Erratum 2:Uhlenbeck Compactness

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April 4, 2009

Thanks to Tim Nguyen some essential mistakes are identified and corrected.

The proof of Theorem 2.3' in case $k = 1$ relied on Theorem 4.3, whose proof had a gap, requiring the estimate in Theorem 2.3" below. This then also allows to prove Theorem 2.3' directly.

Theorem 2.3$'$ *For every $k \in \mathbb{N}$ and $1 < p < \infty$ there exists a constant C such that the following holds: Suppose that $u \in \mathcal{D}(M)$ is a weak solution of the Neumann problem* (wNP) *for $f \in W^{-k,p}(M)$. Then $u \in W^{-k+2,p}(M)$ and*

$$\|u\|_{W^{-k+2,p}} \leq C\bigl(\|f\|_{W^{-k,p}} + |\langle\, u\,,\,1\,\rangle|\bigr).$$

Theorem 2.3$''$ *For every $1 < p < \infty$ there exists a constant C such that*

$$\|\psi\|_{W^{1,p}} \leq C\bigl(\|\Delta\psi\|_{W^{-1,p}} + |\langle\, \psi\,,\,1\,\rangle|\bigr) \qquad \forall \psi \in \mathcal{C}^\infty_\nu(M).$$

Proof: Using Lemma E.3, the injectivity of the map $\psi \mapsto (\Delta\psi, \langle\, \psi, 1\,\rangle)$, and the compactness of the Sobolev embedding $W^{1,p}(M) \hookrightarrow L^p(M)$, it suffices to prove

$$\|\psi\|_{W^{1,p}} \leq C\bigl(\|\Delta\psi\|_{W^{-1,p}} + \|\psi\|_{L^p}\bigr) \qquad \forall \psi \in \mathcal{C}^\infty_\nu(M).$$

From Theorem 2.3 we have

$$\|\psi\|_{W^{2,p}} \leq C\bigl(\|\Delta\psi\|_{L^p} + \|\psi\|_{W^{1,p}}\bigr) \qquad \forall \psi \in \mathcal{C}^\infty_\nu(M).$$

The dual of this estimate is (as shown in the proof of Theorem 2.3' below)

$$\|\psi\|_{L^p} \leq C\bigl(\|\Delta_{(wNP)}\psi\|_{W^{-2,p^*}} + \|\psi\|_{W^{-1,p^*}}\bigr) \qquad \forall \psi \in \mathcal{C}^\infty_\nu(M),$$

where $\Delta_{(wNP)}$ is the operator weakly defined by (wNP). One can interpolate these, as in [S, Thm. 3.1] for a bounded domain $M \subset \mathbb{R}^n$ (with $A = \Delta$, $V = V' = \mathcal{C}^\infty_\nu(M)$, and $N = N'$ the constant functions) to prove the claimed estimate. Note here that Schechters norm

$$|u|_{-1,p} = \mathrm{lub}_{v \in \mathcal{C}^\infty_\nu(M)} \|v\|^{-1}_{W^{1,p^*}} \bigl|\langle\, u, v\,\rangle\bigr|$$

is equivalent to $\|u\|_{W^{-1,p}}$ since $\mathcal{C}^\infty_\nu(M) \subset W^{1,p^*}(M)$ is dense.

More in the spirit of this book, one can deduce the claimed $W^{1,p}$-estimates from the L^p-estimates in Theorem 2.3' for the Neumann problem and Theorem D.2' for the Dirichlet problem, applied to tangential resp. normal derivatives of ψ. That is, for any vector field $X \in \Gamma(TM)$ and $\phi \in \mathcal{C}^\infty(M)$ we calculate using Lemma 5.6

$$\begin{aligned}
&-\int_M \mathcal{L}_X\psi \cdot \Delta\phi \\
&= \int_M \psi \cdot \mathcal{L}_X \Delta\phi + \int_M \mathrm{div}X \cdot \psi \cdot \Delta\phi - \int_{\partial M} g(X,\nu) \cdot \psi \cdot \Delta\phi \\
&= \int_M \psi \cdot \Delta \mathcal{L}_X \phi + \int_M \psi\bigl([\mathcal{L}_X, \Delta]\phi + \mathrm{div}X \Delta\phi\bigr) - \int_{\partial M} g(X,\nu) \cdot \psi \cdot \Delta\phi \\
&= \int_M \Delta\psi \cdot \mathcal{L}_X \phi + \int_M \psi\bigl([\mathcal{L}_X, \Delta]\phi + \mathrm{div}X \Delta\phi\bigr) - \int_{\partial M} \psi\bigl(g(X,\nu)\Delta\phi + \tfrac{\partial}{\partial\nu}(\mathcal{L}_X\phi)\bigr).
\end{aligned}$$

