}, \quad (C, \partial) \mapsto H((C, \partial) \otimes_{\mathbb{Z}} \mathbb{Z}_w). \tag{75}$$

Definition 7.1.2.
- (a) A morphism $f : C \to C$ is called *null-homotopic* if there exists a map $h : C^\bullet \to C^{\bullet-1}$ such that $f = h\partial + \partial h$.
- (b) The *derived category of matrix factorizations* $D\,\mathrm{Fact}(w)$ is the category with the same objects as $\mathrm{Fact}(w)$, and morphisms given by the quotient of $\mathrm{Hom}(\mathrm{Fact}(w))$ by null-homotopic morphisms.
- (c) The *trivial object* in $D\,\mathrm{Fact}(w)$ is the trivial complex $C^0 = C^1 = \{0\}$ equipped with the trivial differential ∂. (Note that $\partial^2 = w\,\mathrm{Id}$, for any w.)

[16] The assumption that C^0, C^1 are free avoids the more complicated localization procedure used in e.g. Hartshorne [12].

Remark 7.1.3. (a) $D\,\mathrm{Fact}(w)$ is naturally a triangulated category, with distinguished "exact" triangles given by the *mapping cone construction*: Given a morphism of matrix factorizations $f:(C^0,\partial^0)\to(C^1,\partial^1)$, its mapping cone is the factorization

$$\mathrm{Cone}(f) := \left(C^0[1]\oplus C^1, \begin{pmatrix} -\partial^0 & f \\ 0 & \partial^1 \end{pmatrix} \right).$$

The exact triangles in $D\,\mathrm{Fact}(w)$ are by definition those isomorphic to triangles

$$\ldots \to C^0 \to C^1 \to \mathrm{Cone}(f) \to C^0[1] \to \ldots.$$

In particular, if $C^0 \xrightarrow{f} C^1 \to C^2 \to C^0[1]$ is an exact triangle then C^2 is (non-canonically) isomorphic to the mapping cone on f. The proofs are the same as for the case $w=0$ of complexes, see e.g. [8].

(b) The cohomology with coefficients functor (75) factors through the derived category to give a functor $D\,\mathrm{Fact}(w)\to \mathrm{Ab}$. Any exact triangle in $D\,\mathrm{Fact}(w)$ gives rise to a long exact sequence of cohomology groups with coefficients in $\mathbb{Z}_w$.

7.2. **Derived Floer theory for a pair of Lagrangians.** The following is a reformulation of results of Oh [29]. Let $D\subset\mathbb{C}$ be the unit disk and fix the base point $1\in\partial D$. Let (M,ω) be a compact monotone symplectic manifold and $L\subset M$ an oriented monotone Lagrangian submanifold. That is, we assume (M1-2) and (L1-2) with $\tau>0$ but not (L3). (Note that, by convention, (L3) is always satisfied in the exact case $\tau=0$.)

For any $J\in\mathcal{J}(M,\omega)$ and submanifold $X\subset L$, let $\mathcal{M}_1^2(L,J,X)$ denote the moduli space of J-holomorphic disks $u:(D,\partial D)\to(M,L)$ with Maslov number 2 and one marked point satisfying $u(1)\in X$, modulo automorphisms of the disk fixing $1\in\partial D$.

Proposition 7.2.1. *For any $\ell\in L$ there exists a subset $\mathcal{J}^{\mathrm{reg}}(\ell)\subset\mathcal{J}(M,\omega)$ of Baire second category such that $\mathcal{M}_1^2(L,J,\{\ell\})$ is a finite set. Any relative spin structure on L induces an orientation on $\mathcal{M}_1^2(L,J,\{\ell\})$. Letting $\epsilon:\mathcal{M}_1^2(L,J,\{\ell\})\to\{\pm1\}$ denote the map comparing the given orientation to the canonical orientation of a point, the disk number of L,*

$$w(L) := \sum_{u\in\mathcal{M}_1^2(L,J,\{\ell\})} \epsilon(u),$$

is independent of $J\in J^{\mathrm{reg}}(\ell)$ and $\ell\in L$.

