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Anti-self-dual instantons with Lagrangian boundary conditions

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Abstract

We consider a nonlocal boundary condition for anti-self-dual instantons on four-manifolds with a space-time splitting of the boundary. This boundary condition naturally arises from making the Chern-Simons functional on a three-manifold with boundary closed: The restriction of the instanton to each time-slice of the boundary is required to lie in a Lagrangian submanifold of the moduli space of flat connections.

We establish the fundamental elliptic regularity and compactness properties of this boundary value problem. Firstly, every weak solution is gauge equivalent to a smooth solution. Secondly, all closed subsets of the moduli space of solutions with an L^p-bound on the curvature for $p > 2$ are compact. We moreover establish the Fredholm property of the linearized operator of this boundary value problem on compact four-manifolds. The proofs are based on a decomposition of the instantons near the boundary. Due to the global nature of the boundary condition the crucial regularity of one of these components has to be established by studying Cauchy-Riemann equations with totally real boundary conditions for functions with values in a complex Banach space.

These results provide the basic analytic set-up for the definition of a Floer homology for pairs consisting of a compact three-manifold with boundary and a Lagrangian submanifold in the moduli space of flat connections over the boundary. Such a Floer homology lies at the center of the program for the proof of the Atiyah-Floer conjecture by Salamon. The program aims to use this Floer homology as intermediate step for the conjectured natural isomorphism between the instanton Floer homology of a homology three-sphere and the symplectic Floer homology of two Lagrangian submanifolds in a moduli space of flat connections arising from a Heegard splitting of the homology three-sphere. These isomorphisms should result from adiabatic limits of the boundary value problem that is studied in this thesis.

Zusammenfassung

Wir betrachten eine nichtlokale Randbedingung für antiselbstduale Instantone auf Viermannigfaltigkeiten, deren Rand eine Raum-Zeit-Aufspaltung trägt. Diese Randbedingung ist die natürliche Bedingung, durch die das Chern-Simons-Funktional einer Dreimannigfaltigkeit mit Rand geschlossen wird: Die Einschränkung des Instantons auf jede Rand-Teilfläche konstanter Zeit liegt in einer Lagrangeschen Untermannigfaltigkeit des Modulraumes der flachen Zusammenhänge.

Wir zeigen die grundlegenden elliptischen Regularitäts- und Kompaktheits-Eigenschaften dieses Randwertproblemes. Jede schwache Lösung ist eichäquivalent zu einer starken Lösung. Weiterhin ist jede abgeschlossene Teilmenge des Modulraumes der Lösungen, die eine L^p-Schranke für die Krümmung mit $p > 2$ hat, kompakt. Zudem zeigen wir die Fredholm-Eigenschaft des linearisierten Operators dieses Randwertproblemes auf kompakten Viermannigfaltigkeiten. Die Beweise basieren auf einer Zerlegung des Instantons in der Nähe des Randes. Die Regularität einer dieser Komponenten macht es wegen der globalen Natur der Randbedingung nötig, Cauchy-Riemann-Gleichungen mit total reellen Randbedingungen für Funktionen mit Werten in einem komplexen Banachraum zu studieren.

Diese Resultate legen die analytischen Grundlagen für die Definition einer Floer-Homologie für Paare bestehend aus einer kompakten Dreimannigfaltigkeit mit Rand und einer Lagrangeschen Untermannigfaltigkeit des Modulraumes der flachen Zusammenhänge über dem Rand. Eine solche Floer-Homologie liegt im Zentrum des Beweisprogrammes von Salamon für die Atiyah-Floer-Vermutung. In diesem dient die erwähnte Floer-Homologie als Zwischenschritt in der Konstruktion des vermuteten natürlichen Isomorphismus' zwischen der Instanton-Floer-Homology einer Homologie-Dreisphäre und der symplektischen Floer-Homologie zweier Lagrangescher Untermannigfaltigkeiten eines Modulraumes flacher Zusammenhänge, die aus einer Heegard-Zerlegung der Homologie-Dreisphäre resultieren. Die zwei Isomorphismen in diesem Programm sollten sich aus adiabatischen Limites des in dieser Doktorarbeit betrachteten Randwertproblemes ergeben.

Contents

Chapter 1

Introduction

We begin by giving a brief introduction to the gauge theory of connections on principal bundles.

Let G be a compact Lie group. A principal G-bundle is a fibre bundle $\pi : P \to M$ with a free, transitive action of G on the fibres $\pi^{-1}(x) \cong \mathrm{G}$. The vertical subbundle $V = \ker(\mathrm{d}\pi) \subset \mathrm{T}P$ is then given by the tangencies $V_p = \{p\xi \mid \xi \in \mathfrak{g}\}$ to the orbits. Here $\mathfrak{g}$ denotes the Lie algebra of G and $p \mapsto p\xi$ is the infinitesimal action of $\xi \in \mathfrak{g}$. Now a horizontal distribution $H \subset \mathrm{T}P$ is an equivariant choice of complements $\mathrm{T}_pP = V_p \oplus H_p$.

Consider a trivialization $P|_U \cong U \times \mathrm{G}$ over $U \subset M$. Then horizontal distributions over U can be identified with $\mathfrak{g}$-valued 1-forms $A \in \Omega^1(U;\mathfrak{g})$ via the following identity: For every $p = (x, g) \in U \times \mathrm{G}$ one identifies $\mathrm{T}_xU \times \mathrm{T}_g\mathrm{G} \cong \mathrm{T}_pP$, then

$$H_p = \{(Y, -gA_x(Y)) \mid Y \in \mathrm{T}_xU\}.$$

Here the 1-form A determines the deviation from the natural horizontal distribution $H_{(g,x)} = \mathrm{T}_xU \times \{0\}$. On a nontrivial bundle there is no such natural choice, but still the horizontal distributions can be identified with the kernels $H_p = \ker A_p$ of certain 1-forms $A \in \mathcal{A}(P)$ called connections. The space of smooth connections $\mathcal{A}(P) \subset \Omega^1(P;\mathfrak{g})$ consists of equivariant $\mathfrak{g}$-valued 1-forms with fixed values on the vertical bundle.

The automorphisms of a principal G-bundle are of the form $p \mapsto p\,u(p)$ with an equivariant smooth map $u : P \to \mathrm{G}$ called a gauge transformation. These form the gauge group $\mathcal{G}(P)$ which acts on the space of connections: The pullback of a connection $A \in \mathcal{A}(P)$ under a G-bundle automorphism given by $u \in \mathcal{G}(P)$ is denoted by u^*A and is called gauge equivalent to A.

For simplicity of notation now consider the trivial G-bundle $P = M \times \mathrm{G}$ over a manifold M. Then the space of connections is $\mathcal{A}(M) := \Omega^1(M;\mathfrak{g})$, the gauge group can be viewed as $\mathcal{G}(M) := \mathcal{C}^\infty(M,\mathrm{G})$, and the action of a gauge transformation $u \in \mathcal{G}(M)$ on a connection $A \in \mathcal{A}(M)$ is given by

$$u^*A = u^{-1}Au + u^{-1}\mathrm{d}u.$$

A connection $A \in \mathcal{A}(M)$ is called flat if the associated horizontal distribution is locally integrable. This is equivalent to $\mathrm{d}_A \circ \mathrm{d}_A = 0$ for the exterior differential $\mathrm{d}_A : \Omega^k(M;\mathfrak{g}) \to \Omega^{k+1}(M;\mathfrak{g})$, $\mathrm{d}_A\omega = \mathrm{d}\omega + [A\wedge\omega]$ associated with the connection. One has $\mathrm{d}_A\mathrm{d}_A\omega = [F_A \wedge \omega]$ for all $\omega \in \Omega^k(M;\mathfrak{g})$, where

$$F_A = \mathrm{d}A + \tfrac{1}{2}[A\wedge A] \;\in\; \Omega^2(X;\mathfrak{g})$$

is called the curvature of A. So a connection is flat if and only if its curvature vanishes. Now the Yang-Mills energy of a connection,

$$\mathcal{YM}(A) = \int_M |F_A|^2,$$

is a gauge invariant measure for the nonintegrability of the horizontal distribution. The extrema of the Yang-Mills functional are the solutions of the Yang-Mills equation $\mathrm{d}_A^*F_A = 0$. Note that the definition of the Yang-Mills functional and more generally of the L^2-inner product of $\mathfrak{g}$-valued differential forms uses a metric on M as well as a G-invariant metric on G. The coderivative d_A^* then is defined as the formal L^2-adjoint of d_A. (More details about gauge theory and these notations can be found in appendix A.)

In the case of a manifold with boundary, the extrema of $\mathcal{YM}$ moreover satisfy the boundary condition $*F_A|_{\partial M} = 0$. So we call $A \in \mathcal{A}(M)$ a Yang-Mills connection if it satisfies the boundary value problem

$$\begin{cases} \mathrm{d}_A^*F_A = 0, \\ *F_A|_{\partial M} = 0. \end{cases}$$

Extrema of $\mathcal{YM}$ actually are weak Yang-Mills connections, that is they satisfy the weak equation

$$\int_M \langle\, F_A \,,\, \mathrm{d}_A\beta \,\rangle = 0 \qquad \forall\, \beta \in \Omega^1(M;\mathfrak{g}),$$

where β runs through all smooth 1-forms. However, one has the following regularity result, which we state in the case of a G-bundle (not necessarily trivial) over a compact 4-manifold M. Here we use the notation $\mathcal{A}^{k,p}(M)$

and $\mathcal{G}^{k,p}(M)$ for the $W^{k,p}$-Sobolev completions of the space of connections and the gauge group.

Theorem (Regularity for Yang-Mills connections)
Let $p > 2$. Then for every weak Yang-Mills connection $A \in \mathcal{A}^{1,p}(M)$ there exists a gauge transformation $u \in \mathcal{G}^{2,p}(M)$ such that u^*A is smooth.

An elementary observation in gauge theory is that the moduli space of gauge equivalence classes of flat connections on a G-bundle over a compact manifold M is compact in the $\mathcal{C}^\infty$-topology. This is obvious from the fact that the gauge equivalence classes of flat connections over M are in one-to-one correspondence with the conjugacy classes of representations of the fundamental group of M, c.f. theorem A.2. The Uhlenbeck compactness theorems are a remarkable generalization of this result. We again state them in the case of a (not necessarily trivial) G-bundle over a compact 4-manifold M. Proofs of these and the above theorem can be found in [U2],[DK], or [We] (explicitly containing the case of manifolds with boundary).

Theorem (Weak Uhlenbeck Compactness)
Let $(A^\nu)_{\nu\in\mathbb{N}} \subset \mathcal{A}^{1,p}(M)$ be a sequence of connections and suppose that $\|F_{A^\nu}\|_p$ is uniformly bounded for some $p > 2$. Then there exists a subsequence (again denoted $(A^\nu)_{\nu\in\mathbb{N}}$) and a sequence of gauge transformations $u^\nu \in \mathcal{G}^{2,p}(M)$ such that ${u^\nu}^*A^\nu$ converges weakly in $\mathcal{A}^{1,p}(M)$.

Theorem (Strong Uhlenbeck Compactness)
Let $(A^\nu)_{\nu\in\mathbb{N}} \subset \mathcal{A}^{1,p}(M)$ be a sequence of weak Yang-Mills connections and suppose that $\|F_{A^\nu}\|_p$ is uniformly bounded for some $p > 2$. Then there exists a subsequence (again denoted $(A^\nu)_{\nu\in\mathbb{N}}$) and a sequence of gauge transformations $u^\nu \in \mathcal{G}^{2,p}(M)$ such that ${u^\nu}^*A^\nu$ converges uniformly with all derivatives to a smooth connection $A \in \mathcal{A}(M)$.

An important application of Uhlenbeck's theorems is the compactification of the moduli space of (gauge equivalence classes of) anti-self-dual instantons over a four-manifold. These compactified moduli spaces are the central ingredients in the construction of the Donaldson invariants of smooth four-manifolds [D2] and of the instanton Floer homology groups of three-manifolds [F1]. Anti-self-dual instantons are special first order solutions of the Yang-Mills equation described in the following.

Let M be a closed oriented 4-manifold, choose a metric, and denote the associated Hodge operator by $*$. Then the curvature $F_A = F_A^+ + F_A^-$ splits into a self-dual and an anti-self-dual part, $F_A^\pm = \frac{1}{2}(F_A \pm *F_A)$. Now an elementary calculation shows that the Yang-Mills energy can be rewritten as

$$\mathcal{YM}(A) = -\int_M \langle\, F_A \wedge F_A \,\rangle \;+\; 2\int_M |F_A^+|^2. \tag{1.1}$$

The first term on the right hand side is a topological invariant of the bundle $P \to M$. (For example, in the case $\mathrm{G} = \mathrm{SU}(2)$ this invariant equals $8\pi^2 c_2(P)$, where $c_2(P)$ is the second Chern number and the inner product on $\mathfrak{su}(2)$ is given by the negative trace of the product of the matrices.) If this invariant is nonnegative, then the minima of the Yang-Mills functional are exactly the anti-self-dual instantons, i.e. connections $A \in \mathcal{A}(M)$ with

$$F_A + *F_A = 0.$$

The formula (1.1) shows that a sequence of anti-self-dual connections on a given bundle automatically has an L^2-bound on the curvature. Uhlenbeck's compactness theorem however does not extend to the case $p = 2$ due to the bubbling phenomenon:

The conformally invariant Yang-Mills energy $\mathcal{YM}(A) = \|F_A\|_{L^2}^2$ can concentrate at single points. If this happens at an interior point, then rescaling near that point yields a sequence of connections on balls of increasing radii whose limit modulo gauge is a nontrivial anti-self-dual instanton that extends to S^4. Its energy equals some positive constant times a characteristic number of the bundle. (This number is nonzero due to the nontriviality of the instanton; for an SU(2)-bundle it is the second Chern number.) So on a closed manifold and for a suitably chosen subsequence this bubbling only occurs at finitely many points. On the complement of these points, Uhlenbeck's compactness theorems ensure $\mathcal{C}^\infty$-convergence on all compact subsets. Now Uhlenbeck's removable singularity theorem [U1] guarantees that the limit connection extends over the 4-manifold (to a connection on a bundle of lower characteristic number). This leads to a compactification of the moduli space of anti-self-dual instantons. In the case of simply connected 4-manifolds with negative definite intersection forms Donaldson used these compactified moduli spaces to prove his famous theorem about the diagonizability of intersection forms [D1].

Now in the case of a 4-manifold with boundary note that the boundary condition $*F_A|_{\partial M} = 0$ for anti-self-dual instantons implies that the curvature vanishes altogether at the boundary. This is an overdetermined boundary value problem comparable to Dirichlet boundary conditions for holomorphic maps. [1] As in the latter case it is natural to consider weaker Lagrangian boundary conditions.

More precisely, a natural boundary condition for holomorphic maps with values in an almost complex manifold (X, J) is to take boundary values in a totally real submanifold. Recall that $J \in \mathrm{End}(\mathrm{T}M)$ is called an almost complex structure if $J^2 = -\mathbb{1}$. A submanifold $L \subset X$ is called totally real if $\mathrm{T}_x L \oplus J\mathrm{T}_x L = \mathrm{T}_x X$ for all $x \in L$. So essentially, it suffices to have Dirichlet boundary conditions for half of the components of the holomorphic map. Special cases of almost complex manifolds are symplectic manifolds (X, ω) with ω-compatible almost complex structures. In this symplectic case the Lagrangian submanifolds are examples of totally real submanifolds.

We consider a version of such Lagrangian boundary conditions for anti-self-dual instantons and prove that they suffice to obtain the analogue of the above regularity and compactness results for Yang-Mills connections.

For that purpose we consider oriented 4-manifolds X with a space-time splitting of the boundary, i.e. each connected component of ∂X is diffeomorphic to $\mathcal{S} \times \Sigma$, where $\mathcal{S}$ is a 1-manifold and Σ is a closed Riemann surface. We shall study a boundary value problem associated to a gauge invariant Lagrangian submanifold $\mathcal{L}$ of the space of flat connections on Σ: The restriction of the anti-self-dual instanton to each time-slice of the boundary is required to belong to $\mathcal{L}$. This boundary condition arises naturally from examining the Chern-Simons functional on a 3-manifold Y with boundary Σ. Namely, the Langrangian boundary condition renders the Chern-Simons

[1] Consider for example the abelian case $\mathrm{G} = S^1$, then the anti-self-duality equation is $\mathrm{d}A = 0$. The gauge freedom is eliminated by going to a local slice, $\mathrm{d}^* A = 0$ and $*A|_{\partial M} = 0$. This already gives the elliptic boundary value problem $\Delta A = 0$ with Dirichlet boundary conditions for the normal component of A and Neumann boundary conditions for the components of A in directions tangential to the boundary. Indeed, let $A = A_0 \mathrm{d}x_0 + \sum_{i=1}^{n-1} A_i \mathrm{d}x_i$, where x_0 is the coordinate normal to the boundary ∂M and the x_i are coordintes of ∂M. Then the local slice boundary condition is $A_0|_{x_0=0} = 0$ and the boundary condition $*F_A|_{\partial M} = 0$ gives $\partial_0 A_i|_{x_0=0} = 0$ for all $i = 1, \ldots, n-1$. So the solutions of this boundary value problem are already unique (up to a constant in the tangential components). But now the anti-self-duality equation gives the additional boundary condition $F_A|_{\partial M} = 0$, in above coordinates $\partial_i A_j|_{x_0=0} = \partial_j A_i|_{x_0=0}$ for all $i, j \in \{1, \ldots, n-1\}$. This makes the boundary value problem overdetermined.

1-form on the space of connections closed, see [Sa1]. The resulting gradient flow equation leads to the boundary value problem studied in this thesis (for the case $X = \mathbb{R} \times Y$), as will be explained in chapter 2. Our main results establish the basic regularity and compactness properties as well as the Fredholm theory, the latter for the compact model case $X = S^1 \times Y$.

Boundary value problems for Yang-Mills connections were already considered by Donaldson in [D3]. He studies the Hermitian Yang-Mills equation for connections induced by Hermitian metrics on holomorphic bundles over a compact Kähler manifold Z with boundary. Here the unique solubility of the Dirichlet problem (prescribing the metric over the boundary) leads to an identification between framed holomorphic bundles over Z (meaning a holomorphic bundle with a fixed trivialization over ∂Z) and Hermitian Yang-Mills connections over Z. In particular, when Z has complex dimension 1 and boundary $\partial Z = S^1$, this links loop groups to moduli spaces of flat connections over Z. This observation suggests an alternative approach to Atiyah's [A1] correspondence between holomorphic curves in the loop group of a compact Lie group G and anti-self-dual instantons on G-bundles over the 4-sphere. The correspondence might be established via an adiabatic limit relating holomorphic spheres in the moduli space of flat connections over the disc to anti-self-dual instantons over the product (of sphere and disc). Our motivation for studying the present boundary value problem lies more in the direction of another such correspondence between holomorphic curves in moduli spaces of flat connections and anti-self-dual instantons – the Atiyah-Floer conjecture for Heegard splittings of a homology-3-sphere.

A Heegard splitting $Y = Y_0 \cup_\Sigma Y_1$ of a closed 3-manifold Y is a decomposition into two handlebodies Y_0 and Y_1 with common boundary Σ. (A handlebody is a 3-ball with a finite number of solid handles attached.) It gives rise to two Floer homologies, i.e. generalized Morse homologies. Firstly, the moduli space M_Σ of gauge equivalence classes of flat connections on the trivial SU(2)-bundle over Σ is a symplectic manifold (with singularities) and the moduli spaces L_{Y_i} of flat connections over Σ that extend to Y_i are Lagrangian submanifolds of M_Σ (see chapter 4). The symplectic Floer homology $\mathrm{HF}_*^{\mathrm{symp}}(M_\Sigma, L_{Y_0}, L_{Y_1})$ is now generated by the intersection points of the Lagrangian submanifolds (as critical points of a generalized Morse theory). The critical points in the case of the instanton Floer homology $\mathrm{HF}_*^{\mathrm{inst}}(Y)$ are the flat connections over Y.