If $X \in \Gamma(TM)$ is tangential to ∂M and such that $[X,\nu]=0$, then the boundary term vanishes for all $\phi \in \mathcal{C}^\infty_\nu(M)$, showing that

$$\begin{aligned}
\bigl|\textstyle\int_M \mathcal{L}_X\psi \cdot \Delta\phi\bigr| &\le \|\Delta\psi\|_{W^{-1,p}} \|\mathcal{L}_X\phi\|_{W^{1,p^*}} + \|\psi\|_{L^p} C'_X \|\phi\|_{W^{2,p^*}} \\
&\le C_X \bigl(\|\Delta\psi\|_{W^{-1,p}} + \|\psi\|_{L^p}\bigr)\|\phi\|_{W^{2,p^*}}
\end{aligned}$$

for some constants C'_X, C_X depending on X. Now the case $k=2$ of Theorem 2.3' provides

$$\|\mathcal{L}_X\psi\|_{L^p} \le C\bigl(\|\Delta_{(wNP)}\mathcal{L}_X\psi\|_{W^{-2,p^*}} + |\langle \mathcal{L}_X\psi, 1\rangle|\bigr) \le C'\bigl(\|\Delta\psi\|_{W^{-1,p}} + \|\psi\|_{L^p}\bigr),$$

where we estimated $|\langle \mathcal{L}_X\psi, 1\rangle| = |\int_M \mathrm{div}X \cdot \psi|$ by $\|\psi\|_{L^p}$.

In order to prove the Theorem it now remains to consider a vector field $X = \tilde\nu$ that restricts to the unit normal $\tilde\nu|_{\partial M} = \nu$ on the boundary. Then the boundary term in the above partial integration is $\int_{\partial M} \psi\bigl(\Delta\phi + \frac{\partial}{\partial\nu}(\mathcal{L}_{\tilde\nu}\phi)\bigr)$. We claim that this term is of first order in ϕ for all $\phi \in \mathcal{C}^\infty_\delta(M)$, i.e. with $\phi|_{\partial M} = 0$. Indeed, if we pick local coordinates near ∂M with $\partial_0 = \tilde\nu$ and $(\partial_j)_{j\ge 1}$ parallel to ∂M, then we can see that in highest order $-\Delta\phi = \partial_0^2\phi + \sum_{j\ge1}\partial_j^2\phi = \partial_0^2\phi = \frac{\partial}{\partial\nu}(\mathcal{L}_{\tilde\nu}\phi)$ since all tangential derivatives of ϕ vanish. Hence we can express this boundary term as the restriction $\Phi|_{\partial M} = \Delta\phi + \frac{\partial}{\partial\nu}(\mathcal{L}_{\tilde\nu}\phi)$ of a function $\Phi \in \mathcal{C}^\infty(M)$ given by first derivatives of ϕ near ∂M, multiplied with a cutoff function. Given any $\varepsilon > 0$ we can choose this cutoff function such that $\|\Phi\|_{L^{p^*}} \le \varepsilon\|\phi\|_{W^{2,p^*}}$ and $\|\Phi\|_{W^{1,p^*}} \le C_\varepsilon\|\phi\|_{W^{2,p^*}}$ with a constant C_ε depending on $\varepsilon > 0$. (To achieve this we multiply a given extension Φ by a cutoff function ζ and use the estimate $\|\zeta\Phi\|_{L^{p^*}} \le C\|\zeta\|_{L^q}\|\Phi\|_{W^{1,p^*}}$, which follows from Hölder and Sobolev estimates with $q=n$ for $p^* < n$, any $q > n$ for $p^* = n$, and $q = p^*$ for $p^* < n$.) With that we obtain