Proof. First, we prove that for generic J and a generic point $m\in M$ there are no J-holomorphic spheres with Chern number one passing through m. For any submanifold $X\subset M$ and almost complex structure $J\in\mathcal{J}(M,\omega)$ let $\mathcal{M}_1^1(M,J,X)$ denote the moduli space of J-holomorphic maps $u:\mathbb{P}^1\to X$ with Chern number one and $u(0)\in X$, modulo holomorphic automorphisms of $\mathbb{P}^1$ fixing $0\in\mathbb{P}^1$. Standard arguments using the Sard-Smale theorem for the universal moduli space show that for J in a subset $\mathcal{J}^{\mathrm{reg}}_{\mathrm{sphere}}(X)\subset\mathcal{J}(M,\omega)$ of Baire second category, the moduli space $\mathcal{M}_1^1(M,J,X)$ is a smooth manifold of dimension $\dim(X)-2$. (Because the Chern number is minimal by monotonicity, multiple covers are impossible, so every map u in the universal moduli space is somewhere injective.) Similarly a parametrized version shows that for a subset $\tilde{\mathcal{J}}^{\mathrm{reg}}_{\mathrm{sphere}}(X)$ of homotopies $\{J_t\}_{t\in[0,1]}$ of Baire second category, the parametrized moduli space $\mathcal{M}_1^1(M,\{J_t\}_{t\in(0,1)},X)$ is a smooth manifold of dimension $\dim(X)-1$. In particular, if X is a finite subset of M then both the ordinary and parametrized moduli spaces are empty.

Next, consider an oriented submanifold $X\subset L$. Another standard argument shows that for J in a subset $\mathcal{J}^{\mathrm{reg}}_{\mathrm{disk}}(L,X)\subset\mathcal{J}(M,\omega)$ of Baire second category the moduli space

$\mathcal{M}_1^2(L, J, X)$ is a smooth manifold of dimension $\dim(X)$. (The results of [22, 21] produce from a J-holomorphic disk that is not somewhere injective a somewhere injective disk of lower energy. By monotonicity and minimality of the Maslov number, this is impossible and so every map u in the universal moduli space is somewhere injective.)

Given a relative spin structure on L one obtains an orientation on $\mathcal{M}_1^2(L, J, X)$ as follows. First, let $\widetilde{\mathcal{M}}_1^2(L, J, X)$ be the moduli space of parametrized J-holomorphic maps $u : (D, \partial D) \to (M, L)$ with Maslov index 2 and $u(1) \in X$. As explained in [7], [46], the orientation on X and relative spin structure on L induce an orientation on $\widetilde{\mathcal{M}}_1^2(L, J, X)$. To obtain an orientation on the quotient $\mathcal{M}_1^2(L, J, X) = \widetilde{\mathcal{M}}_1^2(L, J, X)/\operatorname{Aut}(D, \partial D, 1)$ it suffices to define an orientation on the automorphism group. For that purpose we identify $D \setminus \{1\}$ with the half-space $\mathbb{H} = \{z \in \mathbb{C}, \operatorname{Im}(z) \geq 0\}$. Then we have $\operatorname{Aut}(D, \partial D, 1) \cong (0, \infty) \times \mathbb{R}$, where $(0, \infty)$ acts by dilations on $\mathbb{H}$ and $\mathbb{R}$ acts by translations on $\mathbb{H}$. The standard orientations on the two factors induce the orientation on $\operatorname{Aut}(\mathbb{H})$.

Now suppose that $X = \{\ell\}$ is a point. Then for J in

$$\mathcal{J}^{\mathrm{reg}}(\ell) := J^{\mathrm{reg}}_{\mathrm{sphere}}(\{\ell\}) \cap J^{\mathrm{reg}}_{\mathrm{disk}}(L, \{\ell\})$$