Conjecture (Atiyah, Floer) *Let $Y = Y_0 \cup_\Sigma Y_1$ be a Heegard splitting of a homology 3-sphere. Let L_{Y_i} be the Lagrangian submanifolds of the moduli space M_Σ of flat $\mathrm{SU}(2)$-connections over Σ given by the flat connections over Σ that extend to Y_i. Then there exists a natural isomorphism between the instanton and symplectic Floer homologies*

$$\mathrm{HF}_*^{\mathrm{inst}}(Y) \cong \mathrm{HF}_*^{\mathrm{symp}}(M_\Sigma, L_{Y_0}, L_{Y_1}).$$

The program for the proof by Salamon, [Sa1], is to define the instanton Floer homology $\mathrm{HF}_*^{\mathrm{inst}}(Y, L)$ for 3-manifolds with boundary $\partial Y = \Sigma$ using boundary conditions associated with a Lagrangian submanifold $L \subset M_\Sigma$. Then the conjectured isomorphism might be established in two steps via the intermediate $\mathrm{HF}_*^{\mathrm{inst}}([0,1] \times \Sigma, L_{Y_0} \times L_{Y_1})$ by adiabatic limit type arguments similar to [DS2].

Fukaya was the first to suggest the use of Lagrangian boundary conditions in order to define a Floer homology for 3-manifolds Y with boundary, [Fu1]. He studies a slightly different equation, involving a degeneration of the metric in the anti-self-duality equation, and uses SO(3)-bundles that are nontrivial over the boundary ∂Y. Now there are interesting examples, where one has to work with the trivial bundle. For example, on a handlebody Y there exists no nontrivial G-bundle for connected G. So if one considers any 3-manifold Y with the Lagrangian submanifold $\mathcal{L}_{Y'}$, the space of flat connections on $\partial Y = \partial Y'$ that extend over a handlebody Y', then one also deals with the trivial bundle. So if one wants to use Floer homology on 3-manifolds with boundary to prove the Atiyah-Floer conjecture, then it is crucial to extend this construction to the case of trivial SU(2)-bundles. There are two approaches that suggest themselves for such a generalization. One would be the attempt to extend Fukaya's construction to the case of trivial bundles, and another would be to follow the alternative construction outlined in [Sa1]. The present thesis follows the second route and sets up the basic analysis for this theory. We will only consider trivial G-bundles. However, our main theorems A, B, and C below generalize directly to nontrivial bundles – just the notation becomes more cumbersome.

The main results

We consider the following class of Riemannian 4-manifolds. Here and throughout all Riemann surfaces are closed 2-dimensional manifolds. Moreover, unless otherwise mentioned, all manifolds are allowed to have a smooth boundary.

Definition 1.1 *A* **4-manifold with a boundary space-time splitting** *is a pair (X, τ) with the following properties:*

(i) X is an oriented 4-manifold which can be exhausted by a nested sequence of compact deformation retracts.

(ii) $\tau = (\tau_1, \ldots, \tau_n)$ is an n-tuple of embeddings $\tau_i : \mathcal{S}_i \times \Sigma_i \to X$ with disjoint images, where Σ_i is a Riemann surface and $\mathcal{S}_i$ is either an open interval in $\mathbb{R}$ or is equal to $S^1 = \mathbb{R}/\mathbb{Z}$.

(iii) The boundary ∂X is the union

$$\partial X = \bigcup_{i=1}^{n} \tau_i(\mathcal{S}_i \times \Sigma_i).$$

Definition 1.2 *Let (X, τ) be a 4-manifold with a boundary space-time splitting. A Riemannian metric g on X is called* **compatible** *with τ if for each $i = 1, \ldots n$ there exists a neighbourhood $\mathcal{U}_i \subset \mathcal{S}_i \times [0, \infty)$ of $\mathcal{S}_i \times \{0\}$ and an extension of τ_i to an embedding $\bar{\tau}_i : \mathcal{U}_i \times \Sigma_i \to X$ such that*

$$\bar{\tau}_i^* g = \mathrm{d}s^2 + \mathrm{d}t^2 + g_{s,t}.$$

Here $g_{s,t}$ is a smooth family of metrics on Σ_i and we denote by s the coordinate on $\mathcal{S}_i$ and by t the coordinate on $[0, \infty)$.

We call a triple (X, τ, g) with these properties a **Riemannian 4-manifold with a boundary space-time splitting**.

Remark 1.3 In definition 1.2 the extended embeddings $\bar{\tau}_i$ are uniquely determined by the metric as follows. The restriction $\bar{\tau}_i|_{t=0} = \tau_i$ to the boundary is prescribed, and the paths $t \mapsto \bar{\tau}_i(s, t, z)$ are normal geodesics.

Example 1.4 Let $X := \mathbb{R} \times Y$, where Y is a compact oriented 3-manifold with boundary $\partial Y = \Sigma$, and let $\tau : \mathbb{R} \times \Sigma \to X$ be the obvious inclusion. Given any two metrics g_- and g_+ on Y there exists a metric g on X such that $g = \mathrm{d}s^2 + g_-$ for $s \leq -1$, $g = \mathrm{d}s^2 + g_+$ for $s \geq 1$, and (X, τ, g) satisfies the conditions of definition 1.2. The metric g cannot necessarily be chosen in the form $\mathrm{d}s^2 + g_s$ (one has to homotop the embeddings and the metrics).

Now let (X, τ, g) be a Riemannian 4-manifold with boundary space-time splitting and consider a trivial G-bundle over X for a compact Lie group G. Let $p > 2$, then for each $i = 1, \dots, n$ the Banach space of connections $\mathcal{A}^{0,p}(\Sigma_i)$ carries the symplectic form $\omega(\alpha, \beta) = \int_{\Sigma_i} \langle\, \alpha \wedge \beta \,\rangle$. We fix an n-tuple $\mathcal{L} = (\mathcal{L}_1, \dots, \mathcal{L}_n)$ of Lagrangian submanifolds $\mathcal{L}_i \subset \mathcal{A}^{0,p}(\Sigma_i)$ that are contained in the space of flat connections and that are gauge invariant,

$$\mathcal{L}_i \subset \mathcal{A}^{0,p}_{\text{flat}}(\Sigma_i) \qquad \text{and} \qquad u^*\mathcal{L}_i = \mathcal{L}_i \quad \forall u \in \mathcal{G}^{1,p}(\Sigma_i).$$

Here $\mathcal{A}^{0,p}_{\text{flat}}(\Sigma_i)$ is the space of weakly flat L^p-connections on Σ_i (see chapter 3). A submanifold $\mathcal{L}$ of a symplectic Banach space (Z, ω) is called Lagrangian if it is isotropic, i.e. $\omega|_{\mathcal{L}} \equiv 0$, and is of maximal dimension. To make the latter precise in this infinite dimensional setting, choose a complex structure $J \in \operatorname{End} Z$ that is compatible with ω (i.e. $\omega(\cdot, J\cdot)$ defines a metric on Z). The condition of maximal dimension is now phrased as the topological sum

$$\mathrm{T}_z\mathcal{L} \oplus J\mathrm{T}_z\mathcal{L} = \mathrm{T}_z Z \qquad \forall z \in \mathcal{L}.$$

This condition is independent of the choice of J since the space of compatible complex structures on (Z, ω) is connected [MS1, Proposition 2.48]. Next, the assumptions on the $\mathcal{L}_i$ ensure that the quotients

$$L_i \ := \ \mathcal{L}_i/\mathcal{G}(\Sigma_i) \subset \mathcal{A}^{0,p}_{\text{flat}}(\Sigma_i)/\mathcal{G}^{1,p}(\Sigma_i) \ =: \ M_{\Sigma_i}$$

are (singular) Lagrangian submanifolds in the (singular) moduli space of flat connections, c.f. chapter 4. We consider the following boundary value problem for connections $A \in \mathcal{A}^{1,p}_{\text{loc}}(X)$

$$\left\{ \begin{array}{l} *F_A + F_A = 0, \\ \tau_i^* A|_{\{s\}\times\Sigma_i} \in \mathcal{L}_i \quad \forall s \in \mathcal{S}_i,\ i = 1, \dots, n. \end{array} \right. \tag{1.2}$$

Observe that the boundary condition is meaningful since for every neighbourhood $\mathcal{U} \times \Sigma$ of a boundary component one has the continuous embedding $W^{1,p}(\mathcal{U} \times \Sigma) \subset W^{1,p}(\mathcal{U}, L^p(\Sigma)) \hookrightarrow \mathcal{C}^0(\mathcal{U}, L^p(\Sigma))$. The first nontrivial observation is that every connection in $\mathcal{L}_i$ is gauge equivalent to a smooth connection on Σ_i and hence $\mathcal{L}_i \cap \mathcal{A}(\Sigma)$ is dense in $\mathcal{L}_i$, see theorem 3.1. Moreover, every $W^{1,p}_{\rm loc}$-connection on X satisfying the boundary condition in (1.2) can be locally approximated by smooth connections satisfying the same boundary condition, see corollary 4.2.

Note that the present boundary value problem is a first order equation with first order boundary conditions (flatness in each time-slice). Moreover, the boundary conditions contain some crucial nonlocal (i.e. Lagrangian) information. We moreover emphasize that while $\mathcal{L}_i$ is a smooth submanifold of $\mathcal{A}^{0,p}(\Sigma_i)$, the quotient $\mathcal{L}_i/\mathcal{G}^{1,p}(\Sigma_i)$ is not required to be a smooth submanifold of the moduli space $M_{\Sigma_i} := \mathcal{A}^{0,p}_{\rm flat}(\Sigma_i)/\mathcal{G}^{1,p}(\Sigma_i)$. For example, $\mathcal{L}_i$ could be the set of flat connections on Σ_i that extend to flat connections over a handlebody with boundary Σ_i, see lemma 4.3. To overcome the difficulties arising from the singularities in the quotient, we will work with the (smooth) quotient by the based gauge group.

We will not be interested in existence results for the present boundary value problem but in its elliptic properties. The following two theorems are the main regularity and compactness results for the solutions of (1.2). The regularity theorem is the analogue of the regularity theorem for Yang-Mills connections stated above. The compactness theorem deals with connections satisfying uniform L^p-bounds on the curvature and thus extends Uhlenbeck's strong compactness theorem for anti-self-dual instantons to the present boundary value problem.

Theorem A (Regularity)
Let $p > 2$. Then every solution $A \in \mathcal{A}^{1,p}_{\rm loc}(X)$ of (1.2) is gauge equivalent to a smooth solution, i.e. there exists a gauge transformation $u \in \mathcal{G}^{2,p}_{\rm loc}(X)$ such that $u^*A \in \mathcal{A}(X)$ is smooth.

Theorem B (Compactness)
Let $p > 2$ and let g^ν be a sequence of metrics compatible with τ that uniformly converges with all derivatives on every compact set to a smooth metric. Suppose that $A^\nu \in \mathcal{A}^{1,p}_{\rm loc}(X)$ is a sequence of solutions of (1.2) with respect to the metrics g^ν such that for every compact subset $K \subset X$ there is a uniform bound on the curvature $\|F_{A^\nu}\|_{L^p(K)}$. Then there exists a subsequence (again denoted A^ν) and a sequence of gauge transformations $u^\nu \in \mathcal{G}^{2,p}_{\rm loc}(X)$ such that ${u^\nu}^* A^\nu$ converges uniformly with all derivatives on every compact set to a smooth connection $A \in \mathcal{A}(X)$.

The difficulty of these results lies in the global nature of the boundary condition. This makes it impossible to directly generalize the proof of the regularity and compactness theorems for Yang-Mills connections, where one chooses suitable local gauges, obtains the higher regularity and estimates from an elliptic boundary value problem, and then patches the gauges together. With our global Lagrangian boundary condition one cannot obtain local regularity results.

However, an approach by Salamon can be generalized to manifolds with boundary. One first uses Uhlenbeck's weak compactness theorem to find a weakly $W^{1,p}_{\rm loc}$-convergent subsequence. The limit then serves as reference connection with respect to which a further subsequence can be put into relative Coulomb gauge globally (on large compact sets). Then one has to establish elliptic estimates and regularity results for the given boundary value problem together with the relative Coulomb gauge equations. All the general tools for this approach are established in [We]: We give a detailed proof of the generalization of weak Uhlenbeck compactness to manifolds with boundary that are exhausted by compact deformation retracts. We also generalize Salamon's subtle local slice theorem to compact manifolds with boundary. Moreover, we give a precise formulation of the procedure by Donaldson and Kronheimer that allows to extend regularity and compactness results on compact deformation retracts of noncompact manifolds to the full manifolds.

In this thesis, we concentrate on the last step – the higher regularity and estimates for the boundary value problem in relative Coulomb gauge. Here the crucial point is to establish the higher regularity or estimates for the Σ-component of the connections in a neighbourhood $\mathcal{U}\times\Sigma$ of a boundary component. The global nature of the boundary condition forces us to deal with a Cauchy-Riemann equation on $\mathcal{U}$ with values in the Banach space $\mathcal{A}^{0,p}(\Sigma)$ and with Lagrangian boundary conditions.

The case $2<p\leq 4$, when $W^{1,p}$-functions are not automatically continuous, poses some special difficulties in this last step. Firstly, in order to obtain regularity results from the Cauchy-Riemann equation, one has to straighten out the Lagrangian submanifold by going to suitable coordinates. This requires a $\mathcal{C}^0$-convergence of the connections, which in case $p>4$ is given by a standard Sobolev embedding. In case $p>2$ one still obtains a special compact embedding $W^{1,p}(\mathcal{U}\times\Sigma)\hookrightarrow\mathcal{C}^0(\mathcal{U},L^p(\Sigma))$ that suits our purposes. Secondly, the straightening of the Lagrangian introduces a nonlinearity in the Cauchy-Riemann equation that already poses some problems in case $p>4$.

In case $p \leq 4$ this forces us to deal with the Cauchy-Riemann equation with values in an L^2-Hilbert space and then use some interpolation inequalities for Sobolev norms.

Now as a first step towards the definition of a Floer homology for 3-manifolds with boundary consider the moduli space of finite energy solutions of (1.2),

$$\mathcal{M}(\mathcal{L}) := \left\{ A \in \mathcal{A}^{1,p}_{\rm loc}(X) \,\middle|\, A \text{ satisfies } (1.2), \mathcal{YM}(A) < \infty \right\} / \mathcal{G}^{2,p}_{\rm loc}(X).$$

Theorem A implies that for every equivalence class $[A] \in \mathcal{M}(\mathcal{L})$ one can find a smooth representative $A \in \mathcal{A}(X)$. Theorem B is one step towards a compactness result for $\mathcal{M}(\mathcal{L})$: Every closed subset of $\mathcal{M}(\mathcal{L})$ with a uniform L^p-bound for the curvature is compact. In addition, theorem B allows the metric to vary, which is relevant for the metric-independence of the Floer homology.

Our third main result is a step towards proving that the moduli space $\mathcal{M}(\mathcal{L})$ of solutions of (1.2) is a manifold whose components have finite (but possibly different) dimensions. This also exemplifies our hope that the further analytical details of Floer theory will work out along the usual lines once the right analytic setup has been found in the proof of theorems A and B.

In the context of Floer homology and in Floer-Donaldson theory it is important to consider 4-manifolds with cylindrical ends. This requires an analysis of the asymptotic behaviour which will be carried out elsewhere. We shall restrict the discussion of the Fredholm theory to the compact case. The crucial point is the behaviour of the linearized operator near the boundary; in the interior we are dealing with the usual anti-self-duality equation. Hence it suffices to consider the following model case. Let Y be a compact oriented 3-manifold with boundary $\partial Y = \Sigma$ and suppose that $(g_s)_{s\in S^1}$ is a smooth family of metrics on Y such that

$$X = S^1 \times Y, \qquad \tau : S^1 \times \Sigma \to X, \qquad g = \mathrm{d}s^2 + g_s$$

satisfy the assumptions of definition 1.2. Here the space-time splitting τ of the boundary is the obvious inclusion $\tau : S^1 \times \Sigma \hookrightarrow \partial X = S^1 \times \Sigma$, where $\Sigma = \bigcup_{i=1}^n \Sigma_i$ might be a disjoint union of Riemann surfaces Σ_i. The above n-tuple of Lagrangian submanifolds $\mathcal{L}_i \subset \mathcal{A}^{0,p}(\Sigma_i)$ then defines a gauge invariant Lagrangian submanifold $\mathcal{L} := \mathcal{L}_1 \times \ldots \times \mathcal{L}_n$ of the symplectic Banach space $\mathcal{A}^{0,p}(\Sigma) = \mathcal{A}^{0,p}(\Sigma_1) \times \ldots \times \mathcal{A}^{0,p}(\Sigma_n)$ such that $\mathcal{L} \subset \mathcal{A}^{0,p}_{\rm flat}(\Sigma)$.

In order to linearize the boundary value problem (1.2) together with the local slice condition, fix a smooth connection $A+\Phi \mathrm{d}s \in \mathcal{A}(S^1\times Y)$ such that $A_s := A(s)|_{\partial Y} \in \mathcal{L}$ for all $s\in S^1$. Here $\Phi\in\mathcal{C}^\infty(S^1\times Y,\mathfrak{g})$ and $A\in\mathcal{C}^\infty(S^1\times Y,\mathrm{T}^*Y\otimes\mathfrak{g})$ is an S^1-family of 1-forms on Y (not a 1-form on X as previously). Now let $E^{1,p}_A$ be the space of S^1-families of 1-forms $\alpha\in W^{1,p}(S^1\times Y,\mathrm{T}^*Y\otimes\mathfrak{g})$ that satisfy the boundary conditions

$$*\alpha(s)|_{\partial Y}=0 \qquad \text{and} \qquad \alpha(s)|_{\partial Y}\in \mathrm{T}_{A_s}\mathcal{L} \qquad \text{for all } s\in S^1.$$

Then the linearized operator

$$D_{(A,\Phi)} : E^{1,p}_A \times W^{1,p}(S^1\times Y,\mathfrak{g}) \longrightarrow L^p(S^1\times Y,\mathrm{T}^*Y\otimes\mathfrak{g})\times L^p(S^1\times Y,\mathfrak{g})$$

is given with $\nabla_s=\partial_s+[\Phi,\cdot]$ by

$$D_{(A,\Phi)}(\alpha,\varphi) \;=\; \bigl(\nabla_s\alpha-\mathrm{d}_A\varphi+*\mathrm{d}_A\alpha\,,\,\nabla_s\varphi-\mathrm{d}_A^*\alpha\bigr).$$

The second component of this operator is $-\mathrm{d}^*_{A+\Phi\mathrm{d}s}(\alpha+\varphi\mathrm{d}s)$, and the first boundary condition is $*(\alpha+\varphi\mathrm{d}s)|_{\partial X}=0$, corresponding to the choice of a local slice at $A+\Phi\mathrm{d}s$. In the first component of $D_{(A,\Phi)}$ we have used the global space-time splitting of the metric on $S^1\times Y$ to identify the self-dual 2-forms $*\gamma_s-\gamma_s\wedge\mathrm{d}s$ with families γ_s of 1-forms on Y. The vanishing of this component is equivalent to the linearization $\mathrm{d}^+_{A+\Phi\mathrm{d}s}(\alpha+\varphi\mathrm{d}s)=0$ of the anti-self-duality equation (see chapter 7). Furthermore, the boundary condition $\alpha(s)|_{\partial Y}\in\mathrm{T}_{A_s}\mathcal{L}$ is the linearization of the Lagrangian boundary condition in the boundary value problem (1.2).

Theorem C (Fredholm)
Let Y be a compact oriented 3-manifold with boundary $\partial Y=\Sigma$ and let $S^1\times Y$ be equipped with a product metric $\mathrm{d}s^2+g_s$ that is compatible with $\tau: S^1\times\Sigma\to S^1\times Y$. Let $A+\Phi\mathrm{d}s\in\mathcal{A}(S^1\times Y)$ such that $A(s)|_{\partial Y}\in\mathcal{L}$ for all $s\in S^1$. Then the following holds for all $p>2$.