$$\begin{aligned}
\left|\int_{\partial M} \psi\bigl(\Delta\phi + \tfrac{\partial}{\partial\nu}(\mathcal{L}_{\tilde\nu}\phi)\bigr)\right| &= \left|\int_M \mathrm{d}^*\bigl(\psi \cdot \Phi \cdot g(\tilde\nu, \cdot)\bigr)\right| \\
&\le C_{\tilde\nu}\bigl(\|\psi\|_{L^p}\|\Phi\|_{W^{1,p^*}} + \|\psi\|_{W^{1,p}}\|\Phi\|_{L^{p^*}}\bigr) \\
&\le \bigl(\varepsilon\|\psi\|_{W^{1,p}} + C'_\varepsilon\|\psi\|_{L^p}\bigr)\|\phi\|_{W^{2,p^*}}
\end{aligned}$$

and hence for all $\phi \in \mathcal{C}^\infty_\delta(M)$

$$\left| \int_M \mathcal{L}_{\tilde\nu}\psi \cdot \Delta\phi \right| \leq \bigl(C_{\tilde\nu} \|\Delta\psi\|_{W^{-1,p}} + C'_\varepsilon \|\psi\|_{L^p} + \varepsilon \|\psi\|_{W^{1,p}} \bigr) \|\phi\|_{W^{2,p^*}} .$$

Now the estimate for the Dirichlet problem in case $k = 2$ of Theorem D.2' directly implies

$$\|\mathcal{L}_{\tilde\nu}\psi\|_{L^p} \leq C\bigl(C_{\tilde\nu} \|\Delta\psi\|_{W^{-1,p}} + C'_\varepsilon \|\psi\|_{L^p} + \varepsilon \|\psi\|_{W^{1,p}} \bigr).$$

Finally, we pick a generating set of tangential vector fields X_j and sum over all estimates for $\mathcal{L}_{X_j}\psi$ and $\mathcal{L}_{\tilde\nu}\psi$ to obtain

$$\begin{aligned} \|\psi\|_{W^{1,p}} &\leq C\bigl(\|\psi\|_{L^p} + \|\mathcal{L}_{\tilde\nu}\psi\|_{L^p} + \textstyle\sum_j \|\mathcal{L}_{X_j}\psi\|_{L^p} \bigr) \\ &\leq C(\varepsilon)\bigl(\|\Delta\psi\|_{W^{-1,p}} + \|\psi\|_{L^p} \bigr) + \varepsilon \|\psi\|_{W^{1,p}} \end{aligned}$$

with some constant $C(\varepsilon)$ depending on a free choice of $\varepsilon > 0$. Now picking $\varepsilon = \frac{1}{2}$ and rearranging the inequality proves the theorem. □

Proof of Theorem 2.3′ :
In case $k \geq 2$ this theorem follows directly from Theorems 2.2 and 2.3 by duality: Every $\phi \in \mathcal{C}^\infty(M)$ can be written as $\phi = \Delta\psi + c_\phi$, where $c_\phi = (\mathrm{Vol}\, M)^{-1} \int_M \phi$ and $\psi \in \mathcal{C}^\infty_\nu(M)$ such that $\|\psi\|_{W^{k,p^*}} \leq C' \|\phi - c_\phi\|_{W^{k-2,p^*}}$ for some constant C'. In case $k = 1$ this follows from Theorems 1.5 and 2.3". Now let $c := (\mathrm{Vol}\, M)^{-1} \langle u, 1 \rangle$, then we obtain

$$\begin{aligned} |\langle\, u - c\,,\, \phi \,\rangle| \;&=\; |\langle\, u - c\,,\, \phi - c_\phi \,\rangle| \;=\; \bigl|\langle\, u\,,\, \Delta\psi \,\rangle - c \textstyle\int_M \Delta\psi \bigr| \;=\; |\langle\, f\,,\, \psi \,\rangle| \\ &\leq\; \|f\|_{W^{-k,p}} \|\psi\|_{W^{k,p^*}} \;\leq\; C' \|f\|_{W^{-k,p}} \bigl(\|\phi\|_{W^{k-2,p^*}} + \bigl| \textstyle\int_M \phi \bigr| \bigr) \\ &\leq\; C \|f\|_{W^{-k,p}} \|\phi\|_{W^{k-2,p^*}} . \end{aligned}$$