the moduli space $\mathcal{M}_1^2(M, L, \{\ell\})$ is compact. (For if a sphere or disk bubbled off, then the remaining principal component of the disk would be constant and carry the marked point, therefore taking values in $\{\ell\}$. In the case of a sphere bubble this is impossible since $\mathcal{M}_1^1(M, J, \{\ell\})$ is empty. The case of a disk bubble is excluded since this configuration is not stable.) This moduli space is moreover oriented and zero dimensional, hence the sum in the definition of $w(L)$ is well defined. To see that $w(L)$ is independent of ℓ we consider the moduli space $\mathcal{M}_1^2(L, J, \gamma((0,1)))$ for an embedded path $\gamma : [0,1] \to L$ from $\gamma(0) = \ell_0$ to $\gamma(1) = \ell_1$. This is a smooth, oriented 1-dimensional manifold and a compactness argument similar to the one above shows that $\mathcal{M}_1^2(L, J, \gamma((0,1)))$ gives an oriented cobordism from $\mathcal{M}_1^2(M, L, \ell_0)$ to $\mathcal{M}_1^2(M, L, \ell_1)$, which shows that $w(L)$ is the same for either choice. Similarly, the independence of $w(L)$ from J follows from an oriented cobordism that is provided by the parametrized moduli space $\mathcal{M}_1^1(L, \{J_t\}_{t\in(0,1)}, \{\ell\})$ associated to a generic homotopy from J_0 to J_1. $\square$

We will now extend the definition of Floer cohomology, using the setup and notation from Section 3.

Theorem 7.2.2. *Let M be a compact monotone symplectic manifold and $L^0, L^1 \subset M$ oriented, monotone Lagrangian submanifolds (that is (M1-2) and (L1-2) hold with $\tau > 0$). Suppose that the pair (L^0, L^1) is monotone in the sense of Definition 3.1.2 and relatively spin in the sense of* [46]. *Then, for any $H \in \operatorname{Ham}(L^0, L^1)$ and for J in a subset $\mathcal{J}_t^{\mathrm{reg}}(L^0, L^1; H) \subset \mathcal{J}_t(M, \omega)$ of Baire second category, the Floer differential $\partial : CF(L^0, L^1) \to CF(L^0, L^1)$ satisfies*

$$\partial^2 = (w(L^0) - w(L^1))\,\mathrm{Id}\,.$$

The image $DF(L^0, L^1)$ of $(CF(L^0, L^1), \partial)$ in $D\operatorname{Fact}(w(L^0) - w(L^1))$ is independent of the choice of H and J, up to isomorphism.

Proof. We sketch the proof, following Oh in the case of $\mathbb{Z}_2$ coefficients. For any $x_\pm \in \mathcal{I}(L^0, L^1)$, the zero dimensional component $\mathcal{M}(x_-, x_+)_0$ of Floer trajectories is a finite set, as in Theorem 3.2.2 (b). From part (a) of that theorem we also know that the one-dimensional component $\mathcal{M}(x_-, x_+)_1$ is smooth, but the “compactness modulo breaking” in part (c) does not hold in general: Apart from the breaking of trajectories, a sequence of

Floer trajectories of Maslov index 2 could in the Gromov compactification converge to a constant trajectory and either a sphere bubble of Chern number one or a disk bubble of Maslov number two. All other bubbling effects are excluded by monotonicity. Thus failure of "compactness modulo breaking" occurs only when $x_- = x_+$.

In Theorem 3.2.2, the subset $\mathcal{J}_t^{\rm reg}(L^0, L^1; H)$ consists of those time-dependent almost complex structures $J : [0,1] \to \mathcal{J}(M, \omega)$ for which all $\mathcal{M}(x_-, x_+)$ are smooth and the universal moduli spaces of spheres $\mathcal{M}_1^1(M, \{J(t)\}_{t\in[0,1]}, \{x\})$ are empty for all $x \in \mathcal{I}(L^0, L^1)$. This excludes the Gromov convergence to a constant trajectory and a sphere bubble. We can now restrict to those $J \in \mathcal{J}_t^{\rm reg}(L^0, L^1; H)$ such that $J(k) \in \mathcal{J}^{\rm reg}(L^k, \{x\})$ for $k = 0, 1$ and all $x \in \mathcal{I}(L^0, L^1)$. This still defines a subset in $\mathcal{J}(M, \omega)$ of Baire second category. We claim that now each one-dimensional moduli space $\mathcal{M}(x, x)_1$ of self-connecting trajectories has a compactification as a one-dimensional manifold with boundary