(i) $D_{(A,\Phi)}$ is Fredholm.

(ii) There is a constant C such that for all $\alpha\in E^{1,p}_A$ and $\varphi\in W^{1,p}(S^1\times Y,\mathfrak{g})$

$$\|(\alpha,\varphi)\|_{W^{1,p}}\le C\bigl(\|D_{(A,\Phi)}(\alpha,\varphi)\|_{L^p}+\|(\alpha,\varphi)\|_{L^p}\bigr).$$

(iii) Let $q \geq p^*$ such that $q \neq 2$. Suppose that $\beta \in L^q(S^1 \times Y, \mathrm{T}^*Y \otimes \mathfrak{g})$, $\zeta \in L^q(S^1 \times Y, \mathfrak{g})$, and assume that there exists a constant C such that for all $\alpha \in E^{1,p}_A$ and $\varphi \in W^{1,p}(S^1 \times Y, \mathfrak{g})$

$$\left| \int_{S^1 \times Y} \langle\, D_{(A,\Phi)}(\alpha, \varphi)\, ,\, (\beta, \zeta)\, \rangle \right| \;\leq\; C\, \|(\alpha, \varphi)\|_{L^{q^*}} .$$

Then in fact $\beta \in W^{1,q}(S^1 \times Y, \mathrm{T}^*Y \otimes \mathfrak{g})$ and $\zeta \in W^{1,q}(S^1 \times Y, \mathfrak{g})$.

Here and throughout we use the notation $\frac{1}{p} + \frac{1}{p^*} = 1$ for the conjugate exponent p^* of p. The above inner product $\langle\,\cdot,\cdot\,\rangle$ is the pointwise inner product in $\mathrm{T}^*Y \otimes \mathfrak{g} \times \mathfrak{g}$. The reason for our assumption $q \neq 2$ in theorem C (iii) is a technical problem in dealing with the singularities of $\mathcal{L}/\mathcal{G}^{1,p}(\Sigma)$. We resolve these singularities by dividing only by the based gauge group. This leads to coordinates of $L^p(\Sigma, \mathrm{T}^*\Sigma \otimes \mathfrak{g})$ in a Banach space that comprises based Sobolev spaces $W^{1,p}_z(\Sigma, \mathfrak{g})$ of functions vanishing at a fixed-point $z \in \Sigma$. So these coordinates that straighten out $\mathrm{T}\mathcal{L}$ along $A|_{S^1 \times \partial Y}$ are welldefined only for $p > 2$. Now in order to prove the regularity claimed in theorem C (iii) we have to use such coordinates either for β or for the test 1-forms α, i.e. we have to assume that either $q > 2$ or $q^* > 2$. This is completely sufficient for our purposes – concluding a higher regularity of elements of the cokernel. This will be done via an iteration of theorem C (iii) that can always be chosen such as to jump across $q = 2$. However, we believe that the use of different coordinates should permit to extend this result.

Conjecture *Theorem C (iii) continues to hold for $q = 2$.*

One indication for this conjecture is that the L^2-estimate in theorem C (ii) is true (for $W^{1,p}$-regular α and ϕ with $p > 2$), as will be shown in chapter 7. This L^2-estimate can be proven by a much more elementary method than the general L^p-regularity and -estimates. In fact, it was already stated in [Sa1] as an indication for the wellposedness of the boundary value problem (1.2).

Besides the L^p-compactness and Fredholm theory, the construction of an instanton Floer homology for 3-manifolds with boundary moreover requires an analysis of the bubbling at the boundary for sequences of solutions of (1.2) with bounded energy. Here the standard rescaling technique runs into problems since the local rescaling fails to capture the full information of the boundary condition. The flatness of the connection on each time-slice of the boundary is a local condition, but the Lagrangian condition is global, i.e. it can only be stated for a connection on the full time-slice. The rescaling, however, cuts out small balls in each time-slice. In [Sa1] Salamon expected to obtain anti-self-dual instantons on the half space bubbling off at the boundary, and he conjectured a quantization of their energy. The convergence of the rescaled connections to such instantons on the half space, however, would require some additional information (coming from the Lagrangian boundary condition) on the local behaviour of the finite energy solutions of (1.2) near the boundary.

Alternatively, the analytic framework of the regularity and compactness results obtained in this thesis suggests a global treatment of the time-slices. This might lead to holomorphic discs in the space of connections with Lagrangian boundary conditions, that would bubble off as a result of 2-dimensional rescaling only in the time- and interior direction (preserving the full time-slices). In this thesis we will give some partial results about the possible bubbling phenomena at the boundary.

Outline

This thesis is organized as follows. We give a short introduction to Floer homology and the Atiyah-Floer conjecture in chapter 2. We moreover explain in more detail the program for the proof by Salamon and the motivation for the boundary value problem (1.2).

In chapter 3, we introduce the notion of a weakly flat connection on a general closed manifold. In theorem 3.1, we prove that weakly flat connections are gauge equivalent to smooth connections that are flat in the usual sense. This uses a general technical result, lemma 3.3, that will be applied at several points. It allows to extract regularity results and estimates for individual components of a 1-form from weak equations where boundary conditions are imposed on the test 1-forms. In a subsection we discuss more closely the space of weakly flat connections on a Riemann surface, $\mathcal{A}^{0,p}(\Sigma)$, and its quotients by the gauge group and the based gauge group.

Chapter 4 describes the crucial properties of the Lagrangian submanifolds $\mathcal{L} \subset \mathcal{A}^{0,p}(\Sigma)$ for which the boundary value problem (1.2) will be considered. In particular, corollary 4.2 shows that the space of $W^{1,p}$-connections with Lagrangian boundary conditions is the closure of a space of smooth connections.

Moreover, a subsection introduces the main example $\mathcal{L}_Y$ of such Lagrangian submanifolds, the set of connections on the boundary ∂Y of a handlebody that extend to a flat connection on Y.

Chapter 5 deals with Cauchy-Riemann equations with totally real boundary conditions for functions with values in a complex Banach space. We establish the usual regularity results and estimates under one crucial assumption. In order to obtain L^p-regularity results or estimates, the totally real submanifold (and hence also the complex Banach space) has to be modelled on a closed subspace of an L^p-space. This is the general setting for the key part of the boundary value problem (1.2) and allows to deal with the nonlocal Lagrangian boundary condition. The results in this chapter are central for the proofs of theorems A, B, and C.

In chapter 6, we prove the regularity and compactness result for (1.2), theorems A and B, and chapter 7 establishes the Fredholm theory in the compact case, theorem C. Finally, in chapter 8, we give an outlook on the bubbling analysis at the boundary and prove some partial results.

The appendix gives a more detailed introduction into the basic notations and constructions in gauge theory. This appendix is taken from [We], a forthcoming monograph. The latter is an extensive exposition of the general analytic background of gauge theory. In particular, it contains detailed proofs of Uhlenbeck's compactness and removable singularity theorems as well as the regularity theorem for Yang-Mills connections. The generalizations to manifolds with boundary, noncompact manifolds, and varying metrics seem to not have been written up before. Moreover, this exposition has been set up in such a way as to provide a suitable analytic framework for the treatment of the nonlocal boundary conditions in the present thesis. So in a number of places we will quote results from [We] without proof – these are generally wellknown but cannot be found elsewhere in the precise formulation that we need.

Chapter 2

Floer homology and the Atiyah-Floer conjecture

The Atiyah-Floer conjecture belongs into the general realm of interaction between symplectic geometry and low dimensional topology. Important progress in these areas has been made in the last twenty years starting with the work of Donaldson [D2] on smooth four-manifolds, which was based on anti-self-dual instantons, and with the work of Gromov [Gr] on pseudoholomorphic curves in symplectic manifolds. In both subjects Floer, inspired by Conley and Witten, introduced in the late eighties his idea of a Morse theory for functionals on infinite dimensional spaces, where the critical points have infinite index and coindex, but the relative indices are finite.

As in Morse theory, the chain complex is generated by the critical points of the functional, and the boundary operator is constructed by counting connecting orbits between critical points. Floer's connecting orbits are finite energy solutions of certain nonlinear elliptic equations arising from the gradient flow equation of the functional, which itself is not well-posed. Floer carried out this program in two cases, [F1, F2], which have been developed further by a large number of authors.

The instanton Floer homology $\mathrm{HF}_*^{\mathrm{inst}}(Y)$ of a 3-manifold Y is defined in terms of the Chern–Simons functional on the space $\mathcal{A}(Y)$ of connections over Y (on the trivial SU(2)- or a nontrivial SO(3)-bundle),

$$\mathcal{CS}(A) = \tfrac{1}{2}\int_Y \Big(\langle\, A \wedge \mathrm{d}A \,\rangle + \tfrac{1}{3}\langle\, [A \wedge A] \wedge A \,\rangle \Big). \tag{2.1}$$

In this case the critical points are gauge equivalence classes (using only the identity component of the gauge group) of flat connections, and the connecting orbits are the anti-self-dual instantons on $\mathbb{R}\times Y$ with finite Yang-Mills energy.

In the symplectic case Floer considered two Lagrangian submanifolds L_0 and L_1 in a symplectic manifold (M,ω). The critical points of the symplectic action functional on the space of paths connecting L_0 to L_1 are the constant paths, that is the intersection points of L_0 and L_1. The connecting orbits are pseudoholomorphic strips $u:\mathbb{R}\times[0,1]\to M$ of finite energy $\frac{1}{2}\int|\nabla u|^2$, whose boundary arcs $s\mapsto u(s,0)$ and $s\mapsto u(s,1)$ lie in L_0 and L_1 respectively. Here one has to choose an almost complex structure $J\in\mathrm{End}(\mathrm{T}M)$, $J^2=-\mathbb{1}$ that is compatible with the symplectic form, i.e. $\omega(\cdot,J\cdot)$ is a metric on M. Then a pseudoholomorphic curve is a solution of

$$\partial_s u + J(u)\,\partial_t u = 0.$$

Under certain monotonicity assumptions this gives rise to Floer homology groups $\mathrm{HF}_*^{\mathrm{symp}}(M,L_0,L_1)$. Similarly, one can define Floer homology groups $\mathrm{HF}_*^{\mathrm{symp}}(M,\varphi)$ for symplectomorphisms $\varphi\in\mathrm{Diff}(M,\omega)$, [F3]. Here the critical points are the fixed-points of φ, and the connecting orbits are finite energy pseudoholomorphic curves $u:\mathbb{R}^2\to M$ satisfying a twist condition associated with φ.

Atiyah and Floer conjectured the existence of natural isomorphisms between the instanton Floer homology of 3-manifolds and the symplectic Floer homology of Lagrangians or symplectomorphisms on moduli spaces of flat connections associated with the 3-manifold. The starting point for these conjectures is the Atiyah-Bott [AB, §9] picture of the moduli space of flat connections over a Riemann surface as a symplectic quotient.

The moduli space of flat connections over a Riemann surface

Let $P\to\Sigma$ be a G-bundle over a Riemann surface Σ with a compact Lie group G. The space of connections on P is an affine space $\mathcal{A}(P)=\tilde{A}+\Omega^1(\Sigma;\mathfrak{g}_P)$ for any fixed reference connection $\tilde{A}$. It carries a symplectic structure: For $\alpha,\beta\in\mathrm{T}_A\mathcal{A}(P)=\Omega^1(\Sigma;\mathfrak{g}_P)$

$$\omega(\alpha,\beta) := \int_\Sigma \langle\, \alpha\wedge\beta \,\rangle. \tag{2.2}$$

The action of the gauge group $\mathcal{G}(P)$ on $\mathcal{A}(P)$ can be viewed as Hamiltonian action of an infinite dimensional Lie group. The Lie algebra of $\mathcal{G}(P)$ is $\Omega^0(\Sigma;\mathfrak{g}_P)$ and the infinitesimal action of $\xi \in \Omega^0(\Sigma;\mathfrak{g}_P)$ is given by the vector field

$$X_\xi : \begin{array}{ccc} \mathcal{A}(P) & \longrightarrow & \Omega^1(\Sigma;\mathfrak{g}_P) = \mathrm{T}_A\mathcal{A}(P) \\ A & \longmapsto & \mathrm{d}_A\xi. \end{array}$$

This is the Hamiltonian vector field of the function $A \mapsto \langle\, \mu(A)\,,\, \xi \,\rangle$ on $\mathcal{A}(P)$. Here $\langle\, \zeta, \xi \,\rangle = \int_\Sigma \langle\, \zeta, \xi \,\rangle$ denotes the inner product on the Lie algebra $\Omega^0(\Sigma;\mathfrak{g}_P)$ and μ is the moment map of the gauge action,

$$\mu : \begin{array}{ccc} \mathcal{A}(P) & \longrightarrow & \Omega^0(\Sigma;\mathfrak{g}_P) \\ A & \longmapsto & *F_A. \end{array}$$

Indeed, one has for all $\beta \in \Omega^1(\Sigma;\mathfrak{g}_P)$

$$\omega(X_\xi(A)\,,\,\beta) \;=\; \int_\Sigma \langle\, \mathrm{d}_A\xi \wedge \beta \,\rangle \;=\; -\langle\, *\mathrm{d}_A\beta\,,\,\xi\,\rangle,$$

where $*\mathrm{d}_A$ is the differential of the moment map at $A \in \mathcal{A}(P)$. For a flat connection $A \in \mathcal{A}_{\mathrm{flat}}(P)$ one has a chain complex

$$\begin{array}{ccccc} \mathrm{Lie}\,\mathcal{G}(P) & \longrightarrow & \mathrm{T}_A\mathcal{A} & \longrightarrow & \mathrm{Lie}\,\mathcal{G}(P) \\ \| & & \| & & \| \\ \Omega^0(\Sigma;\mathfrak{g}_P) & \xrightarrow{\mathrm{d}_A} & \Omega^1(\Sigma;\mathfrak{g}_P) & \xrightarrow{*\mathrm{d}_A} & \Omega^0(\Sigma;\mathfrak{g}_P) \end{array}$$

Here the second arrow is the infinitesimal action at A and the third arrow is the differential of the moment map at A. If A is moreover irreducible[1], then the quotient $\mu^{-1}(0)/\mathcal{G}(P) = \mathcal{A}_{\mathrm{flat}}(P)/\mathcal{G}(P) =: M_P$ is a smooth manifold near the equivalence class of A. (This can be made precise in the setting of Banach manifolds – using the Sobolev completions of the space of connections and the gauge group.) So the moduli space of connections M_P is a smooth manifold with singularities at the reducible connections. It can moreover be seen as the symplectic quotient of the Hamiltonian gauge action on the symplectic space of connections,

$$M_P \;=\; \mathcal{A}(P)/\!/\mathcal{G}(P) \;=\; \mu^{-1}(0)/\mathcal{G}(P).$$

So M_P is a (singular) symplectic manifold with the symplectic structure induced by (2.2).

[1] A connection $A \in \mathcal{A}_{\mathrm{flat}}(P)$ is called irreducible if its isotropy subgroup of $\mathcal{G}(P)$ (the group of gauge transformations that leave A fixed) is discrete, i.e. $\mathrm{d}_A|_{\Omega^0}$ is injective. For a closed Riemann surface this is equivalent to $\mathrm{d}_A^*|_{\Omega^1}$ being surjective.

The Atiyah–Floer conjecture for mapping tori

A first version of the Atiyah–Floer conjecture for mapping tori was confirmed by Dostoglou and Salamon [DS2]: Consider a nontrivial SO(3)-bundle $P \to \Sigma$ over a Riemann surface Σ with an orientation preserving automorphism $f : P \to P$. Similarly as above, the moduli space $M_P := \mathcal{A}_{\text{flat}}(P)/\mathcal{G}_0(P)$ of flat connections on P modulo the identity component $\mathcal{G}_0(P)$ of the gauge group is a compact symplectic manifold, and the pullback of connections under f obviously induces a symplectomorphism $\varphi_f : M_P \to M_P$. So one has a symplectic Floer homology $\mathrm{HF}_*^{\text{symp}}(\varphi_f)$. On the other hand, the automorphism f induces an orientation preserving diffeomorphism $f : \Sigma \to \Sigma$ (by abuse of notation), which gives rise to an oriented closed 3-manifold, the mapping cylinder $Y_f := \mathbb{R} \times \Sigma / \sim$ with $(t+1, z) \sim (t, f(z))$ for all $t \in \mathbb{R}$ and $z \in \Sigma$. The same mapping cylinder construction yields a nontrivial SO(3)-bundle P_f over Y_f. This can be used to define the instanton Floer homology $\mathrm{HF}_*^{\text{inst}}(Y_f)$.

Theorem 2.1 (Dostoglou,Salamon) *Let $f : P \to P$ be an orientation preserving automorphism of a nontrivial* SO(3)*-bundle $P \to \Sigma$ over a Riemann surface Σ. There is a natural isomorphism*

$$\mathrm{HF}_*^{\text{inst}}(Y_f) \cong \mathrm{HF}_*^{\text{symp}}(\varphi_f).$$

The proof relates pseudoholomorphic sections of a bundle over a cylinder with fibre M_P to anti-self-dual instantons on the corresponding 4-manifold (in which the fibre is replaced by Σ itself). The relation is established by an adiabatic limit argument in which the metric on Σ converges to zero. More precisely, a flow line of the symplectic Floer homology is a pseudoholomorphic cylinder, $u : \mathbb{R}^2 \to M_P$ with the twist condition $u(s, t+1) = \varphi_f(u(s,t))$ and

$$\partial_s u + J(u)\partial_t u = 0.$$

Here the almost complex structure J on M_P is induced by the Hodge operator on Σ. One can lift u to a map $A : \mathbb{R}^2 \to \mathcal{A}_{\text{flat}}(P)$ with $A(s, t+1) = f^* A(s,t)$. Then the pseudoholomorphic equation is equivalent to

$$\partial_s A + *\partial_t A = \mathrm{d}_A \Phi + *\mathrm{d}_A \Psi,$$

where $\Phi, \Psi : \mathbb{R}^2 \to \Omega^0(\Sigma, \mathfrak{g}_P)$ are uniquely determined by A and satisfy the same twist condition as A. Now one can view $A + \Phi \mathrm{d}s + \Psi \mathrm{d}t$ as connection on $\mathbb{R}\times P_f$ which satisfies

$$\begin{aligned}\partial_s A - \mathrm{d}_A\Phi + *\partial_t A - *\mathrm{d}_A\Psi &= 0,\\ *F_A &= 0.\end{aligned} \tag{2.3}$$

On the other hand, a flow line of the instanton Floer homology is an (equivalence class of an) anti-self-dual instanton on $\mathbb{R}\times P_f$, i.e. a connection $A+\Phi\mathrm{d}s+\Psi\mathrm{d}t$ that satisfies

$$\begin{aligned}\partial_s A - \mathrm{d}_A\Phi + *\partial_t A - *\mathrm{d}_A\Psi &= 0,\\ \partial_t\Phi - \partial_s\Psi - [\Phi,\Psi] - \varepsilon^{-2} * F_A &= 0.\end{aligned} \tag{2.4}$$

Here one can fix any $\varepsilon > 0$ by rescaling the metric on Σ by the factor ε^2, since the instanton Floer homology is independent of the metric on the base manifold. An adaptation of Uhlenbeck's compactness theorems shows that sequences of such anti-self-dual instantons for $\varepsilon\to 0$ converge (modulo gauge) to solutions of (2.3). Now the gauge equivalence classes of these solutions are exactly the pseudoholomorphic cylinders. Conversely, an implicit function argument shows that for sufficiently small $\varepsilon > 0$ near every solution of (2.3) one finds a solution of (2.4). So this gives a bijection between the flow lines of the symplectic and the instanton Floer homology. Moreover, the critical points of both Floer homologies can be naturally identified as follows. A connection $A+\Psi\mathrm{d}t\in\mathcal{A}(\mathbb{R}\times P)$ is flat if $F_A=0$ and $\dot{A}-\mathrm{d}_A\Psi=0$. So if one makes Ψ vanish by a gauge transformation $u\in\mathcal{G}(\mathbb{R}\times P)$ then one obtain a constant path $u^*(A+\Psi\mathrm{d}t)\equiv A_0\in\mathcal{A}_{\text{flat}}(P)$. Now such a constant path comes from a flat conection $A+\Psi\mathrm{d}t$ on $P_f=\mathbb{R}\times P/\sim$ precisely when f^*A_0 is equivalent by a gauge transformation in the identity component of the gauge group to A_0, that is the gauge equivalence class of A_0 is a fixed-point of the map ϕ_f induced by f^* on the moduli space M_P. This indentifies the critical points of the instanton Floer homology with those of the symplectic Floer homology.