Here the new constant C arises from $\bigl| \int_M \phi \bigr| = |\, \langle\, \phi, 1 \,\rangle \,| \leq \|\phi\|_{W^{k-2,p^*}} \|1\|_{W^{2-k,p}}$. Since the above estimate holds for all $\phi \in \mathcal{C}^\infty(M)$, it proves that $u - c$ and hence also u lie in $\bigl(W^{k-2,p^*}(M) \bigr)^* = W^{-k+2,p}(M)$ with

$$\begin{aligned} \|u\|_{W^{2-k,p}} &\leq \|u - c\|_{W^{2-k,p}} + \|c\|_{W^{2-k,p}} \\ &\leq C \|f\|_{W^{-k,p}} + (\mathrm{Vol}\, M)^{-1} \|1\|_{W^{2-k,p}} |\, \langle\, u\,,\, 1 \,\rangle \,| . \end{aligned}$$

□

In Theorem 4.7, the closedness of $\mathrm{im}\, \tilde D$ was shown by hiding the required estimate in a seemingly obvious functional analytic statement. However, the sum $\ker \nabla'_{q^*} \oplus \mathrm{im}\, \nabla_{q^*} \subset L^{q^*}(M, T^*M \otimes E)$ of the two closed subspaces on page 65 is closed only if we can establish an estimate $\|\nabla_{q^*} u\|_{q^*} \leq C \|\tau + \nabla_{q^*} u\|_{q^*}$ for all $\tau \in \ker \nabla'_{q^*}$. That is, we require an estimate $\|\nabla_{q^*} u\|_{q^*} \leq C \|\nabla'_{q^*} \nabla_{q^*} u\|_{W^{-1,q}}$, which will follow from Theorem 2.3". However, this estimate allows to drop all direct sum considerations and we are left with a much shorter proof.

Proof of Theorem 4.7′ :
We begin by deducing that $\operatorname{im}\tilde{D}$ is closed from Lemma E.3 (i), the compactness of the embedding $W^{1,q}\hookrightarrow L^q$, and the estimate for all $u\in W^{1,q}(M,E)$

$$\|u\|_{W^{1,q}} \;\leq\; C\bigl(\|\tilde{D}u\|_{W^{-1,q}}+\|u\|_q\bigr).$$

If ∇ is the trivial connection on a trivial bundle E, then $\tilde{D}u=\Delta u$ for all $u\in\mathcal{C}^\infty_\nu(M,E)$, and hence the estimate follows from Theorem 2.3" for the $W^{1,q}$-closure of $\mathcal{C}^\infty_\nu(M,E)$, i.e. for all $u\in W^{1,q}(M,E)$. For nontrivial bundles we use local trivializations and cutoff functions, and nontrivial connections introduce lower order terms. All of these can be estimated by the lower order term $\|u\|_q$. This finishes the proof of closedness of $\operatorname{im}\tilde{D}$.

Now we proceed as in the original proof: Let $u\in\ker\tilde{D}$, then lemma 4.1 asserts for all $\psi\in W^{2,q^*}(M,E)$

$$0 \;=\; \nabla'\nabla u(\psi) \;=\; \int_M \langle\,\nabla u\wedge *\nabla\psi\,\rangle \;=\; \int_M \langle\,u\,,\,\nabla^*\nabla\psi\,\rangle + \int_{\partial M}\langle\,u\,,\,\nabla_\nu\psi\,\rangle\,.$$

Thus $(u,u|_{\partial M})\in(\operatorname{im}D)^\perp$ with the operator D of theorem 4.6 for $p=q^*$, and this implies that $u\in\mathrm{H}^0(M,\nabla)$. On the other hand every horizontal section obviously lies in the kernel of $\tilde{D}$, so $\ker\tilde{D}=\mathrm{H}^0(M,\nabla)$ and this is of finite dimension as before in theorem 4.6.