$$\partial\overline{\mathcal{M}(x,x)_1} \;\cong\; \bigcup_y \big(\mathcal{M}(x,y)_0 \times \mathcal{M}(y,x)_0\big) \;\cup\; \mathcal{M}_1^2(L^0, J(0), \{x\})^- \;\cup\; \mathcal{M}_1^2(L^1, J(1), \{x\})$$

and that furthermore the orientations on these moduli spaces induced by the relative spin structures are compatible with the inclusion of the boundary. Here $\mathcal{M}_1^2(L^0, \{x\})^-$ denotes the moduli space $\mathcal{M}_1^2(L^0, \{x\})$ with orientation reversed. The proof of the claim uses a gluing theorem of non-transverse type for pseudoholomorphic maps with Lagrangian boundary conditions, which can be adapted from [26, Chapter 10] as follows: We replace L^0 with its translate under the Hamiltonian flow of H, then $\mathcal{I}(L^0, L^1) = L^0 \pitchfork L^1$ and the Floer trajectories are unperturbed J-holomorphic strips (where J has suffered some Hamiltonian transformation, too). Pick $v_k \subset \widetilde{\mathcal{M}}_1^2(L^k, J(k), \{x\})$, then the gluing construction gives maps for $k = 0, 1$

$$(T, \infty) \times (-T^{-1}, T^{-1}) \longrightarrow \widetilde{\mathcal{M}}^2(x, x) \tag{76}$$

to the moduli space of parametrized Floer trajectories of index 2. This construction identifies v_k with a map $v_k : \mathbb{H} \to M$ on the half space $\mathbb{H} \cong D \setminus \{1\}$; then for $(\tau, \sigma) \in (T, \infty) \times (-T^{-1}, T^{-1})$ it shifts this map by σ and outside of a half disk of radius $\frac{1}{2}\tau^{1/2}$ around 0, interpolates it to the constant solution x outside of the half disk of radius $\tau^{1/2}$ (using a slowly varying cutoff function in submanifold coordinates of L^0 and L^1 near $x \in M$). Then it rescales this map by τ to a half-disk of radius $\tau^{-1/2}$ and glues it to the constant solution on the infinite strip $\mathbb{R} \times [0, 1]$ on a half disk of radius $\tau^{-1/2}$ around the point $(0, k)$. The resulting map $u : \mathbb{R} \times [0, 1] \to M$ is an approximate Floer trajectory. An application of the implicit function theorem gives an exact solution for T sufficiently large.

It remains to examine the effect of the gluing on orientations. The gluing construction can equivalently be described by viewing the domain $(T, \infty) \times (-T^{-1}, T^{-1})$ of the gluing map as a subset of the automorphism group $\mathrm{Aut}(D, \partial D, 1) \cong (0, \infty) \times \mathbb{R}$ and identifying it with its image $U := \big((T, \infty) \times (-T^{-1}, T^{-1})\big) \cdot v_0 \subset \widetilde{\mathcal{M}}_1^2(L^k, J, \{x\})$ on v_0. The resulting map $U \to \widetilde{\mathcal{M}}^2(x, x)$ is simply the parametrized gluing map (with a fixed gluing parameter) on pairs of disks (after capping off the strip-like ends) and so orientation preserving by the definitions of [46]. Now the infinitesimal translation action of $(-T^{-1}, T^{-1})$ on U approximately agrees under the gluing with the infinitesimal translation action on $\widetilde{\mathcal{M}}^2(x, x)$ for $k = 0$, resp. its inverse for $k = 1$. So, after quotienting by the translations the gluing map induces an embedding $(T, \infty) \to \mathcal{M}(x, x)_1 = \widetilde{\mathcal{M}}^2(x, x)/\mathbb{R}$ for sufficiently large T, and taking the order of factors into account, we see that this embedding is orientation preserving resp. reversing for $k = 1$ resp. $k = 0$. Summing over the boundary of the one-dimensional manifold $\partial\overline{\mathcal{M}(x,x)_1}$ thus proves $\partial^2 - w(L^0)\,\mathrm{Id} + w(L^1)\,\mathrm{Id} = 0$.