The Atiyah–Floer conjecture for homology 3-spheres

The original version in [A2] of the Atiyah–Floer conjecture for Heegard splittings of homology 3-spheres is still open: A Heegard splitting $Y=Y_0\cup_\Sigma Y_1$ of a homology 3-sphere is a decomposition into two handlebodies Y_0, Y_1 with common boundary Σ. It gives rise to two Floer homologies. The moduli space

M_Σ of flat connections on the trivial SU(2)-bundle over Σ is a finite dimensional symplectic manifold (with singularities) and the moduli spaces L_{Y_i} of flat connections over Σ that extend to Y_i are Lagrangian submanifolds of M_Σ (see chapter 4). Atiyah and Floer conjectured that the resulting Lagrangian Floer homology should be isomorphic to the instanton Floer homology of Y.

Conjecture 2.2 (Atiyah,Floer) *Let $Y = Y_0 \cup_\Sigma Y_1$ be a Heegard splitting of a homology 3-sphere. Then there exists a natural isomorphism*

$$\mathrm{HF}_*^{\mathrm{inst}}(Y) \cong \mathrm{HF}_*^{\mathrm{symp}}(M_\Sigma, L_{Y_0}, L_{Y_1}).$$

Taubes [T] proved that the Euler characteristics agree. Salamon [Sa1] outlined a program for a proof of the Atiyah–Floer conjecture for homology 3-spheres. The central point of this program is a Lagrangian boundary value problem for anti-self-dual instantons that is motivated by the Chern-Simons functional.

Chern-Simons functional on 3-manifolds with boundary

In this subsection we give a geometric motivation for the boundary value problem that is treated in this thesis. The map

$$\mathcal{F}_A : \alpha \mapsto \int_Y \langle F_A \wedge \alpha \rangle$$

defines a 1-form $\mathcal{F}$ on the space of connections $\mathcal{A}(Y)$ on the trivial G-bundle over a compact 3-manifold Y. Near a connection $A \in \mathcal{A}(Y)$ with trivial isotropy group the space of connections $\mathcal{A}(Y)$ can be seen as $\mathcal{G}(Y)$-bundle over $M_Y := \mathcal{A}(Y)/\mathcal{G}(Y)$. The 1-form $\mathcal{F}$ on this bundle is gauge invariant, and if Y is closed, then it is also horizontal, i.e. it vanishes on the fibres: For all vertical tangent vectors $\alpha = \mathrm{d}_A\xi$ with $\xi \in \Omega^0(Y;\mathfrak{g})$ Stokes' theorem gives

$$\mathcal{F}_A(\mathrm{d}_A\xi) \;=\; -\int_Y \langle\, \mathrm{d}_A F_A \,,\, \xi \,\rangle + \int_{\partial Y} \langle\, F_A \,,\, \xi \,\rangle \;=\; 0. \tag{2.5}$$

So for closed manifolds Y this 1-form descends to a 1-form on the (singular) moduli space M_Y, and moreover $\mathcal{F}$ is closed. Indeed, one can view $\alpha, \beta \in \Omega^1(Y;\mathfrak{g}) = \mathrm{T}_A\mathcal{A}(Y)$ as (constant) vector fields on $\mathcal{A}(Y)$, then their

Lie bracket vanishes and one obtains by Stokes' theorem and due to $\partial Y = \emptyset$

$$\begin{aligned}
\mathrm{d}\mathcal{F}(\alpha,\beta) &= \nabla_\alpha(\mathcal{F}(\beta)) - \nabla_\beta(\mathcal{F}(\alpha)) \\
&= \int_Y \langle\, \mathrm{d}_A\alpha \wedge \beta\,\rangle - \int_Y \langle\, \mathrm{d}_A\beta \wedge \alpha\,\rangle \\
&= \int_{\partial Y} \langle\, \alpha \wedge \beta\,\rangle \;=\; 0.
\end{aligned} \tag{2.6}$$

In fact, for closed Y the 1-form $\mathcal{F}$ is even exact – it is the differential of the Chern-Simons functional (2.1). If Y has nonempty boundary $\partial Y = \Sigma$ then the differential (2.6) does not vanish but is equal to the standard symplectic structure ω defined in (2.2) on $\mathcal{A}(\Sigma)$. To render $\mathcal{F}$ closed, it is natural to pick a Lagrangian submanifold $\mathcal{L} \subset \mathcal{A}(\Sigma)$ and restrict $\mathcal{F}$ to the space

$$\mathcal{A}(Y,\mathcal{L}) := \{A \in \mathcal{A}(Y) \,|\, A|_\Sigma \in \mathcal{L}\}.$$

(If $\mathcal{L} \subset \mathcal{A}(\Sigma)$ is any submanifold, then the closedness of $\mathcal{F}$ is equivalent to $\omega|_{\mathcal{L}} \equiv 0$, and the maximal such submanifolds are precisely the Lagrangian submanifolds.) In order that $\mathcal{F}$ again descends to a 1-form on the moduli space $\mathcal{A}(Y,\mathcal{L})/\mathcal{G}(Y)$ one has to assume that $\mathcal{L}$ is gauge invariant and that $\mathcal{L} \subset \mathcal{A}_{\text{flat}}(\Sigma)$ lies in the space of flat connections. [2] The first assumption ensures that $\mathcal{G}(Y)$ acts on $\mathcal{A}(Y,\mathcal{L})$ and the second assumption renders $\mathcal{F}$ horizontal, c.f. (2.5). Under these assumptions $\mathcal{L}$ descends to a (singular) Lagrangian submanifold in the (singular) moduli space of flat connections,

$$L := \mathcal{L}/\mathcal{G}(\Sigma) \subset M_\Sigma := \mathcal{A}_{\text{flat}}(\Sigma)/\mathcal{G}(\Sigma).$$

In order to obtain a well defined Floer homology for such Lagrangians we shall moreover assume that L is simply connected. (This ensures a monotonicity property that gives control on the energies of flow lines between fixed critical points.) Now in general, $\mathcal{L}$ is not simply connected, but its fundamental group cancels with that of $\mathcal{G}(\Sigma)$. This is the reason why $\mathcal{F}$ is not exact but can only be written as the differential of the multi-valued Chern-Simons functional

$$\mathcal{CS}_{\mathcal{L}}(A) = \tfrac{1}{2}\int_Y \Big(\langle\, A \wedge \mathrm{d}A\,\rangle + \tfrac{1}{3}\langle\, [A \wedge A] \wedge A\,\rangle \Big) + \int_0^1 \int_\Sigma \langle\, A_0(t) \wedge \dot{A}_0(t)\,\rangle \,\mathrm{d}t.$$

[2] These two assumptions are equivalent if G is connected, simply connected, and has a discrete center – as for example G = SU(2).

Here one has to choose a path $A_0(t) \in \mathcal{L}$ with $A_0(0) = 0$ (or any other fixed reference connection in $\mathcal{L}$) and $A_0(1) = A|_\Sigma$. Unless $\mathcal{L}$ is simply connected the homotopy class of this path is not unique and hence the right-hand side is only well defined up to some positive integer. From now on we take $G = \mathrm{SU}(2)$, then this defines a functional $\mathcal{CS}_\mathcal{L} : \mathcal{A}(Y, \mathcal{L}) \to \mathbb{R}/4\pi^2\mathbb{Z}$. A negative gradient flow line of this functional is a path $A : \mathbb{R} \to \mathcal{A}(Y)$ satisfying

$$\partial_s A + *F_A = 0, \qquad A(s)|_\Sigma \in \mathcal{L} \quad \forall s \in \mathbb{R}.$$

Equivalently, one can view this path as connection $\tilde{A} = A + \Phi \mathrm{d}s \in \mathcal{A}(\mathbb{R} \times Y)$ in the special gauge $\Phi \equiv 0$. Then the above equation is the anti-self-duality equation for $\tilde{A}$. So the gauge equivalence classes of gradient flow lines of the Chern-Simons functional are in one-to-one correspondence with the gauge equivalence classes of solutions of the following boundary value problem for connections $\tilde{A} \in \mathcal{A}(\mathbb{R} \times Y)$

$$\begin{cases} F_{\tilde{A}} + *F_{\tilde{A}} = 0, \\ \quad \tilde{A}|_{\{s\}\times\Sigma} \in \mathcal{L} \qquad \forall s \in \mathbb{R}. \end{cases} \tag{2.7}$$

This is precisely the boundary value problem that we study in this thesis.

Floer homology for 3-manifolds with boundary

Fukaya set up a program to define Floer homology groups for 3-manifolds Y with boundary $\partial Y = \Sigma$ using Lagrangian boundary conditions in the moduli space M_Σ of flat connections, [Fu1, Fu2]. For the definition of Floer connecting orbits he uses a degeneration of the metric on Y to couple the anti-self-duality equation for instantons on the interior of $\mathbb{R} \times Y$ to the pseudoholomorphic equation for strips in M_Σ with a Lagrangian boundary condition. In order that the moduli space M_Σ is a smooth symplectic manifold, Fukaya uses SO(3)-bundles over Y that are nontrivial over the boundary ∂Y.

The above discussion of the Chern-Simons functional suggests an alternative approach by Salamon [Sa1] that allows to use trivial SU(2)-bundles. This approach uses the solutions of (2.7) to define the Floer homology groups $\mathrm{HF}_*^{\mathrm{inst}}(Y, L)$ for a 3-manifold with boundary $\partial Y = \Sigma$ and a Lagrangian submanifold $L = \mathcal{L}/\mathcal{G}(\Sigma) \subset M_\Sigma$. This is the starting point of his program for the proof of the Atiyah-Floer conjecture 2.2. Fukaya's approach would also fit into this program, however, this would require to extend his definition of the Floer homology groups $\mathrm{HF}_*^{\mathrm{inst}}(Y,L)$ to trivial bundles and thus singular moduli spaces. This thesis follows Salamon's approach and sets up the basic analysis for the boundary value problem (2.7) that is required for the definition of Floer homology groups.

The Floer complex will be generated by the critical points,

$$\mathrm{CF}(Y,L) := \bigoplus_{[A]\in\mathcal{A}^*_{\mathrm{flat}}(Y,\mathcal{L})/\mathcal{G}(Y)} \mathbb{Z}\langle\,[A]\,\rangle.$$

Here $\mathcal{A}^*_{\mathrm{flat}}(Y,\mathcal{L})$ denotes the set of irreducible flat connections $A\in\mathcal{A}_{\mathrm{flat}}(Y)$ with Lagrangian boundary conditions $A|_\Sigma\in\mathcal{L}$. [3] For any two such connections A^+, A^- one then has to study the moduli space of Floer connecting orbits,

$$\mathcal{M}(A^-,A^+) = \bigl\{\tilde{A}\in\mathcal{A}(\mathbb{R}\times Y)\;\big|\;\tilde{A}\text{ satisfies (2.7)},\ \lim_{s\to\pm\infty}\tilde{A}=A^\pm\bigr\}/\mathcal{G}(\mathbb{R}\times Y).$$

The analysis of the asymptotic behaviour of solutions of (2.7) should show that the convergence at infinity is equivalent to the Yang-Mills energy of the instanton being finite, hence these moduli spaces consist of connected components of

$$\mathcal{M}(\mathcal{L}) = \bigl\{\tilde{A}\in\mathcal{A}(\mathbb{R}\times Y)\;\big|\;(2.7),\mathcal{YM}(\tilde{A})<\infty\bigr\}/\mathcal{G}(\mathbb{R}\times Y).$$

Firstly, theorem A shows that the spaces of smooth connections and gauge transformations in the definition of these moduli spaces can be replaced by suitable Sobolev completions. Now $\mathcal{M}(\mathcal{L})$ can be identified with the zeros of a section of a Banach space bundle. Theorem C then is one step towards showing that this moduli space is a smooth manifold. It asserts that in the compact model case, the linearizations of the section at its zeros are Fredholm operators. The analysis of the asymptotic behaviour of the solutions should be combined with this Fredholm theory (and a perturbation of the equations (2.7)) to show that $\mathcal{M}(A^-,A^+)$ is a disjoint union of smooth manifolds of the dimension $\mu(A^-)-\mu(A^+)+8\mathbb{Z}$. (These connected components are distinguished by the homotopy class of the path in $\mathcal{A}(Y,\mathcal{L})$ running from

[3] There should be no reducible flat connections with Lagrangian boundary conditions other than the gauge orbit of the trivial connection. This will be guaranteed by certain conditions on Y and L, for example this is the case when $L=L_{Y'}$ for a handlebody Y' with $\partial Y'=\bar{\Sigma}$ such that $Y\cup_\Sigma Y'$ is a homology-3-sphere.

A^- to A^+.) Here $\mu : \mathcal{A}^*_{\rm flat}(Y,\mathcal{L}) \to \mathbb{Z}/8\mathbb{Z}$ is a (mod 8)-grading on the Floer complex. Now a monotonicity property provides a fixed Yang-Mills energy for the connections in the k-dimensional part $\mathcal{M}^k(A^-,A^+)$ of the space of connecting orbits.

Theorem B is one step towards a compactification of these moduli spaces $\mathcal{M}^k(A^-,A^+)$. It proves their compactness under the assumption of an L^p-bound on the curvature for $p > 2$, whereas the Yang-Mills energy is only the L^2-norm. So an analysis of the possible bubbling phenomena should then lead to the required compactification. In the case of index difference $\mu(A^-) - \mu(A^+) = 1$ (mod 8), this will show that $\mathcal{M}^1(A^-,A^+)/\mathbb{R}$ is a finite set (after dividing out the time-shifts). Thus the Floer boundary operator can be defined by

$$\partial\langle\,[A^-]\,\rangle := \sum_{\mu(A^+)\cong\mu(A^-)-1} \#\big(\mathcal{M}^1(A^-,A^+)/\mathbb{R}\big)\;\langle\,[A^+]\,\rangle.$$

As usual in Floer homologies, $\partial\circ\partial = 0$ should then follow from a glueing theorem that identifies the broken flow lines $\bigcap_{[A]}\mathcal{M}^1(A^-,A)/\mathbb{R}\times\mathcal{M}^1(A,A^+)/\mathbb{R}$ with the boundary of the compactification of $\mathcal{M}^2(A^-,A^+)/\mathbb{R}$. (The number of broken flow lines from A^- via some $A\in\mathcal{A}^*_{\rm flat}(Y,\mathcal{L})$ to A^+ gives the factor in front of $\langle\,[A^+]\,\rangle$ in $\partial\partial\langle\,[A^-]\,\rangle$. This number is even, so it vanishes in $\mathbb{Z}_2$, since it is the number of boundary points of a compact 1-manifold. If counted with signs according to certain choices of orientations, then it also vanishes in $\mathbb{Z}$.) One then obtains the Floer homology groups $\mathrm{HF}^{\rm inst}_*(Y,L) := H_*(\mathrm{CF},\partial)$.

In order to prove that these are independent of the metric on Y, one defines a chain homomorphism between the Floer complexes corresponding to different metrics g^- and g^+ on Y. This is done completely analogous to the definition of ∂ by counting the solutions of (2.7) for a metric on $\mathbb{R}\times Y$ that interpolates between g^- and g^+ (see example 1.4). Finally, one has to prove that the thus defined isomorphism of Floer homologies is independent of the chosen interpolating metric, i.e. find a chain homotopy equivalence between the chain homomorphisms of two different interpolating metrics. This homomorphism of the Floer complexes is again defined by counting elements of a moduli space. This time however, one chooses a path between the two interpolating metrics, then the moduli space contains pairs of metrics in this path and solutions of (2.7) with respect to this metric. The compactification of these moduli spaces is the reason for considering varying metrics in theorem B.

The program for a proof of the Atiyah-Floer conjecture and the significance of this thesis

The first step of the program by Salamon [Sa1] for the proof of the Atiyah-Floer conjecture is to define the instanton Floer homology $\mathrm{HF}_*^{\mathrm{inst}}(Y,L)$ for a 3-manifold with boundary $\partial Y = \Sigma$ and a Lagrangian submanifold $L \subset M_\Sigma$ in the moduli space of flat connections. The basic analytic framework for this Floer theory is set up in this thesis. The regularity theorem A shows that the boundary value problem whose solutions will be the connecting orbits is wellposed. The compactness theorem B is a major step towards the compactification of the moduli spaces of connecting orbits. It remains to analyse the bubbling phenomena at the boundary – these remaining obstacles for the compactification of the moduli spaces are discussed in chapter 8. Theorem C sets up the Fredholm theory (leading to the smoothness of the moduli spaces) for the compact model case. Here it remains to analyse the asymptotic behaviour of finite energy solutions of the boundary value problem (2.7) – which should be a straightforward analogon of the closed case – and combine this with theorem C to a Fredholm theory in the noncompact case. Suitable perturbations of the anti-self-duality equation as in the closed case then should lead to the required transversality result giving the moduli spaces the structure of finite dimensional smooth manifolds. Together with the compactification of the moduli spaces this allows to define the Floer boundary operator ∂. Finally, the proof of $\partial \circ \partial = 0$ and the metric independence of the resulting homology require several glueing theorems that should work analogously to Floer's original glueing theorem [F1].

The next step in the program for the proof of the Atiyah-Floer conjecture is to consider a Heegard splitting $Y = Y_0 \cup_\Sigma Y_1$ of a homology 3-sphere, and replace the instanton Floer homology of Y by the Floer homology of the 3-manifold $[0,1] \times \Sigma$ with boundary $\bar{\Sigma} \cup \Sigma$ and the Lagrangian submanifold $L_{Y_0} \times L_{Y_1} \subset M_{\bar{\Sigma}\cup\Sigma}$ in the moduli space of flat connections over $\bar{\Sigma} \cup \Sigma$. Here $\partial Y_0 = \Sigma$ and $\partial Y_1 = \bar{\Sigma}$ is the same Riemann surface with opposite orientation. Let $\mathcal{L}_{Y_i}$ be the space of flat connections over Σ that extend to flat connections over Y_i. Then $\mathcal{L} := \mathcal{L}_{Y_0} \times \mathcal{L}_{Y_1} \subset \mathcal{A}(\bar{\Sigma}) \times \mathcal{A}(\Sigma) = \mathcal{A}(\bar{\Sigma} \cup \Sigma)$ is a Lagrangian submanifold as considered in (2.7) and $\mathcal{L}/\mathcal{G}(\bar{\Sigma}\cup\Sigma) = L_{Y_0} \times L_{Y_1}$ with $L_{Y_i} = \mathcal{L}_{Y_i}/\mathcal{G}(\Sigma)$.

In [Sa1] it is conjectured that these two Floer homologies should be isomorphic.

Conjecture 2.3 *If* $Y = Y_0 \cup_\Sigma Y_1$ *is a homology 3-sphere then there is a natural isomorphism*

$$\mathrm{HF}_*^{\mathrm{inst}}(Y) \cong \mathrm{HF}_*^{\mathrm{inst}}([0,1] \times \Sigma, L_{Y_0} \times L_{Y_1}).$$

Firstly, the Floer complexes can be identified as follows. Consider a flat connection $A \in \mathcal{A}_{\mathrm{flat}}(Y)$. Thicken $\Sigma \subset Y$ and slightly shrink the Y_i such that one obtains a disjoint union $Y = Y_0 \cup [0,1] \times \Sigma \cup Y_1$. Now $A|_{[0,1]\times\Sigma} \in \mathcal{A}_{\mathrm{flat}}([0,1] \times \Sigma)$ with the boundary values $A|_{\{0\}\times\Sigma} = A|_{\partial Y_0} \in \mathcal{L}_{Y_0}$ and $A|_{\{1\}\times\Sigma} = A|_{\partial Y_1} \in \mathcal{L}_{Y_1}$. So the main task is to identify the connecting orbits.