The same argument can be used to show that $(\operatorname{im}\tilde{D})^\perp=\mathrm{H}^0(M,\nabla)$: Let $u\in(\operatorname{im}\tilde{D})^\perp\subset W^{1,q^*}(M,E)$, i.e. $\tilde{D}\psi(u)=0$ for all $\psi\in W^{1,q}(M,E)$. Then for all $\psi\in W^{2,q}(M,E)$ by lemma 4.1

$$0 \;=\; \nabla'\nabla\psi(u) \;=\; \int_M \langle\,\nabla\psi\wedge *\nabla u\,\rangle \;=\; \int_M \langle\,u\,,\,\nabla^*\nabla\psi\,\rangle + \int_{\partial M}\langle\,u\,,\,\nabla_\nu\psi\,\rangle\,.$$

This shows $(u,u|_{\partial M})\in(\operatorname{im}D)^\perp$ with $p=q$, and thus theorem 4.6 asserts that $u\in\mathrm{H}^0(M,\nabla)$. Conversely, every $u\in\mathrm{H}^0(M,\nabla)$ satisfies

$$\tilde{D}\psi(u) \;=\; \int_M \langle\,\nabla\psi\wedge *\nabla u\,\rangle \;=\; 0$$

for all $\psi\in W^{1,q}(M,E)$. So we have established $(\operatorname{im}\tilde{D})^\perp=\mathrm{H}^0(M,\nabla)$. Since $\operatorname{im}\tilde{D}$ is closed, the quotient norm is well defined on the cokernel $W^{1,q}(M,E)/\operatorname{im}\tilde{D}$ and makes it a Banach space. The cokernel has the same dimension as its dual space, which is isomorphic to $(\operatorname{im}\tilde{D})^\perp$. Thus $\operatorname{codim}\operatorname{im}\tilde{D}=\dim\mathrm{H}^0(M,\nabla)=\dim\ker\tilde{D}$ proving the Fredholm property and index 0 of $\tilde{D}$.

To determine the image of $\tilde{D}$ explicitly note that $\operatorname{im}\tilde{D}\subset\operatorname{im}\nabla'_q$ since ∇ maps $W^{1,q}(M,E)$ to $L^q(M,\mathrm{T}^*M\otimes E)$. On the other hand, by the definition of ∇' one has $\operatorname{im}\nabla'_q\subset\mathrm{H}^0(M,\nabla)^\perp=(\operatorname{im}\tilde{D})^{\perp\perp}=\operatorname{im}\tilde{D}$. Hence indeed $\operatorname{im}\tilde{D}=\operatorname{im}\nabla'_q$ as claimed. □

Finally, we correct some more missing boundary terms in a partial integration. These were only missing in the case $k\geq 1$ for $\alpha\in W^{k,p}(M,T^*M)$.

Proof of Theorem 5.3 (i)
Let $\alpha^\nu \in \mathcal{C}^\infty(M, \mathrm{T}^*M)$ be an L^p-approximating sequence for α such that $\alpha^\nu \equiv 0$ in a neighbourhood of ∂M. Then one obtains for all $\phi \in \mathcal{T}$