The proof that the image of $(CF(L^0, L^1), \partial)$ in the derived category of matrix factorizations is independent of all choices up to isomorphism is essentially the same as that of Proposition 4.3.1, which produces a pair of chain maps whose compositions are null homotopic. □

Definition 7.2.3. Let $L^0, L^1 \subset M$ be a pair of Lagrangian submanifolds as in Theorem 7.2.2.

(a) We define the *derived Floer factorization* $DF(L^0, L^1)$ to be the image of the Floer matrix factorization $(CF(L^0, L^1), \partial)$ in $\mathrm{Obj}(D\,\mathrm{Fact}(w(L^0) - w(L^1)))$.

(b) We define the *Floer cohomology with coefficients in* $\mathbb{Z}_w$, $w := w(L^0) - w(L^1)$ to be the image

$$HF(L^0, L^1; \mathbb{Z}_w) := H\big((CF(L^0, L^1), \partial) \otimes_{\mathbb{Z}} \mathbb{Z}_w\big),$$

of $(CF(L^0, L^1), \partial)$ under the cohomology with coefficients functor (75).

Remark 7.2.4. (a) In the case $w = w(L^0) - w(L^1) = 0$ this definition coincides with the usual definition of Floer cohomology with $\mathbb{Z}$ coefficients in the sense that the functor taking cohomology with $\mathbb{Z}$ coefficients from $\mathcal{D}\,\mathrm{Fact}(0)$ to the category of finitely generated $\mathbb{Z}_2$-graded abelian groups induces a bijection between isomorphism classes of objects.

(b) Theorem 7.2.2 and Remark 7.1.3 show that the Floer cohomology $HF(L^0, L^1; \mathbb{Z}_w)$ is independent of all choices up to isomorphism in the category of abelian groups.

(c) The differential for a monotone pair $(L, \psi(L))$ with any symplectomorphism $\psi \in \mathrm{Symp}(M)$ always squares to zero, since $w(L) = w(\psi(L))$ by Proposition 7.2.1.

Remark 7.2.5. One advantage of the derived category is that duals and tensor products behave as expected. The following identities are immediate from the definitions.

(a) Suppose that (L^0, L^1) is a monotone, relatively spin pair of compact, oriented Lagrangian submanifolds. Then $DF(L^0, L^1) = DF(L^1, L^0)^\vee$.

(b) Suppose that (L_0^0, L_0^1) and (L_1^0, L_1^1) are monotone, relatively spin pairs of compact, oriented Lagrangian submanifolds in compact, monotone symplectic manifolds M_0 and M_1. Then $DF(L_0^0 \times L_1^0, L_0^1 \times L_1^1) = DF(L_0^0, L_0^1) \otimes DF(L_1^0, L_1^1)$.

In our main result 1.0.1 the assumption (L3) on the minimal Maslov number was needed only for the definition of the Floer cohomologies. The bijection between the trajectory spaces for small widths and for the composed Lagrangian correspondence in Theorem 5.0.5 only requires that the minimal Maslov number of the Lagrangians is at least two (which is automatic in the monotone orientable case). The comparison of orientations in [46] is also independent of Maslov indices, and the morphisms between Floer theories with strips of different widths also yield isomorphisms between the derived Floer factorizations. Hence we have the following generalized version of Theorem 1.0.1. (This also extends further to general sequences of Lagrangian correspondences as in Theorem 5.0.3.)

Theorem 7.2.6. *Let M_0, M_1, M_2 be compact symplectic manifolds satisfying (M1-2) with the same monotonicity constant $\tau > 0$, let*

$$L_0 \subset M_0, \quad L_{01} \subset M_0^- \times M_1, \quad L_{12} \subset M_1^- \times M_2, \quad L_2 \subset M_2^-$$

be Lagrangian submanifolds satisfying (L1-2), and assume that the pair $(L_0 \times L_{12}, L_{01} \times L_2)$ is monotone and relatively spin. Suppose that $L_{01} \circ L_{12}$ is smooth, embedded into $M_0^- \times M_2$ by

π_{02}, also satisfies (L1), and the pair $(L_0 \times L_2, L_{01} \circ L_{12})$ is monotone. Then, with respect to the induced relative spin structures, there exists a canonical isomorphism in $D\,\mathrm{Fact}$ between the derived objects $DF(L_0 \times L_{12}, L_{01} \times L_2)$ and $DF(L_0 \times L_2, L_{01} \circ L_{12})$.