The idea of the proof is to choose an embedding $(0,1) \times \Sigma \hookrightarrow Y$ starting from a tubular neighbourhood of $\Sigma \subset Y$ at $t = \frac{1}{2}$ and shrinking $\{t\} \times \Sigma$ to the 1-skeleton of Y_t for $t = 0, 1$. Then the anti-self-dual instantons on $\mathbb{R} \times Y$ pull back to anti-self-dual instantons on $\mathbb{R} \times [0,1] \times \Sigma$ with a degenerate metric for $t = 0$ and $t = 1$. On the other hand, one can consider anti-self-dual instantons on $\mathbb{R} \times [\varepsilon, 1-\varepsilon] \times \Sigma$ with boundary values in $\mathcal{L}_{Y_0}$ and $\mathcal{L}_{Y_1}$. As $\varepsilon \to 0$, one should be able to pass from this genuine boundary value problem to solutions on the closed manifold Y.

The final part of the proof of the Atiyah–Floer conjecture would be to establish the following conjecture [Sa1].

Conjecture 2.4 *If* $Y = Y_0 \cup_\Sigma Y_1$ *is a homology 3-sphere then there is a natural isomorphism*

$$\mathrm{HF}_*^{\mathrm{inst}}([0,1] \times \Sigma, L_{Y_0} \times L_{Y_1}) \cong \mathrm{HF}_*^{\mathrm{symp}}(M_\Sigma, L_{Y_0}, L_{Y_1}).$$

Again, one sees easily that the Floer complexes (generated by the critical points) are naturally isomorphic as follows. In the instanton Floer homology the critical points are the equivalence classes of flat connections on $[0,1] \times \Sigma$ with boundary values in $\mathcal{L}_{Y_0}$ and $\mathcal{L}_{Y_1}$ at $t = 0$ and $t = 1$ respectively. They can be written as $A + \Psi \mathrm{d}t$ with $A(t) \in \Omega^1(\Sigma; \mathfrak{g})$ and $\Psi(t) \in \Omega^0(\Sigma; \mathfrak{g})$ for all $t \in [0,1]$. One can make Ψ vanish by a gauge transformation, then the flatness condition $\dot{A} - \mathrm{d}_A \Psi = 0$ implies that A is t-independent, so $A(0) = A(1) \in \mathcal{L}_{Y_0} \cap \mathcal{L}_{Y_1}$. Thus the critical points here can be identified with intersection points of the Lagrangian submanifolds L_{Y_0} and L_{Y_1} in the moduli space M_Σ – which are exactly the critical points of the symplectic Floer homology.

So the proof of this conjecture requires an adaptation of the adiabatic limit argument in [DS2] to boundary value problems for anti-self-dual instantons and pseudoholomorphic curves respectively in order to identify the moduli spaces of connecting orbits. Here one again deals with the boundary value problem (2.7) studied in this thesis. As the metric on Σ converges to zero, the solutions, i.e. anti-self-dual instantons on $\mathbb{R} \times [0,1] \times \Sigma$ with Lagrangian boundary conditions in $\mathcal{L}_{Y_0}, \mathcal{L}_{Y_1} \subset M_\Sigma$ should be in one-to-one correspondence with connections on $\mathbb{R} \times [0,1] \times \Sigma$ that descend to pseudoholomorphic strips in M_Σ with boundary values in L_{Y_0} and L_{Y_1}.

Chapter 3

Weakly flat connections

In this chapter we consider the trivial G-bundle over a closed manifold Σ of dimension $n \geq 2$, where G is a compact Lie group. Fix $p > n$. Then a connection $A \in \mathcal{A}^{0,p}(\Sigma)$ is called **weakly flat** if

$$\int_\Sigma \langle\, A \,,\, \mathrm{d}^*\omega - \tfrac{1}{2}(-1)^n * [A \wedge *\omega] \,\rangle = 0 \qquad \forall \omega \in \Omega^2(\Sigma, \mathfrak{g}). \tag{3.1}$$

For sufficiently regular connections, (3.1) is equivalent to the connection being flat, that is $F_A = \mathrm{d}A + \frac{1}{2}[A \wedge A] = 0$. We denote the space of weakly flat L^p-connections over Σ by

$$\mathcal{A}^{0,p}_{\mathrm{flat}}(\Sigma) := \bigl\{ A \in \mathcal{A}^{0,p}(\Sigma) \bigm| A \text{ satisfies (3.1)} \bigr\}.$$

One can check that this space is invariant under the action of the gauge group $\mathcal{G}^{1,p}(\Sigma)$. Here $p \geq 2$ is required in order that (3.1) is welldefined, and $\mathcal{G}^{1,p}(\Sigma)$ is welldefined for $p > n$, see e.g. [We, Appendix B].

The next theorem shows that the quotient $\mathcal{A}^{0,p}_{\mathrm{flat}}(\Sigma)/\mathcal{G}^{1,p}(\Sigma)$ can be identified with the usual moduli space of flat connections $\mathcal{A}_{\mathrm{flat}}(\Sigma)/\mathcal{G}(\Sigma)$ – smooth flat connections modulo smooth gauge transformations.

Theorem 3.1 *For every weakly flat connection $A \in \mathcal{A}^{0,p}_{\mathrm{flat}}(\Sigma)$ there exists a gauge transformation $u \in \mathcal{G}^{1,p}(\Sigma)$ such that $u^*A \in \mathcal{A}_{\mathrm{flat}}(\Sigma)$ is smooth.*

The proof will be based on the following L^p-version of the local slice theorem, a proof of which can be found in [We, Theorem 9.3].

Proposition 3.2 *Fix a reference connection $\hat{A} \in \mathcal{A}^{0,p}(\Sigma)$. Then there exists a constant $\delta > 0$ such that for every $A \in \mathcal{A}^{0,p}(\Sigma)$ with $\|A - \hat{A}\|_p \le \delta$ there exists a gauge transformation $u \in \mathcal{G}^{1,p}(\Sigma)$ such that*

$$\int_\Sigma \big\langle u^*A - \hat{A}\,,\, \mathrm{d}_{\hat{A}}\eta \big\rangle = 0 \qquad \forall \eta \in \mathcal{C}^\infty(\Sigma, \mathfrak{g}). \tag{3.2}$$

Equivalently, one has for $v = u^{-1} \in \mathcal{G}^{1,p}(\Sigma)$

$$\int_\Sigma \big\langle v^*\hat{A} - A\,,\, \mathrm{d}_A\eta \big\rangle = 0 \qquad \forall \eta \in \mathcal{C}^\infty(\Sigma, \mathfrak{g}).$$

The weak flatness together with the weak Coulomb gauge condition (3.2) form an elliptic system, so theorem 3.1 then is a consequence of the following lemma. For closed manifolds $M = \Sigma$, this regularity result is essentially due to the Hodge decomposition of L^p-regular 1-forms. In the case when $M = \Sigma$ is in fact a Riemann surface, this result directly follows from the regularity theory for the weak Laplace equation. However, the lemma also holds on manifolds with boundary, and it yields componentwise regularity results. This will be useful for the proof of theorem C in chapter 7. Here we use the following notation:

$$\mathcal{C}^\infty_\delta(M) = \big\{ \phi \in \mathcal{C}^\infty(M) \bigm| \phi|_{\partial M} = 0 \big\},$$
$$\mathcal{C}^\infty_\nu(M) = \big\{ \phi \in \mathcal{C}^\infty(M) \bigm| \tfrac{\partial\phi}{\partial\nu}|_{\partial M} = 0 \big\}.$$

Lemma 3.3 *Let (M, g) be a compact Riemannian manifold (possibly with boundary), let $k \in \mathbb{N}_0$ and $1 < p < \infty$. Let $X \in \Gamma(\mathrm{T}M)$ be a smooth vector field that is either perpendicular to the boundary, i.e. $X|_{\partial M} = h \cdot \nu$ for some $h \in \mathcal{C}^\infty(\partial M)$, or tangential, i.e. $X|_{\partial M} \in \Gamma(\mathrm{T}\partial M)$. In the first case let $\mathcal{T} = \mathcal{C}^\infty_\delta(M)$, in the latter case let $\mathcal{T} = \mathcal{C}^\infty_\nu(M)$. Then there exists a constant C such that the following holds:*

Let $f \in W^{k,p}(M)$, $\gamma \in W^{k,p}(M, \Lambda^2\mathrm{T}^*M)$, and suppose that the 1-form $\alpha \in W^{k,p}(M, \mathrm{T}^*M)$ satisfies

$$\int_M \langle\, \alpha\,,\, \mathrm{d}\eta \,\rangle = \int_M f \cdot \eta \qquad \forall \eta \in \mathcal{C}^\infty(M),$$
$$\int_M \langle\, \alpha\,,\, \mathrm{d}^*\omega \,\rangle = \int_M \langle\, \gamma\,,\, \omega \,\rangle \qquad \forall \omega = \mathrm{d}(\phi \cdot \iota_X g)\,,\ \phi \in \mathcal{T}.$$

Then $\alpha(X) \in W^{k+1,p}(M)$ and

$$\|\alpha(X)\|_{W^{k+1,p}} \le C\big(\|f\|_{W^{k,p}} + \|\gamma\|_{W^{k,p}} + \|\alpha\|_{W^{k,p}} \big).$$

Remark 3.4 In the case $k=0$ let $\frac{1}{p}+\frac{1}{p^*}=1$, then the weak equations for α can be replaced by the following: There exist constants c_1 and c_2 such that

$$\begin{aligned} \left|\int_M \langle\, \alpha\,,\, \mathrm{d}\eta\,\rangle\right| &\leq c_1\, \|\eta\|_{p^*} && \forall \eta \in \mathcal{C}^\infty(M), \\ \left|\int_M \langle\, \alpha\,,\, \mathrm{d}^*\mathrm{d}(\phi\cdot\iota_X g)\,\rangle\right| &\leq c_2\, \|\phi\|_{W^{1,p^*}} && \forall \phi \in \mathcal{T}. \end{aligned}$$

The estimate then becomes $\|\alpha(X)\|_{W^{1,p}} \leq C\bigl(c_1+c_2+\|\alpha\|_p\bigr)$.

The regularity and estimates claimed here will follow from weak Laplace equations. We abbreviate $\Delta := \mathrm{d}^*\mathrm{d}$. Then the proof will use the following standard elliptic theory for the Laplace operator with Dirichlet or (inhomogenous) Neumann boundary conditions, which can for example be found in [GT] and [We, Theorems 2.3',3.2,D.2].

Proposition 3.5 *Let $k\in\mathbb{N}$, then there exists a constant C such that the following holds. Let $f\in W^{k-1,p}(M)$ and $G\in W^{k,p}(M)$ and suppose that $u\in W^{k,p}(M)$ is a weak solution of the Dirichlet problem (or the Neumann problem with inhomogenous boundary conditions), that is for all $\psi\in\mathcal{C}^\infty_\delta(M)$ (or for all $\psi\in\mathcal{C}^\infty_\nu(M)$)*

$$\int_M u\cdot\Delta\psi \;=\int_M f\cdot\psi+\int_{\partial M} G\cdot\psi.$$

Then $u\in W^{k+1,p}(M)$ and

$$\|u\|_{W^{k+1,p}} \leq C\bigl(\|f\|_{W^{k-1,p}}+\|G\|_{W^{k,p}}+\|u\|_{W^{k,p}}\bigr).$$

In the special case $k=0$ there exists a constant C such that the following holds: Suppose that $u\in L^p(M)$ and that there exists a constant c such that for all $\psi\in\mathcal{C}^\infty_\delta(M)$ (or for all $\psi\in\mathcal{C}^\infty_\nu(M)$)

$$\int_M u\cdot\Delta\psi \;\leq c\|\psi\|_{W^{1,p^*}}.$$

Then $u\in W^{1,p}(M)$ and

$$\|u\|_{W^{1,p}} \leq C\bigl(c+\|u\|_{L^p}\bigr).$$

Proof of lemma 3.3 and remark 3.4 :
Let $\alpha^\nu \in \mathcal{C}^\infty(M, \mathrm{T}^*M)$ be an L^p-approximating sequence for α such that $\alpha^\nu \equiv 0$ near ∂M. Then one obtains for all $\phi \in \mathcal{T}$

$$\begin{aligned}
\int_M \alpha(X)\cdot\Delta\phi &= \lim_{\nu\to\infty}\Bigl(\int_M \langle\, \mathcal{L}_X\alpha^\nu \,,\, \mathrm{d}\phi \,\rangle - \int_M \langle\, \iota_X \mathrm{d}\alpha^\nu \,,\, \mathrm{d}\phi\,\rangle\Bigr) \\
&= \lim_{\nu\to\infty}\Bigl(-\int_M \langle\, \alpha^\nu \,,\, \mathcal{L}_X \mathrm{d}\phi\,\rangle - \int_M \langle\, \alpha^\nu \,,\, \mathrm{div}X\cdot\mathrm{d}\phi\,\rangle \\
&\qquad - \int_M \langle\, \alpha^\nu \,,\, \iota_{Y_{\mathrm{d}\phi}}\mathcal{L}_X g\,\rangle - \int_M \langle\, \mathrm{d}\alpha^\nu \,,\, \iota_X g\wedge \mathrm{d}\phi\,\rangle\Bigr) \\
&= \int_M \langle\, \alpha \,,\, \mathrm{d}(-\mathcal{L}_X\phi - \mathrm{div}X\cdot\phi)\,\rangle - \int_M \langle\, \alpha \,,\, \mathrm{d}^*(\iota_X g\wedge\mathrm{d}\phi)\,\rangle \\
&\quad + \int_M \langle\, \alpha \,,\, \phi\cdot\mathrm{d}(\mathrm{div}X) - \iota_{Y_{\mathrm{d}\phi}}\mathcal{L}_X g\,\rangle \\
&= \int_M \langle\, f \,,\, -\mathcal{L}_X\phi - \mathrm{div}X\cdot\phi\,\rangle + \int_M \langle\, \gamma \,,\, \mathrm{d}(\phi\cdot\iota_X g)\,\rangle \\
&\quad - \int_M \langle\, \alpha \,,\, \mathrm{d}^*(\phi\cdot\mathrm{d}\iota_X g)\,\rangle + \int_M \langle\, \alpha \,,\, \phi\cdot\mathrm{d}(\mathrm{div}X) - \iota_{Y_{\mathrm{d}\phi}}\mathcal{L}_X g\,\rangle.
\end{aligned}$$

Here the vector field $Y_{\mathrm{d}\phi}$ is given by $\iota_{Y_{\mathrm{d}\phi}} g = \mathrm{d}\phi$. In the case $k \geq 1$ further partial integration yields for all $\phi \in \mathcal{T}$

$$\int_M \alpha(X)\cdot\Delta\phi = \int_M F\cdot\phi + \int_{\partial M} G\cdot\phi,$$

where $F \in W^{k-1,p}(M)$, $G \in W^{k,p}(M)$, and for some constant C

$$\|F\|_{W^{k-1,p}} + \|G\|_{W^{k,p}} \leq C\bigl(\|f\|_{W^{k,p}} + \|\gamma\|_{W^{k,p}} + \|\alpha\|_{W^{k,p}}\bigr).$$

So the regularity proposition 3.5 for the weak Laplace equation with either Neumann ($\mathcal{T} = \mathcal{C}^\infty_\nu(M)$) or Dirichlet ($\mathcal{T} = \mathcal{C}^\infty_\delta(M)$) boundary conditions proves that $\alpha(X) \in W^{k+1,p}(M)$ with the according estimate.

In the case $k = 0$ one works with the following inequality: Let $\frac{1}{p^*} + \frac{1}{p} = 1$, then there is a constant C such that for all $\phi \in \mathcal{T}$

$$\left|\int_M \alpha(X)\cdot\Delta\phi\right| \leq C\bigl(\|f\|_p + \|\gamma\|_p + \|\alpha\|_p\bigr)\|\phi\|_{W^{1,p^*}}.$$

(Under the assumptions of remark 3.4, one simply replaces $\|f\|_p$ and $\|\gamma\|_p$ by c_1 and c_2 respectively.) The regularity and estimate for $\alpha(X)$ then follow from proposition 3.5. □

Proof of theorem 3.1 :
Consider a weakly flat connection $A \in \mathcal{A}^{0,p}_{\text{flat}}(\Sigma)$. Let $\delta > 0$ be the constant from proposition 3.2 for the reference connection A and choose a smooth connection $\tilde{A} \in \mathcal{A}(\Sigma)$ such that $\|\tilde{A} - A\|_p \leq \delta$. Then by proposition 3.2 there exists a gauge transformation $u \in \mathcal{G}^{1,p}(\Sigma)$ such that

$$\int_\Sigma \big\langle u^*A - \tilde{A}\,,\, \mathrm{d}_{\tilde{A}}\eta \big\rangle = 0 \qquad \forall \eta \in \mathcal{C}^\infty(\Sigma, \mathfrak{g}).$$

Now lemma 3.3 asserts that $\alpha := u^*A - \tilde{A} \in L^p(\Sigma, \mathrm{T}^*\Sigma \otimes \mathfrak{g})$ is in fact smooth. (By the definition of Sobolev spaces via coordinate charts it suffices to prove the regularity and estimate for $\alpha(X)$, where $X \in \Gamma(\mathrm{T}\Sigma)$ is any smooth vector field on Σ.) This is due to the weak equations

$$\begin{aligned} \int_\Sigma \langle\, \alpha\,,\, \mathrm{d}\eta\,\rangle &= -\int_\Sigma \langle\, *[\alpha \wedge *\tilde{A}]\,,\, \eta\,\rangle & \forall \eta \in \mathcal{C}^\infty(\Sigma, \mathfrak{g}), \\ \int_\Sigma \langle\, \alpha\,,\, \mathrm{d}^*\omega\,\rangle &= -\int_\Sigma \langle\, \mathrm{d}\tilde{A} + \tfrac{1}{2}[u^*A \wedge u^*A]\,,\, \omega\,\rangle & \forall \omega \in \Omega^2(\Sigma, \mathfrak{g}). \end{aligned}$$

Firstly, the inhomogeneous terms are of class $L^{\frac{p}{2}}$, hence the lemma asserts $W^{1,\frac{p}{2}}$-regularity of α and u^*A. Now if $p \leq 2n$, then the Sobolev embedding gives L^{p_1}-regularity of u^*A with $p_1 := \frac{np}{2n-p}$ (in case $p = 2n$ one can choose any $p_1 > 2n$). This is iterated to obtain L^{p_j}-regularity for the sequence $p_{j+1} = \frac{np_j}{2n-p_j}$ (or any $p_{j+1} > 2n$ in case $p_j \geq 2n$) with $p_0 = p$. One checks that $p_{j+1} \geq \theta p_j$ with $\theta = \frac{n}{2n-p} > 1$ due to $p > n$. So after finitely many steps this yields $W^{1,q}$-regularity for some $q = \frac{p_N}{2} > n$. The same is the case if $p > 2n$ at the beginning. Next, if u^*A is of class $W^{k,q}$ for some $k \in \mathbb{N}$, then the inhomogeneous terms also are of class $W^{k,q}$ and the lemma asserts the $W^{k+1,q}$-regularity of α and hence u^*A. Iterating this argument proves the smoothness of $u^*A = \tilde{A} + \alpha$. □

Weakly flat connections over a Riemann surface

Now we consider more closely the special case when Σ is a Riemann surface. Theorem 3.1 shows that the injection $\mathcal{A}_{\text{flat}}(\Sigma)/\mathcal{G}(\Sigma) \hookrightarrow \mathcal{A}^{0,p}_{\text{flat}}(\Sigma)/\mathcal{G}^{1,p}(\Sigma)$ in fact is a bijection. These moduli spaces are identified and denoted by M_Σ. Furthermore, the holonomy induces a natural bijection from M_Σ to the space of conjugacy classes of homomorphisms from $\pi_1(\Sigma)$ to G, see theorem A.2

$$M_\Sigma \;:=\; \mathcal{A}^{0,p}_{\text{flat}}(\Sigma)/\mathcal{G}^{1,p}(\Sigma) \;\cong\; \mathcal{A}_{\text{flat}}(\Sigma)/\mathcal{G}(\Sigma) \;\cong\; \mathrm{Hom}(\pi_1(\Sigma), \mathrm{G})/\sim\,.$$

From this one sees that M_Σ is a finite dimensional singular manifold. If we assume $\mathrm{G}=\mathrm{SU}(2)$, then M_Σ has singularities at the product connection and at the further reducible connections – corresponding to the connections for which the holonomy group is not $\mathrm{SU}(2)$ but only $\{\mathbb{1}\}$ or is conjugate to the maximal torus $S^1\subset\mathrm{SU}(2)$. [1] Away from these singularities, the dimension of M_Σ is $6g-6$, where g is the genus of Σ. (The arguments in [DS1, §4] show that $\mathrm{T}_{[A]}M_\Sigma\cong\ker\mathrm{d}_A/\mathrm{im}\,\mathrm{d}_A=h^1_A$ has dimension $3\cdot(2g-2)$ at irreducible connections A.)