$$
\begin{aligned}
&\int_M \alpha(X) \cdot \Delta\phi = \lim_{\nu\to\infty} \int_M \mathrm{d}\iota_X \alpha^\nu \cdot \mathrm{d}\phi \\
&= \lim_{\nu\to\infty} \Big(\int_M \langle\, \mathcal{L}_X \alpha^\nu \,,\, \mathrm{d}\phi \,\rangle - \int_M \langle\, \iota_X \mathrm{d}\alpha^\nu \,,\, \mathrm{d}\phi \,\rangle \Big) \\
&= \lim_{\nu\to\infty} \Big(-\int_M \langle\, \alpha^\nu \,,\, \mathcal{L}_X \mathrm{d}\phi \,\rangle - \int_M \langle\, \alpha^\nu \,,\, \mathrm{div}X \cdot \mathrm{d}\phi \,\rangle \\
&\qquad\qquad + \int_M \langle\, \iota_{Y_{\alpha^\nu}} \mathcal{L}_X g \,,\, \mathrm{d}\phi \,\rangle - \int_M \langle\, \mathrm{d}\alpha^\nu \,,\, \iota_X g \wedge \mathrm{d}\phi \,\rangle \Big) \\
&= \lim_{\nu\to\infty} \Big(-\int_M \langle\, \alpha^\nu \,,\, \mathrm{d}(\mathcal{L}_X \phi) \,\rangle - \int_M \langle\, \alpha^\nu \,,\, \mathrm{d}^*(\iota_X g \wedge \mathrm{d}\phi) \,\rangle \\
&\qquad\qquad + \int_M \big\langle \big(\iota_{Y_{\alpha^\nu}} \mathcal{L}_X g \,-\, \mathrm{div}X \cdot \alpha^\nu \big) \,,\, \mathrm{d}\phi \big\rangle \Big) \\
&= -\int_M \langle\, \alpha \,,\, \mathrm{d}(\mathcal{L}_X \phi) \,\rangle - \int_M \langle\, \alpha \,,\, \mathrm{d}^*(\iota_X g \wedge \mathrm{d}\phi) \,\rangle \\
&\quad + \int_M \big\langle\, \mathrm{d}^*\big(\iota_{Y_\alpha} \mathcal{L}_X g \,-\, \mathrm{div}X \cdot \alpha \big) \,,\, \phi \,\big\rangle + \int_{\partial M} *\big(\iota_{Y_\alpha} \mathcal{L}_X g \,-\, \mathrm{div}X \cdot \alpha \big) \cdot \phi \\
&= \int_M \big(-f_1 - f_2 + \mathrm{d}^*\big(\iota_{Y_\alpha} \mathcal{L}_X g \,-\, \mathrm{div}X \cdot \alpha \big) \big)\, \phi \,+ \int_{\partial M} \big(\mathcal{L}_X g(Y_\alpha, \nu) \,-\, \mathrm{div}X \cdot \alpha(\nu) - h \big) \cdot \phi
\end{aligned}
$$

Here we used Cartan's formula $\mathcal{L}_X \alpha = \mathrm{d}\iota_X \alpha + \iota_X \mathrm{d}\alpha$, and the vector field Y_α is given by $\iota_{Y_\alpha} g = \alpha$. In case $\mathcal{T} = \mathcal{C}^\infty_\delta(M)$ the boundary vanishes and we obtain regularity and estimates for $\alpha(X)$ as before. In case $\mathcal{T} = \mathcal{C}^\infty_\nu(M)$ the above calculation shows that $\alpha(X)$ is a weak solution of the inhomogenous Neumann problem (3.4) for $f \in W^{k-1,p}(M)$ as before and with the boundary condition

$$
h - \mathcal{L}_X g(Y_\alpha, \nu) + \mathrm{div}X \cdot \alpha(\nu) \in W^{k,p}_\partial(M).
$$

So the regularity theorem 3.2 asserts that $\alpha(X) \in W^{k+1,p}(M)$ with the estimate

$$
\begin{aligned}
\|\alpha(X)\|_{W^{k+1,p}} &\leq C\big(\big\| -f_1 - f_2 + \mathrm{d}^*\big(\iota_{Y_\alpha} \mathcal{L}_X g \,-\, \mathrm{div}X \cdot \alpha \big) \big\|_{W^{k-1,p}} \\
&\qquad + \| h - \mathcal{L}_X g(Y_\alpha, \nu) + \mathrm{div}X \cdot \alpha(\nu) \|_{W^{k,p}_\delta} \big) \\
&\leq C\big(\|f_1\|_{W^{k-1,p}} + \|f_2\|_{W^{k-1,p}} + \|h\|_{W^{k,p}_\delta} + \|\alpha\|_{W^{k,p}} \big).
\end{aligned}
$$

Again, in the first estimate, the constant from theorem 3.2 depends continuously on the metric in the $W^{k,\infty}$-topology, but in the second inequality, the derivatives of g and X lead to continuous $W^{k+1,\infty}$-dependence of the constant on the metric and the vector field.

References

[S] M. Schechter, On Lp Estimates and Regularity I, *American Journal of Mathematics*, Vol. 85, No. 1 (1963), pp. 1–13.