7.3. **Derived relative invariants.** Given a quilted surface $\underline{S}$, a collection $\underline{M}$ of compact, monotone symplectic manifolds (satisfying (M1-2) with a fixed constant $\tau > 0$) and a collection $\underline{L}$ of compact, oriented, monotone Lagrangian boundary conditions that satisfy (L1-2) and are monotone and relatively spin in the sense of Section 4.1 , one obtains a derived invariant

$$D\Phi_{\underline{S}} : \bigotimes_{\underline{e} \in \mathcal{E}_+(\underline{S})} DF(\underline{L}_{\underline{e}}) \to \bigotimes_{\underline{e} \in \mathcal{E}_-(\underline{S})} DF(\underline{L}_{\underline{e}}),$$

where the derived objects are the images of the corresponding chain groups in the derived category of matrix factorizations. The proof is exactly the same as in the Floer cohomology case, since in fact we used the assumption on the minimal Maslov number only to make the Floer cohomology groups well-defined.

Example 7.3.1. (a) Given an admissible Lagrangian $L \subset M$ one obtains an *identity morphism* $I_L : \mathbb{Z} \to DF(L, L)$, where $\mathbb{Z}$ is the trivial complex in degree 0. The differential for $CF(L, L)$ automatically squares to zero by Remark 7.2.4, so after passing to cohomology the identity morphism I_L induces the identity object $1_L \in HF(L, L)$.

(b) Given an admissible triple L^0, L^1, L^2 of Lagrangians one obtains a *derived composition* morphism in $D\,\mathrm{Fact}$

$$D\mu_2 : DF(L^0, L^1) \times DF(L^1, L^2) \to DF(L^0, L^2).$$

The derived composition morphism is also associative.

One has a derived analog of Theorem 5.4.1, by the same arguments. The statement is left to the reader.

7.4. **Donaldson-Fukaya category of Lagrangians.** Let (M, ω) be a compact, monotone symplectic manifold, equipped with a Maslov cover $\mathrm{Lag}^N(M) \to \mathrm{Lag}(M)$ and background class $b \in H^2(M, \mathbb{Z}_2)$. One can define a category-like structure $\mathrm{Don}(M)$ by taking as objects the set of Lagrangian branes as in Definition 6.1.4, but without the assumption on the minimal Maslov number. To any pair of objects (L^0, L^1), the morphism object $\mathrm{Hom}(L^0, L^1) := DF(L^0, L^1)$ is an object in $D\,\mathrm{Fact}$. Composition is given by $D\mu_2$, which is a morphism in $D\,\mathrm{Fact}$.

$\mathrm{Don}(M)$ might be called a category enriched in the derived category of matrix factorizations, except that the morphism object is not a set; or a category object in the derived category of matrix factorizations, except that only the morphisms are objects in this category. The results of Section 6 hold with appropriate modifications of categories to categories enriched in $D\,\mathrm{Fact}$. In particular, one can define an enriched 2-category whose objects are compact monotone symplectic manifolds, morphisms are Lagrangian correspondences, and for each pair of 1-morphisms we have as 2-morphisms an object of $D\,\mathrm{Fact}$. The standard representation becomes a 2-functor to the 2-category whose objects are categories enriched in $D\,\mathrm{Fact}$, 1-morphisms are functors, and 2-morphisms are natural transformations.

The matrix factorizations here appear in [7] in the following guise. A *weak A_∞ category* consists of a collection of objects and composition maps $\mu_d, d \geq 0$, satisfying the A_∞ associativity relations modified to include an operation μ_0 which assigns to any object X

an element $\mu_0(X) \in \operatorname{Hom}(X,X)$. The first A_∞ relation is $(\mu_1)^2 = \mu_2(\mu_0 \otimes 1 - 1 \otimes \mu_0)$. Special features of the Fukaya category in the monotone case are that $\mu_2(\mu_0 \otimes 1 - 1 \otimes \mu_0)$ is a multiple of the identity morphism for any object X, and a homotopy of perturbation data produces a chain homotopy for μ_1, due to lack of disk bubbling.

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