For the same reasons the space of weakly flat connections $\mathcal{A}^{0,p}_{\mathrm{flat}}(\Sigma)$ is in general not a Banach submanifold of $\mathcal{A}^{0,p}(\Sigma)$ but a principal bundle over a singular base manifold. To be more precise fix a point $z\in\Sigma$ and consider the space of based gauge transformations, defined as

$$\mathcal{G}^{1,p}_z(\Sigma):=\left\{u\in\mathcal{G}^{1,p}(\Sigma)\,\middle|\,u(z)=\mathbb{1}\right\}.$$

This Lie group acts freely on $\mathcal{A}^{0,p}_{\mathrm{flat}}(\Sigma)$. The quotient space $\mathcal{A}^{0,p}_{\mathrm{flat}}(\Sigma)/\mathcal{G}^{1,p}_z(\Sigma)$ can be identified with $\mathrm{Hom}(\pi_1(\Sigma),\mathrm{G})$ via the holonomy based at z. This based holonomy map $\rho_z:\mathcal{A}^{0,p}_{\mathrm{flat}}(\Sigma)\to\mathrm{Hom}(\pi_1(\Sigma),\mathrm{G})$ is defined by first choosing a based gauge transformation that makes the connection smooth and then computing the holonomy. Now ρ_z gives $\mathcal{A}^{0,p}_{\mathrm{flat}}(\Sigma)$ the structure of a principal bundle with fibre $\mathcal{G}^{1,p}_z(\Sigma)$ over the finite dimensional singular manifold $\mathrm{Hom}(\pi_1(\Sigma),\mathrm{G})$,

$$\mathcal{G}^{1,p}_z(\Sigma)\hookrightarrow\mathcal{A}^{0,p}_{\mathrm{flat}}(\Sigma)\xrightarrow{\rho_z}\mathrm{Hom}(\pi_1(\Sigma),\mathrm{G}).$$

Note that this discussion does not require the Riemann surface Σ to be connected. Only when fixing a base point for the holonomy map and the based gauge transformations one has to adapt the definition. Whenever $\Sigma=\bigcup_{i=1}^n\Sigma_i$ has several connected components Σ_i, then 'fixing a point $z\in\Sigma$' implicitly means that one fixes a point $z_i\in\Sigma_i$ in each connected component. The group of based gauge transformations then becomes

$$\mathcal{G}^{1,p}_z\big(\textstyle\bigcup_{i=1}^n\Sigma_i\big):=\left\{u\in\mathcal{G}^{1,p}(\Sigma)\,\middle|\,u(z_i)=\mathbb{1}\quad\forall i=1,\dots,n\right\}.$$

[1] The holonomy group of a connection is given by the holonomies of all loops in Σ, c.f. appendix A. Now the isotropy subgroup of $\mathcal{G}(\Sigma)$ of the connection is isomorphic to the centralizer of the holonomy group, see [DK, Lemma 4.2.8].

Chapter 4

Lagrangians in the space of connections

Consider the trivial G-bundle over a (possibly disconnected) Riemann surface Σ of (total) genus g. Here again G is assumed to be a compact Lie group and $\mathfrak{g}$ denotes its finite dimensional Lie algebra. There is a gauge invariant symplectic form ω on the space of connections $\mathcal{A}^{0,p}(\Sigma)$ for $p > 2$ defined as follows. For tangent vectors $\alpha, \beta \in L^p(\Sigma, \mathrm{T}^*\Sigma \otimes \mathfrak{g})$ to the affine space $\mathcal{A}^{0,p}(\Sigma)$

$$\omega(\alpha, \beta) = \int_\Sigma \langle\, \alpha \wedge \beta \,\rangle. \tag{4.1}$$

The action of the infinite dimensional gauge group $\mathcal{G}^{1,p}(\Sigma)$ on the symplectic Banach space $(\mathcal{A}^{0,p}(\Sigma), \omega)$ is Hamiltonian with moment map $A \mapsto *F_A$ (more precisely, the equivalent weak expression in $(W^{1,p^*}(\Sigma, \mathfrak{g}))^*$). So M_Σ can be viewed as the symplectic quotient $\mathcal{A}^{0,p}(\Sigma)/\!/\mathcal{G}^{1,p}(\Sigma)$ as was first observed by Atiyah and Bott [AB], c.f. chapter 2. However, 0 is not a regular value of the moment map, so M_Σ is a singular symplectic manifold. Due to these singularities at the reducible connections we prefer to work in the infinite dimensional setting.

A **Lagrangian submanifold** $\mathcal{L} \subset \mathcal{A}^{0,p}(\Sigma)$ is a Banach submanifold that is isotropic, $\omega|_{\mathrm{T}_A\mathcal{L}} = 0$ for all $A \in \mathcal{L}$, and is of maximal dimension. In this infinite dimensional setting, the latter condition can be phrased as

$$\mathrm{T}_A\mathcal{A}^{0,p}(\Sigma) = \mathrm{T}_A\mathcal{L} \oplus J\,\mathrm{T}_A\mathcal{L} \qquad \forall A \in \mathcal{L}.$$

Here one has to choose an ω-compatible complex structure $J \in \operatorname{End} \mathrm{T}\mathcal{A}^{0,p}(\Sigma)$, i.e. such that $J^2 = -\mathbb{1}$ and $\omega(\cdot, J\cdot)$ defines a metric. (The latter condition is in fact independent of the choice of J since the space of ω-compatible complex structures is connected [MS1, Proposition 2.48].)

Now note that the Hodge $*$ operator is an ω-compatible complex structure since $\omega(\cdot,*\cdot)$ is the L^2-inner product. For all $\alpha,\beta\in L^p(\Sigma,\mathrm{T}^*\Sigma\otimes\mathfrak{g})$

$$\omega(\alpha,*\beta) \;=\; \int_S \langle\,\alpha\wedge *\beta\,\rangle \;=\; \langle\,\alpha\,,\,\beta\,\rangle_{L^2}. \tag{4.2}$$

Hence a Banach submanifold $\mathcal{L}\subset\mathcal{A}^{0,p}(\Sigma)$ is Lagrangian if and only if it is isotropic, $\omega|_{\mathcal{L}}\equiv 0$, and totally real, i.e. for all $A\in\mathcal{L}$

$$L^p(\Sigma,\mathrm{T}^*\Sigma\otimes\mathfrak{g}) \;=\; \mathrm{T}_A\mathcal{L}\oplus *\mathrm{T}_A\mathcal{L}. \tag{4.3}$$

Suppose that $\mathcal{L}$ is Lagrangian and in addition $\mathcal{L}\subset\mathcal{A}^{0,p}_{\mathrm{flat}}(\Sigma)$, then one also has the twisted Hodge decomposition (see e.g. [Wa, Theorem 6.8])

$$L^p(\Sigma,\mathrm{T}^*\Sigma\otimes\mathfrak{g}) \;=\; \mathrm{im}\,\mathrm{d}_A\oplus\mathrm{im}\,\mathrm{d}_A^*\oplus h^1_A, \tag{4.4}$$

where $h^1_A=\ker\mathrm{d}_A\cap\ker\mathrm{d}_A^*$. The assumption $\mathcal{L}\subset\mathcal{A}^{0,p}_{\mathrm{flat}}(\Sigma)$ also ensures that $\mathcal{L}$ is gauge invariant if G is connected and simply connected. On the other hand, the gauge invariance of $\mathcal{L}$ implies $\mathcal{L}\subset\mathcal{A}^{0,p}_{\mathrm{flat}}(\Sigma)$ if the Lie bracket on G is nondegenerate (i.e. the center of G is discrete). So for example in the case $\mathrm{G}=\mathrm{SU}(2)$ both conditions are equivalent.

In general we will consider Lagrangian submanifolds $\mathcal{L}\subset\mathcal{A}^{0,p}(\Sigma)$ and assume that they are both gauge invariant and contained in $\mathcal{A}^{0,p}_{\mathrm{flat}}(\Sigma)$. In that case $\mathcal{L}$ descends to a (singular) submanifold of the (singular) moduli space of flat connections,

$$L \;:=\; \mathcal{L}/\mathcal{G}^{1,p}(\Sigma)\subset\mathcal{A}^{0,p}_{\mathrm{flat}}(\Sigma)/\mathcal{G}^{1,p}(\Sigma) \;=:\; M_\Sigma.$$

This submanifold is obviously isotropic, i.e. the symplectic structure induced by (4.1) on M_Σ vanishes on L. Moreover, its tangent spaces have half of the dimension of those of M_Σ, so $L\subset M_\Sigma$ is a Lagrangian submanifold. Indeed, in the Hodge decomposition (4.4) $\mathrm{im}\,\mathrm{d}_A^*$ is the complement of $\ker\mathrm{d}_A=\mathrm{T}_A\mathcal{A}^{0,p}_{\mathrm{flat}}(\Sigma)$, $\mathrm{im}\,\mathrm{d}_A$ is the tangent space to the orbit of $\mathcal{G}^{1,p}(\Sigma)$ through A, and so $h^1_A\cong\mathrm{T}_{[A]}M_\Sigma$. Now compare this with the decomposition (4.3). Here $\mathrm{im}\,\mathrm{d}_A\subset\mathrm{T}_A\mathcal{L}$ and $\mathrm{im}\,\mathrm{d}_A^*\subset *\mathrm{T}_A\mathcal{L}$ are mapped to each other by the complex structure $*$, and thus the intersection of h^1_A with $\mathrm{T}_A\mathcal{L}$ is of half dimension – this is exactly $\mathrm{T}_{[A]}\mathcal{L}$.

Moreover, our assumptions on the Lagrangian submanifold ensure that the holonomy map $\rho_z : \mathcal{L} \to \mathrm{Hom}(\pi_1(\Sigma), \mathrm{G})$ based at $z \in \Sigma$ is welldefined and invariant under the action of the based gauge group $\mathcal{G}^{1,p}_z(\Sigma)$. Note that $\mathrm{Hom}(\pi_1(\Sigma), \mathrm{G})$ naturally embeds into $\mathrm{Hom}(\pi_1(\Sigma \setminus \{z\}), \mathrm{G})$, which is a smooth manifold diffeomorphic to G^{2g}. This gives $\mathrm{Hom}(\pi_1(\Sigma), \mathrm{G})$ a differentiable structure (that is in fact independent of $z \in \Sigma$), however, it is a manifold with singularities. In the following lemma we list some crucial properties of the Lagrangian submanifolds $\mathcal{L}$ that we will deal with. Here we use the notation

$$W^{1,p}_z(\Sigma, \mathfrak{g}) := \left\{ \xi \in W^{1,p}(\Sigma, \mathfrak{g}) \,\middle|\, \xi(z) = 0 \right\}$$

for the Lie algebra $\mathrm{T}_{\mathbb{1}}\mathcal{G}^{1,p}_z(\Sigma)$ of the based gauge group. (If Σ is not connected then as before one fixes a base point in each connected component and modifies the definition of $W^{1,p}_z(\Sigma, \mathfrak{g})$ accordingly.) Moreover, in this chapter, we will denote the differential of a map ϕ at a point x by $\mathrm{T}_x\phi$ in order to distinguish it from the exterior differential on differential forms, d_A, associated with a connection A.

Lemma 4.1 *Let $\mathcal{L} \subset \mathcal{A}^{0,p}(\Sigma)$ be a Lagrangian submanifold and fix $z \in \Sigma$. Suppose that $\mathcal{L} \subset \mathcal{A}^{0,p}_{\mathrm{flat}}(\Sigma)$ and that $\mathcal{L}$ is invariant under the action of $\mathcal{G}^{1,p}_z(\Sigma)$. Then the following holds:*

(i) $L := \mathcal{L}/\mathcal{G}^{1,p}_z(\Sigma)$ is a smooth manifold of dimension $m = g \cdot \dim \mathrm{G}$ and the holonomy induces a diffeomorphism $\rho_z : L \to M$ to a submanifold $M \subset \mathrm{Hom}(\pi_1(\Sigma), \mathrm{G})$.

(ii) $\mathcal{L}$ has the structure of a principal $\mathcal{G}^{1,p}_z(\Sigma)$-bundle over M,

$$\mathcal{G}^{1,p}_z(\Sigma) \hookrightarrow \mathcal{L} \xrightarrow{\rho_z} M.$$

(iii) Fix $A \in \mathcal{L}$. Then there exists a local section $\phi : V \to \mathcal{L}$ over a neighbourhood $V \subset \mathbb{R}^m$ of 0 such that $\phi(0) = A$ and $\rho_z \circ \phi$ is a diffeomorphism to a neighbourhood of $\rho_z(A)$. This gives rise to Banach submanifold coordinates for $\mathcal{L} \subset \mathcal{A}^{0,p}(\Sigma)$, namely a smooth embedding

$$\Theta : \mathcal{W} \to \mathcal{A}^{0,p}(\Sigma)$$

defined on a neighbourhood $\mathcal{W} \subset W^{1,p}_z(\Sigma, \mathfrak{g}) \times \mathbb{R}^m \times W^{1,p}_z(\Sigma, \mathfrak{g}) \times \mathbb{R}^m$ of zero by

$$\Theta(\xi_0, v_0, \xi_1, v_1) := \exp(\xi_0)^*\phi(v_0) + *\mathrm{d}_A\xi_1 + *\mathrm{T}_0\phi(v_1).$$

Moreover, if A is smooth, then the local section can be chosen such that the image is smooth, $\phi : V \to \mathcal{L} \cap \mathcal{A}(\Sigma)$. Now the same map Θ is a diffeomorphism between neighbourhoods of zero in $W^{1,p}_z(\Sigma, \mathfrak{g}^2) \times \mathbb{R}^{2m}$ and neighbourhoods of A in $\mathcal{A}^{0,p}(\Sigma)$ for all $p > 2$.

One reason for this detailed description of the Banach submanifold charts is the following approximation result for $W^{1,p}$-connections with Lagrangian boundary values.

Corollary 4.2 *Let $\mathcal{L} \subset \mathcal{A}^{0,p}(\Sigma)$ be as in lemma 4.1 and let*

$$\Omega \subset \mathbb{H} := \{(s,t) \in \mathbb{R}^2 \,\big|\, t \geq 0\}$$

be a compact submanifold. Suppose that $A \in \mathcal{A}^{1,p}(\Omega \times \Sigma)$ satisfies the boundary condition

$$A|_{(s,0)\times\Sigma} \in \mathcal{L} \qquad \forall (s,0) \in \partial\Omega. \tag{4.5}$$

Then there exists a sequence of smooth connections $A^\nu \in \mathcal{A}(\Omega \times \Sigma)$ that satisfy (4.5) and converge to A in the $W^{1,p}$-norm.

Proof of corollary 4.2:
We decompose $A = \Phi \mathrm{d}s + \Psi \mathrm{d}t + B$ into two functions $\Phi, \Psi \in W^{1,p}(\Omega \times \Sigma, \mathfrak{g})$ and a family of 1-forms $B \in W^{1,p}(\Omega \times \Sigma, \mathrm{T}^*\Sigma \otimes \mathfrak{g})$ on Σ such that $B(s,0) \in \mathcal{L}$ for all $(s,0) \in \partial\Omega$. Then it suffices to find an approximating sequence for B with Lagrangian boundary conditions on a neighbourhood of $\Omega \cap \partial\mathbb{H}$. This can be patched together with any smooth $W^{1,p}$-approximation of B on the rest of Ω and can be combined with standard approximations of the functions Φ and Ψ to obtain the required approximation of A.

So fix any $(s_0, 0) \in \Omega \cap \partial\mathbb{H}$ and use theorem 3.1 to find $u_0 \in \mathcal{G}^{1,p}(\Sigma)$ such that $A_0 := u_0^* B(s_0, 0)$ is smooth. Lemma 4.1 (iii) gives a diffeomorphism $\Theta : \mathcal{W} \to \mathcal{V}$ between neighbourhoods $\mathcal{W} \subset W^{1,p}_z(\Sigma, \mathfrak{g}^2) \times \mathbb{R}^{2m}$ of zero and $\mathcal{V} \subset \mathcal{A}^{0,p}(\Sigma)$ of A_0. This was constructed such that $\mathcal{C}^\infty(\Sigma, \mathfrak{g}^2) \times \mathbb{R}^{2m}$ is mapped to $\mathcal{A}(\Sigma)$ and such that $\Theta : \mathcal{W} \cap W^{2,p}_z(\Sigma, \mathfrak{g}^2) \times \mathbb{R}^{2m} \to \mathcal{V} \cap \mathcal{A}^{1,p}(\Sigma)$ also is a diffeomorphism. Now note that $B \in \mathcal{C}^0(\Omega, \mathcal{A}^{0,p}(\Sigma))$. Hence there exists a neighbourhood $U \subset \Omega$ of $(s_0, 0)$ and one can choose a smooth gauge transformation $u \in \mathcal{G}(\Sigma)$ that is $W^{1,p}$-close to u_0 such that $u^* B(s,t) \in \mathcal{V}$ for all $(s,t) \in U$. Now we define $\xi = (\xi_0, \xi_1) : U \to W^{1,p}_z(\Sigma, \mathfrak{g}^2)$ and $v = (v_0, v_1) : U \to \mathbb{R}^{2m}$ by $\Theta(\xi(s,t), v(s,t)) = u^* B(s,t)$. Recall that B is of class $W^{1,p}$ on $U \times \Sigma$, hence it lies in both $W^{1,p}(U, \mathcal{A}^{0,p}(\Sigma)$ and $L^p(U, \mathcal{A}^{1,p}(\Sigma)$.

Thus $\xi \in W^{1,p}(U, W^{1,p}_z(\Sigma, \mathfrak{g}^2)) \cap L^p(U, W^{2,p}_z(\Sigma, \mathfrak{g}^2))$ and $v \in W^{1,p}(U, \mathbb{R}^{2m})$, and these satisfy the boundary conditions $\xi_1|_{t=0} = 0$ and $v_1|_{t=0} = 0$ due to the Lagrangian boundary condition for B. Now there exist $\xi^\nu \in \mathcal{C}^\infty(U \times \Sigma, \mathfrak{g}^2)$ and $v^\nu \in \mathcal{C}^\infty(U, \mathbb{R}^{2m})$ such that $\xi^\nu \to \xi$ and $v^\nu \to v$ in all these spaces, $\xi^\nu(\cdot, z) \equiv 0$, $\xi^\nu_1|_{t=0} = 0$, and $v^\nu_1|_{t=0} = 0$. (These are constructed with the help of mollifiers as in lemma 5.8; also see remark 5.9. One first reflects ξ at the boundary and mollifies it with respect to U to obtain approximations in $\mathcal{C}^\infty(U, W^{2,p}_z(\Sigma, \mathfrak{g}^2))$ with zero boundary values. Next, one mollifies on Σ, and finally one corrects the value at z.) It follows that $B^\nu(s,t) := (u^{-1})^*\Theta(\xi^\nu(s,t), v^\nu(s,t))$ is a sequence of smooth maps from U to $\mathcal{A}(\Sigma)$ which satisfies the Lagrangian boundary condition and converges to B in the $W^{1,p}$-norm.

Now $\Omega \cap \partial\mathbb{H}$ is compact, so it is covered by finitely many such neighbourhoods U_i on which there exist smooth $W^{1,p}$-approximations of B with Lagrangian boundary values. These can be patched together in a finite procedure since the above construction allows to interpolate in the coordinates between ξ^ν, v^ν and other smooth approximations ξ', v' (arising from approximations of B on another neighbourhood U' in different coordinates) of ξ and v respectively. This gives the required approximation of B in a neighbourhood of $\Omega \cap \partial\mathbb{H}$ and thus finishes the proof. □

Proof of lemma 4.1:

Fix $A \in \mathcal{L}$ and consider the following two decompositions:

$$\begin{aligned} L^p(\Sigma, \mathrm{T}^*\Sigma \otimes \mathfrak{g}) &= \mathrm{T}_A\mathcal{L} \oplus *\mathrm{T}_A\mathcal{L} \\ &= \mathrm{d}_A W^{1,p}_z(\Sigma, \mathfrak{g}) \oplus *\mathrm{d}_A W^{1,p}_z(\Sigma, \mathfrak{g}) \oplus \tilde{h}_A, \end{aligned} \tag{4.6}$$

where $\tilde{h}_A$ is a complement of the image of the following Fredholm operator:

$$D_A : \begin{array}{ccc} W^{1,p}_z(\Sigma, \mathfrak{g}) \times W^{1,p}_z(\Sigma, \mathfrak{g}) & \longrightarrow & L^p(\Sigma, \mathrm{T}^*\Sigma \otimes \mathfrak{g}) \\ (\xi, \zeta) & \longmapsto & \mathrm{d}_A\xi + *\mathrm{d}_A\zeta. \end{array}$$

To see that D_A is Fredholm note that for every $A \in \mathcal{A}^{0,p}_{\mathrm{flat}}(\Sigma)$ the operator D_A is injective and is a compact perturbation of D_0. Hence moreover the dimension of $\operatorname{coker} D_A$ (and thus of $\tilde{h}_A$) is the same as that of $\operatorname{coker} D_0$. In the case $A = 0$ one can choose the space of $\mathfrak{g}$-valued harmonic 1-forms $h^1 = \ker \mathrm{d} \cap \ker \mathrm{d}^*$ as complement $\tilde{h}_0$. So $\tilde{h}_A$ must always have the dimension $\dim \tilde{h}_A = \dim h^1 = 2g \cdot \dim \mathrm{G} = 2m$. (Note that in general one can choose $\tilde{h}_A$ to contain h^1_A, but this might not exhaust the whole complement.)

Due to the $\mathcal{G}^{1,p}_z(\Sigma)$-invariance of $\mathcal{L}$ the splittings (4.6) now imply that there exists an m-dimensional subspace $L_A \subset \tilde{h}_A$ such that

$$\mathrm{T}_A\mathcal{L} = \mathrm{d}_A W^{1,p}_z(\Sigma, \mathfrak{g}) \oplus L_A.$$

So $\mathrm{T}_A\mathcal{L}$ is isomorphic to the Banach space $W^{1,p}_z(\Sigma,\mathfrak{g}) \times \mathbb{R}^m$ via $\mathrm{d}_A \oplus F$ for some isomorphism $F : \mathbb{R}^m \to L_A$. Here we have used the fact that d_A is injective when restricted to $W^{1,p}_z(\Sigma,\mathfrak{g})$. Now choose a coordinate chart $\Phi : \mathrm{T}_A\mathcal{L} \to \mathcal{L}$ defined near $\Phi(0) = A$, then the following map is defined for a sufficiently small neighbourhood $V \subset \mathbb{R}^m$ of 0,

$$\Psi : \begin{array}{ccc} \mathcal{G}^{1,p}_z(\Sigma) \times V & \longrightarrow & \mathcal{L} \\ (u,v) & \longmapsto & u^*\big(\Phi \circ (\mathrm{T}_0\Phi)^{-1} \circ F(v)\big). \end{array}$$

We will show that this is an embedding and a submersion (and thus a diffeomorphism to its image). Firstly, $\mathrm{T}_{(\mathbb{1},0)}\Psi : (\xi, w) \mapsto \mathrm{d}_A\xi + Fw$ is an isomorphism. Next, note that $\Psi(u,v) = u^*\Psi(\mathbb{1},v)$ and use this to calculate for all $u \in \mathcal{G}^{1,p}_z(\Sigma)$, $\xi \in W^{1,p}_z(\Sigma,\mathfrak{g})$, and $v, w \in \mathbb{R}^m$

$$\mathrm{T}_{(u,v)}\Psi : (\xi u, w) \mapsto u^{-1}\big(\mathrm{d}_{\Psi(\mathbb{1},v)}\xi + \mathrm{T}_{(\mathbb{1},v)}\Psi(0,w)\big)u.$$

One sees that $u(\mathrm{T}_{(u,v)}\Psi)u^{-1}$ is a small perturbation of $\mathrm{T}_{(\mathbb{1},0)}\Psi$, hence one can choose V sufficiently small (independently of u) such that $\mathrm{T}_{(u,v)}\Psi$ also is an isomorphism for all $v \in V$. So it remains to check that Ψ in fact is globally injective.

Suppose that $u, u' \in \mathcal{G}^{1,p}_z(\Sigma)$ and $v, v' \in V$ such that $\Psi(u,v) = \Psi(u',v')$. Rewrite this as $\Psi(\mathbb{1},v) = \Psi(\tilde{u},v')$ with $\tilde{u} := u'u^{-1} \in \mathcal{G}^{1,p}_z(\Sigma)$. Now by the choice of a sufficiently small V the norm $\|\Psi(\mathbb{1},v) - \Psi(\mathbb{1},v')\|_p$ can be made arbitrarily small. Then the identity $\Psi(\mathbb{1},v) = \tilde{u}^*\Psi(\mathbb{1},v')$ automatically implies that $\tilde{u}$ is $\mathcal{C}^0$-close to $\mathbb{1}$. (Otherwise one would find a sequence of L^p-connections $A^\nu \to A$ and $u^\nu \in \mathcal{G}^{1,p}_z(\Sigma)$ such that $\|{u^\nu}^*A^\nu - A^\nu\|_p \to 0$ but $d_{\mathcal{C}^0}(u^\nu,\mathbb{1}) \geq \Delta > 0$. However, from $(u^\nu)^{-1}\mathrm{d}u^\nu = {u^\nu}^*A^\nu - (u^\nu)^{-1}A^\nu u^\nu$ one obtains an L^p-bound on $\mathrm{d}u^\nu$ and thus finds a weakly $W^{1,p}$-convergent subsequence of the u^ν. Its limit $u \in \mathcal{G}^{1,p}_z(\Sigma)$ would have to satisfy $u^*A = A$, hence $u \equiv \mathbb{1}$ in contradiction to $d_{\mathcal{C}^0}(u,\mathbb{1}) \geq \Delta > 0$.) So one can write $\tilde{u} = \exp(\xi)$ where $\xi \in W^{1,p}_z(\Sigma,\mathfrak{g})$ is small in the L^∞-norm. Next, the identity

$$\tilde{u}^{-1}\mathrm{d}\tilde{u} = \Psi(\mathbb{1},v) - \tilde{u}^{-1}\Psi(\mathbb{1},v')\tilde{u}$$

shows that $\|\xi\|_{W^{1,p}}$ will be small if V is small (and thus $\tilde{u}$ is $\mathcal{C}^0$-close to $\mathbb{1}$). Hence if V is sufficiently small, then $(\tilde{u},v')$ and $(\mathbb{1},v)$ automatically lie in a neighbourhood of $(\mathbb{1},0)$ on which Ψ is injective, and hence $u = u'$ and $v = v'$.

We have thus shown that $\Psi : \mathcal{G}^{1,p}_z(\Sigma) \times V \to \mathcal{L}$ is a diffeomorphism to its image. This provides manifold charts $\psi : V \to \mathcal{L}/\mathcal{G}^{1,p}_z(\Sigma)$, $v \mapsto [\Psi(\mathbb{1}, v)]$ for $L := \mathcal{L}/\mathcal{G}^{1,p}_z(\Sigma)$. Now fix $2g$ generators of the fundamental group $\pi_1(\Sigma)$, then the corresponding holonomy map $\rho_z : L \to \mathrm{G} \times \cdots \times \mathrm{G}$ is an embedding, so its image $M \subset \mathrm{Hom}(\pi_1(\Sigma), \mathrm{G})$ is a smooth submanifold. This proves (i). For (ii) the diffeomorphism Ψ gives a bundle chart over $\mathcal{U} := \rho_z(\psi(V)) \subset M$, namely

$$\Psi \circ \big(\mathrm{id} \times (\rho_z \circ \psi)^{-1}\big) \;:\; \mathcal{G}^{1,p}_z(\Sigma) \times \mathcal{U} \longrightarrow \mathcal{L}.$$

Furthermore, the local section for (iii) is given by $\phi(v) := \Psi(\mathbb{1}, v)$. However, this is a map $\phi : V \to \mathcal{L}$; it does not necessarily take values in the smooth connections. Now if $A \in \mathcal{L} \cap \mathcal{A}(\Sigma)$ is smooth, then for a sufficiently small neighbourhood V this section can be modified by gauge transformations such that $\phi : V \to \mathcal{L} \cap \mathcal{A}(\Sigma)$. To see this, note that the gauge transformations in the local slice theorem are given by an implicit function theorem: One solves $D(v, \xi) = 0$ for $\xi = \xi(v) \in W^{1,p}(\Sigma, \mathfrak{g})$ with the following operator:

$$D : \begin{array}{ccccc} V \times W^{1,p}(\Sigma, \mathfrak{g}) & \longrightarrow & \mathrm{im}\, \mathrm{d}'_A & \subset & \big(W^{1,p^*}(\Sigma, \mathfrak{g})\big)^* \\ (v, \xi) & \longmapsto & \mathrm{d}'_A\big(\exp(\xi)^* \phi(v) - A\big) & . & \end{array}$$

Here d'_A denotes the dual operator of d_A on $W^{1,p^*}(\Sigma, \mathfrak{g})$. One has $D(0,0) = 0$ and checks that $\partial_2 D(0,0) : \xi \to \mathrm{d}'_A \mathrm{d}_A \xi$ is a surjective map to $\mathrm{im}\, \mathrm{d}'_A$, see e.g. [We, Lemma 9.5]. The implicit function theorem [L, XIV,Theorem 2.1] then gives the required gauge transformations $\exp(\xi(v)) \in \mathcal{G}^{1,p}(\Sigma)$ that bring $\phi(v)$ into local Coulomb gauge and thus make it smooth. (By construction $\phi(v)$ is weakly flat, then see the proof of theorem 3.1.) This modification by gauge transformations does not affect the topological direct sum decomposition $\mathrm{T}_A\mathcal{L} = \mathrm{d}_A W^{1,p}_z(\Sigma, \mathfrak{g}) \oplus \mathrm{im}\, \mathrm{T}_0\phi$.

To see that the given map Θ is a diffeomorphism between neighbourhoods of 0 and A just note that the inverse of $\mathrm{T}_0\Theta$ is given by the splitting

$$\begin{aligned} L^p(\Sigma, \mathrm{T}^*\Sigma \otimes \mathfrak{g}) &= \mathrm{T}_A\mathcal{L} \oplus *\mathrm{T}_A\mathcal{L} \\ &= \mathrm{d}_A W^{1,p}_z(\Sigma, \mathfrak{g}) \oplus \mathrm{im}\, \mathrm{T}_0\phi \oplus *\mathrm{d}_A W^{1,p}_z(\Sigma, \mathfrak{g}) \oplus *\mathrm{im}\, \mathrm{T}_0\phi \end{aligned}$$

composed with the inverses of $\mathrm{d}_A|_{W^{1,p}_z(\Sigma,\mathfrak{g})}$ and $\mathrm{T}_0\phi$. □

Now observe that the choice of $p > 2$ for the Lagrangian submanifolds in the above lemma is accidental. All connections $A \in \mathcal{L}$ are gauge equivalent to a smooth connection, and the L^q-completion of $\mathcal{L} \cap \mathcal{A}(\Sigma)$ is a Lagrangian submanifold in $\mathcal{A}^{0,q}(\Sigma)$ for all $q > 2$. In fact, this simply is the restricted $(q > p)$ or completed $(q < p)$ $\mathcal{G}^{1,q}_z(\Sigma)$-bundle over M.

The main example

Suppose that G is connected and simply connected and that $\Sigma = \partial Y$ is the boundary of a handlebody Y. (Again, the handlebody and thus its boundary might consist of several connected components.) The crucial property of a handle body Y is that the inclusion $\iota : \Sigma \to Y$ induces an isomorphism $\pi_1(Y) \cong \pi_1(\Sigma)/\partial\pi_2(Y,\Sigma)$. This is since Y retracts onto its 1-skeleton, which can be chosen to lie in Σ, so we have the exact sequence

$$0 = \pi_2(Y) \to \pi_2(Y,\Sigma) \xrightarrow{\partial} \pi_1(\Sigma) \xrightarrow{\iota} \pi_1(Y) \to \pi_1(Y,\Sigma) = 0.$$

The assumptions on G together with the fact that $\pi_2(\mathrm{G}) = 0$ for any Lie group G (see e.g. [B, Proposition 7.5]) ensures that the gauge group $\mathcal{G}^{1,p}(\Sigma)$ is connected and that every gauge transformation on Σ can be extended to Y.

Let $p > 2$ and let $\mathcal{L}_Y$ be the $L^p(\Sigma)$-closure of the set of smooth flat connections on Σ that can be extended to a flat connection on Y,

$$\mathcal{L}_Y \;:=\; \mathrm{cl}\left\{ A \in \mathcal{A}_{\mathrm{flat}}(\Sigma) \;\middle|\; \exists \tilde{A} \in \mathcal{A}_{\mathrm{flat}}(Y) : \tilde{A}|_\Sigma = A \right\} \;\subset\; \mathcal{A}^{0,p}(\Sigma).$$

This set has the following properties.

Lemma 4.3

(i) $\mathcal{L}_Y = \left\{ u^*(A|_\Sigma) \;\middle|\; A \in \mathcal{A}_{\mathrm{flat}}(Y), u \in \mathcal{G}^{1,p}(\Sigma) \right\}$

(ii) $\mathcal{L}_Y \subset \mathcal{A}^{0,p}(\Sigma)$ *is a Lagrangian submanifold.*

(iii) $\mathcal{L}_Y \subset \mathcal{A}^{0,p}_{\mathrm{flat}}(\Sigma)$ *and* $\mathcal{L}_Y$ *is invariant under the action of* $\mathcal{G}^{1,p}(\Sigma)$.

(iv) *Fix any* $z \in \Sigma$. *Then*

$$\mathcal{L}_Y = \left\{ A \in \mathcal{A}^{0,p}_{\mathrm{flat}}(\Sigma) \;\middle|\; \rho_z(A) \in \mathrm{Hom}(\pi_1(Y),\mathrm{G}) \subset \mathrm{Hom}(\pi_1(\Sigma),\mathrm{G}) \right\},$$

where we identify

$$\mathrm{Hom}(\pi_1(Y),\mathrm{G}) \cong \left\{ \rho \in \mathrm{Hom}(\pi_1(\Sigma),\mathrm{G}) \;\middle|\; \rho(\partial\pi_2(Y,\Sigma)) = \{\mathbb{1}\} \right\}.$$

So $\mathcal{L}_Y$ *obtains the structure of a* $\mathcal{G}^{1,p}_z(\Sigma)$*-bundle over the* g*-fold product* $M = \mathrm{G} \times \cdots \times \mathrm{G} \cong \mathrm{Hom}(\pi_1(Y),\mathrm{G})$,

$$\mathcal{G}^{1,p}_z(\Sigma) \hookrightarrow \mathcal{L}_Y \xrightarrow{\rho_z} \mathrm{Hom}(\pi_1(Y),\mathrm{G}).$$

Proof: Firstly, $\mathcal{L}_Y \subset \mathcal{A}^{0,p}_{\text{flat}}(\Sigma)$ follows from the fact that weak flatness is an L^p-closed condition for $p > 2$. The holonomy $\rho_z : \mathcal{A}^{0,p}_{\text{flat}}(\Sigma) \to \mathrm{G} \times \cdots \times \mathrm{G}$ is continuous with respect to the L^p-topology. Thus for every $A \in \mathcal{L}_Y$ the holonomy vanishes on those loops in Σ that are contractible in Y. On the other hand, in view of theorem 3.1, every $A \in \mathcal{A}^{0,p}_{\text{flat}}(\Sigma)$ whose holonomy descends to $\mathrm{Hom}(\pi_1(Y), \mathrm{G})$ can be written as $A = u^*\tilde{A}$, where $u \in \mathcal{G}^{1,p}_z(\Sigma)$ and the holonomy of $\tilde{A} \in \mathcal{A}_{\text{flat}}(\Sigma)$ also vanishes along the loops that are contractible in Y. Thus $\tilde{A}$ can be extended to a flat connection on Y and smooth approximation of u proves that $A \in \mathcal{L}_Y$. This proves the alternative definitions of $\mathcal{L}_Y$ in (iv) and (i). Then (iii) is a consequence of (i).

To prove the second assertion in (iv) we explicitly construct local sections of $\mathcal{L}_Y$. Let the loops $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g \subset \Sigma$ represent the standard generators of $\pi_1(\Sigma)$ such that $\alpha_1, \ldots, \alpha_g$ generate $\pi_1(Y)$ and such that the only nonzero intersections are $\alpha_i \cap \beta_i$. [1] Now fix $A \in \mathcal{L}_Y$. In order to change its holonomy along α_i by some $g \in \mathrm{G}$ close to $\mathbb{1}$, one gauge transforms A in a small neighbourhood of β_i in Σ with a smooth gauge transformation that equals $\mathbb{1}$ and g respectively near the two boundary components of that ring about β_i. That way one obtains a smooth local section $\phi : V \to \mathcal{L}_Y$ defined on a neighbourhood $V \subset \mathfrak{g}^g$ of 0, such that $\phi(0) = A$ and $\rho_z \circ \phi : V \to \mathrm{Hom}(\pi_1(Y), \mathrm{G})$ is a bijection onto a neighbourhood of $\rho_z(A)$. This leads to a bundle chart

$$\Psi : \begin{array}{ccc} \mathcal{G}^{1,p}_z(\Sigma) \times V & \longrightarrow & \mathcal{L}_Y \\ (u, v) & \longmapsto & u^*\phi(v). \end{array}$$

Note that for smooth $A \in \mathcal{L}_Y \cap \mathcal{A}(\Sigma)$ the local section ϕ constructed above in fact is a section in the smooth part $\mathcal{L}_Y \cap \mathcal{A}(\Sigma)$ of the Lagrangian. Using these bundle charts one also checks that $\mathcal{L}_Y \subset \mathcal{A}^{0,p}(\Sigma)$ is indeed a Banach submanifold. A submanifold chart near $\Psi(u, v) \in \mathcal{A}^{0,p}(\Sigma)$ is given by $(\xi, w) \mapsto \Psi(\exp(\xi)u, v + w) + *\mathrm{T}_{(u,v)}\Psi(\xi, w)$. As in lemma 4.1 one checks that this is a local diffeomorphism.

To verify the Lagrangian condition it suffices to consider ω on $\mathrm{T}_A\mathcal{A}^{0,p}(\Sigma)$ for smooth $A \in \mathcal{L}_Y$. This is because both ω and $\mathcal{L}_Y$ are invariant under the gauge action. So pick some $A \in \mathcal{L}_Y \cap \mathcal{A}(\Sigma)$ and find $\tilde{A} \in \mathcal{A}_{\text{flat}}(Y)$ such that $A = \tilde{A}|_\Sigma$. Let $\alpha, \beta \in \mathrm{T}_A\mathcal{L}_Y$, then by the characterization of $\mathcal{L}_Y$ in (i) we find $\xi, \zeta \in W^{1,p}(\Sigma, \mathfrak{g})$ and paths $\tilde{A}^\alpha, \tilde{A}^\beta : [-1, 1] \to \mathcal{A}_{\text{flat}}(Y)$ such that

[1] $\pi_1(\Sigma)$ is the quotient of the free group generated by elements $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g$ by the relation $\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1} \ldots \alpha_g\beta_g\alpha_g^{-1}\beta_g^{-1} = \mathbb{1}$, whereas $\pi_1(Y)$ is the free group generated by $\alpha_1, \ldots, \alpha_g$.

$\tilde{A}^\alpha(0) = \tilde{A}^\beta(0) = \tilde{A}$ and

$$\alpha \;=\; \mathrm{d}_A\xi + \tfrac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0}\tilde{A}^\alpha(s)|_\Sigma, \qquad \beta \;=\; \mathrm{d}_A\zeta + \tfrac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0}\tilde{A}^\beta(s)|_\Sigma.$$

Now firstly Stokes' theorem on Σ with $\partial\Sigma = \emptyset$ proves

$$\omega(\mathrm{d}_A\xi\,,\,\mathrm{d}_A\zeta) \;=\; \lim_{\nu\to\infty}\int_\Sigma \langle\, \mathrm{d}_A\xi^\nu \wedge \mathrm{d}_A\zeta^\nu \,\rangle \;=\; \lim_{\nu\to\infty}\int_\Sigma \mathrm{d}\langle\, \xi^\nu \wedge \mathrm{d}_A\zeta^\nu \,\rangle \;=\; 0.$$

Here we have used smooth $W^{1,p}$-approximations ξ^ν and ζ^ν of ξ and ζ respectively.

Similarly, one obtains $\omega(\mathrm{d}_A\xi, \frac{\mathrm{d}}{\mathrm{d}s}\tilde{A}^\beta|_\Sigma) = 0$ and $\omega(\frac{\mathrm{d}}{\mathrm{d}s}\tilde{A}^\alpha|_\Sigma, \mathrm{d}_A\zeta) = 0$ since $\mathrm{d}_A\big(\frac{\mathrm{d}}{\mathrm{d}s}\tilde{A}^\alpha|_\Sigma\big) = \frac{\mathrm{d}}{\mathrm{d}s}F_{\tilde{A}^\alpha}\big|_\Sigma = 0$. Finally, Stokes' theorem with $\partial Y = \Sigma$ yields due to $F_{\tilde{A}^\alpha(s)} = 0$ for all s

$$\begin{aligned}\omega(\alpha\,,\,\beta) \;&=\; \int_\Sigma \langle\, \tfrac{\mathrm{d}}{\mathrm{d}s}\tilde{A}^\alpha|_\Sigma \wedge \tfrac{\mathrm{d}}{\mathrm{d}s}\tilde{A}^\beta|_\Sigma \,\rangle \;=\; \int_Y \mathrm{d}\langle\, \tfrac{\mathrm{d}}{\mathrm{d}s}\tilde{A}^\alpha \wedge \tfrac{\mathrm{d}}{\mathrm{d}s}\tilde{A}^\beta \,\rangle \\ &=\; \int_Y \langle\, \tfrac{\mathrm{d}}{\mathrm{d}s}F_{\tilde{A}^\alpha} \wedge \tfrac{\mathrm{d}}{\mathrm{d}s}\tilde{A}^\beta \,\rangle - \int_Y \langle\, \tfrac{\mathrm{d}}{\mathrm{d}s}\tilde{A}^\alpha \wedge \tfrac{\mathrm{d}}{\mathrm{d}s}F_{\tilde{A}^\beta} \,\rangle \;=\; 0.\end{aligned}$$

This proves that $\omega|_{\mathrm{T}_A\mathcal{L}_Y} = 0$ and recalling (4.2) one moreover sees that $\mathrm{T}_A\mathcal{L}_Y$ and $*\mathrm{T}_A\mathcal{L}_Y$ are L^2-orthogonal. In fact, we even have the topological decomposition $L^p(\Sigma, \mathrm{T}^*\Sigma\otimes\mathfrak{g}) = \mathrm{T}_A\mathcal{L}_Y \oplus *\mathrm{T}_A\mathcal{L}_Y$, and this proves the Lagrangian property of $\mathcal{L}_Y$. To see that this direct sum indeed exhausts the whole space consider the Hodge type decomposition as in the proof of lemma 4.1,

$$L^p(\Sigma, \mathrm{T}^*\Sigma\otimes\mathfrak{g}) = \mathrm{d}_A W^{1,p}_z(\Sigma,\mathfrak{g}) \oplus *\mathrm{d}_A W^{1,p}_z(\Sigma,\mathfrak{g}) \oplus \tilde{h}_A.$$

Here we have $\dim\tilde{h}_A = 2g\cdot\dim\mathrm{G}$, and we have already seen that $\mathcal{L}_Y$ is a $\mathcal{G}^{1,p}_z(\Sigma)$-bundle over the $(g\cdot\dim\mathrm{G})$-dimensional manifold $\mathrm{Hom}(\pi_1(Y),\mathrm{G})$. So $\mathrm{d}_A W^{1,p}_z(\Sigma,\mathfrak{g}) \subset \mathrm{T}_A\mathcal{L}_Y$ is the tangent space to the fibre through A, and then for dimensional reasons $\mathrm{T}_A\mathcal{L}_Y \oplus *\mathrm{T}_A\mathcal{L}_Y$ also exhausts $\tilde{h}_A$ and thus all of $L^p(\Sigma, \mathrm{T}^*\Sigma\otimes\mathfrak{g})$. □

Chapter 5

Cauchy-Riemann equations in Banach spaces

In this chapter we give a general regularity result for Cauchy-Riemann equations in complex Banach spaces with totally real boundary conditions. One component of the boundary value problem (1.2) will take this form, and the regularity result for this component will be the key point of the proof of theorems A and B.

So consider a Banach space X with a complex structure $J \in \operatorname{End} X$, i.e. such that $J^2 = -\mathbb{1}$. Let $\mathcal{L} \subset X$ be a totally real submanifold, i.e. a Banach submanifold such that $X = \mathrm{T}_x\mathcal{L} \oplus J\,\mathrm{T}_x\mathcal{L}$ for all $x \in \mathcal{L}$.

Example 5.1 In our application the Banach space will be $\mathcal{A}^{0,p}(\Sigma)$ with the complex structure given by the Hodge operator $J = *$ on 1-forms for any metric on Σ. Then J is compatible with the symplectic structure ω on $\mathcal{A}^{0,p}(\Sigma)$ given by (4.1), since $\omega(\cdot, J\cdot)$ is an L^2-inner product on the 1-forms. Hence any Lagrangian submanifold $\mathcal{L} \subset \mathcal{A}^{0,p}(\Sigma)$ is totally real.

Recall that the gauge invariant Lagrangian submanifolds $\mathcal{L} \subset \mathcal{A}^{0,p}_{\mathrm{flat}}(\Sigma)$ in the previous chapter were modelled on $W^{1,p}_z(\Sigma, \mathfrak{g}) \oplus \mathbb{R}^m$. We will use a similar assumption for the general totally real submanifolds $\mathcal{L} \subset X$.

Assumption: Throughout this chapter we suppose that the totally real submanifold $\mathcal{L} \subset X$ is – as a Banach manifold – modelled on a closed subspace $Y \subset Z$ of an L^p-space $Z = L^p(M, \mathbb{R}^m)$ for some $p > 1$, $m \in \mathbb{N}$, and a closed manifold M.

Example 5.2

(i) Every finite dimensional space $\mathbb{R}^m$ is isometric to the subspace of constants in $L^p(M,\mathbb{R}^m)$ for $\operatorname{Vol} M = 1$.

(ii) The Sobolev space $W^{\ell,p}(M)$ (and thus every closed subspace thereof) is bounded isomorphic to a closed subspace of $L^p(M,\mathbb{R}^m)$.

To see this, choose vector fields $X_1,\dots,X_k \in \Gamma(\mathrm{T}M)$ that span T_xM for all $x \in M$. Then the map $u \mapsto (u, \nabla_{X_1}u, \dots, \nabla^\ell_{X_k}u)$ running through all derivatives of u up to order ℓ gives a bounded isomorphism between $W^{\ell,p}(M)$ and a closed subspace of $L^p(M,\mathbb{R}^m) =: Z$ for $m = 1 + k + \ldots + k^\ell$. Estimates in Z then also yield $W^{\ell,p}$-estimates since the norms are equivalent.

(iii) Finite products of closed subspaces in $L^p(M_i,\mathbb{R}^{m_i})$ are isometric to a closed subspace of $L^p(\bigcup M_i, \mathbb{R}^{\max\{m_i\}})$.

Now let $\mathbb{H}$ be the half space

$$\mathbb{H} := \{(s,t) \in \mathbb{R}^2 \mid t \geq 0\}.$$

Let $\Omega \subset \mathbb{H}$ be a compact 2-dimensional submanifold, i.e. Ω has smooth boundary that might intersect $\partial\mathbb{H} = \{t = 0\}$. We consider a map $u : \Omega \to X$ that solves the following boundary value problem:

$$\begin{cases} \partial_s u + J_{s,t}\partial_t u = G, \\ u(s,0) \in \mathcal{L} \quad \forall (s,0) \in \partial\Omega. \end{cases} \tag{5.1}$$

Here $J : \Omega \to \operatorname{End} X$ will be a suitably regular family of complex structures on X. Now assume that $u, G \in W^{k,q}(\Omega, X)$ for some $k \in \mathbb{N}$ and

$$q := \begin{cases} p & ; k \geq 2, \\ 2p & ; k = 1. \end{cases}$$

In both cases the following theorem then gives $W^{k+1,p}$-regularity for u. Here and throughout the interior of Ω is defined with respect to the topology of $\mathbb{H}$, so $\operatorname{int}\Omega$ still contains $\partial\Omega \cap \partial\mathbb{H}$.

Theorem 5.3 *Fix $1 < p < \infty$ and a compact subset $K \subset \operatorname{int}\Omega$. Let $u_0 \in \mathcal{C}^\infty(\Omega, X)$ be such that $u_0(s,0) \in \mathcal{L}$ for all $(s,0) \in \partial\Omega$, and let $J_0 \in \mathcal{C}^\infty(\Omega, \operatorname{End} X)$ be a smooth family of complex structures on X. Then there exists a constant $\delta_1 > 0$ and for all $k \in \mathbb{N}$ there exist constants $\delta_2 > 0$ and C such that the following holds:*

Let $J \in \mathcal{C}^{k+1}(\Omega, \operatorname{End} X)$ be a family of complex structures on X. Suppose that $u, G \in W^{k,q}(\Omega, X)$ (with q as above) solve (5.1) and that

$$\|u - u_0\|_{L^\infty(\Omega,X)} \le \delta_1, \qquad \|J - J_0\|_{\mathcal{C}^{k+1}(\Omega,\operatorname{End} X)} \le \delta_2.$$

Then $u \in W^{k+1,p}(K, X)$ and

$$\|u - u_0\|_{W^{k+1,p}(K,X)} \le C\bigl(1 + \|G\|_{W^{k,q}(\Omega,X)} + \|u - u_0\|_{W^{k,q}(\Omega,X)}\bigr).$$

Note that the u_0 in this theorem is not a solution of the equation. It only satisfies the boundary condition and will be required for the choice of coordinates near $\mathcal{L}$ in the proof.

Theorem 5.3 will be essential for the nonlinear regularity and compactness results in chapter 6. For the Fredholm theory in chapter 7 we will moreover need the following regularity and estimate for the linearization of the boundary value problem (5.1). In order to state the corresponding weak equation we introduce the dual operator $J^* \in \operatorname{End} X^*$ of the complex structure $J \in \operatorname{End} X$, where X^* denotes the dual space of X.

Theorem 5.4 *Fix $1 < p < \infty$ and a compact subset $K \subset \operatorname{int}\Omega$. Fix a path $x \in \mathcal{C}^\infty(\mathbb{R}, \mathcal{L})$ in $\mathcal{L}$ and let $J \in \mathcal{C}^\infty(\Omega, \operatorname{End} X)$ be a family of complex structures on X. Then there is a constant C such that the following holds:*

Suppose that $u \in L^p(\Omega, X)$ and that there exists a constant c_u such that for all $\psi \in \mathcal{C}^\infty(\Omega, X^*)$ with $\operatorname{supp}\psi \subset \operatorname{int}\Omega$ and $\psi(s,0) \in (J(s,0)\mathrm{T}_{x(s)}\mathcal{L})^\perp$ for all $(s,0) \in \partial\Omega$

$$\left| \int_\Omega \langle\, u \,,\, \partial_s\psi + \partial_t(J^*\psi) \,\rangle \right| \;\le\; c_u \|\psi\|_{L^{p^*}(\Omega,X^*)}.$$

Then $u \in W^{1,p}(K, X)$ and

$$\|u\|_{W^{1,p}(K,X)} \le C\bigl(c_u + \|u\|_{L^p(\Omega,X)}\bigr).$$

Corollary 5.5 *Under the assumptions of theorem 5.4 there exists a constant C such that the following holds: Suppose that $u \in W^{1,p}(\Omega, X)$ satisfies $u(s,0) \in \mathrm{T}_{x(s)}\mathcal{L}$ for all $(s,0) \in \partial\Omega$, then*

$$\|u\|_{W^{1,p}(K,X)} \le C\bigl(\|\partial_s u + J\partial_t u\|_{L^p(\Omega,X)} + \|u\|_{L^p(\Omega,X)}\bigr).$$

Proof of corollary 5.5:
Let $u \in W^{1,p}(\Omega, X)$ and $\psi \in \mathcal{C}^\infty(\Omega, X^*)$ such that $\operatorname{supp}\psi \subset \operatorname{int}\Omega$ and with the boundary conditions $u(s,0) \in \mathrm{T}_{x(s)}\mathcal{L}$ and $\psi(s,0) \in (J(s,0)\mathrm{T}_{x(s)}\mathcal{L})^\perp$ for all $(s,0) \in \partial\Omega$. Then one obtains the weak estimate, where the boundary term vanishes,

$$\left|\int_\Omega \langle\, u\,,\, \partial_s\psi + \partial_t(J^*\psi)\,\rangle\right| = \left|\int_\Omega \langle\, \partial_s u + J\partial_t u\,,\, \psi\,\rangle - \int_{\partial\Omega\cap\partial\mathbb{H}} \langle\, Ju\,,\, \psi\,\rangle\right|$$
$$\leq \|\partial_s u + J\partial_t u\|_{L^p(\Omega,X)}\|\psi\|_{L^{p^*}(\Omega,X^*)}.$$

This holds for all ψ as above, so the estimate follows from theorem 5.4. □

Remark 5.6 Theorem 5.4 and corollary 5.5 also hold when one considers compact domains $K \subset \operatorname{int}\Omega \subset S^1 \times [0,\infty)$ in the half cylinder and a loop $x \in \mathcal{C}^\infty(S^1, \mathcal{L})$.

To see this, identify $S^1 \cong \mathbb{R}/\mathbb{Z}$, identify K with a compact subset $K' \subset \mathbb{H}$ in $[0,1] \times [0,\infty)$, and periodically extend x and u for $s \in [-1,2]$. Then u is defined and satisfies the weak equation on some open domain $\Omega' \subset \mathbb{H}$ such that $K' \subset \operatorname{int}\Omega'$, so the results for $\mathbb{H}$ apply.

Recall that the totally real submanifold $\mathcal{L}$ is modelled on the Banach space Y. The idea for the proof of theorem 5.3 is to straighten out the boundary condition by going to local coordinates in $Y \times Y$ near $u_0 \in X$ such that $Y \times \{0\}$ corresponds to the submanifold $\mathcal{L}$ and the complex structure becomes standard along $Y \times \{0\}$. For theorem 5.4 one chooses $\mathbb{R}$-dependent coordinates for X that identify $Y \times \{0\}$ with $\mathrm{T}_{x(s)}\mathcal{L}$ along the path $x : \mathbb{R} \to \mathcal{L}$.

Then the boundary value problem (5.1) yields Dirichlet and Neumann boundary conditions for the two components and one can use regularity results for the Laplace equation with such boundary conditions. However, there are two difficulties. Firstly, by straightening out the totally real submanifold, the complex structure J becomes explicitly dependent on u, so one has to deal carfully with nonlinearities in the equation. Secondly, this approach requires a Caldéron-Zygmund inequality for functions with values in a Banach space. In general, the Caldéron-Zygmund inequality is only true for values in Hilbert spaces. In our case we only need the L^p-inequality for functions with values in L^p-Sobolev spaces. In that case, the Caldéron-Zygmund inequality holds, as can be seen by integrating over the real valued inequality. This will be made precise in the following lemma. We abbreviate $\Delta := \mathrm{d}^*\mathrm{d}$, denote by

Z^* the dual space of any Banach space Z and by $\langle \cdot, \cdot \rangle$ the pairing of Z and Z^*. The Sobolev spaces of Banach space valued functions considered below are all defined as completions of the smooth functions with respect to the respective Sobolev norm. Moreover, we use the notation

$$\mathcal{C}^\infty_\delta(\Omega, Z^*) := \{\psi \in \mathcal{C}^\infty(\Omega, Z^*) \mid \psi|_{\partial\Omega} = 0\},$$
$$\mathcal{C}^\infty_\nu(\Omega, Z^*) := \{\psi \in \mathcal{C}^\infty(\Omega, Z^*) \mid \tfrac{\partial\psi}{\partial\nu}|_{\partial\Omega} = 0\}.$$

Lemma 5.7 *Let Ω be a compact Riemannian manifold with boundary, let $1 < p < \infty$ and $k \in \mathbb{N}$. Let $Z = L^p(M)$ for some closed manifold M. Then there exists a constant C such that the following holds.*

(i) Let $f \in W^{k-1,p}(\Omega, Z)$ and suppose that $u \in W^{k,p}(\Omega, Z)$ solves

$$\int_\Omega \langle u , \Delta\psi \rangle = \int_\Omega \langle f , \psi \rangle \qquad \forall \psi \in \mathcal{C}^\infty_\delta(\Omega, Z^*).$$

Then $u \in W^{k+1,p}(\Omega, Z)$ and $\|u\|_{W^{k+1,p}} \leq C\|f\|_{W^{k-1,p}}$.

(ii) Let $f \in W^{k-1,p}(\Omega, Z)$, $g \in W^{k,p}(\Omega, Z)$, and suppose that $u \in W^{k,p}(\Omega, Z)$ solves

$$\int_\Omega \langle u , \Delta\psi \rangle = \int_\Omega \langle f , \psi \rangle + \int_{\partial\Omega} \langle g , \psi \rangle \qquad \forall \psi \in \mathcal{C}^\infty_\nu(\Omega, Z^*).$$

Then $u \in W^{k+1,p}(\Omega, Z)$ and

$$\|u\|_{W^{k+1,p}} \leq C\big(\|f\|_{W^{k-1,p}} + \|g\|_{W^{k,p}} + \|u\|_{L^p}\big).$$

(iii) Suppose that $u \in L^p(\Omega, Z)$ and there exists a constant c_u such that

$$\left| \int_{\Omega\times M} u \cdot \Delta_\Omega \psi \right| \;\leq\; c_u \|\psi\|_{W^{1,p^*}(\Omega, Z^*)} \qquad \forall \psi \in \mathcal{C}^\infty_\delta(\Omega \times M).$$

Then $u \in W^{1,p}(\Omega, Z)$ and $\|u\|_{W^{1,p}(\Omega,Z)} \leq C c_u$.

(iv) Suppose that $u \in L^p(\Omega, Z)$ and there exists a constant c_u such that

$$\left| \int_{\Omega\times M} u \cdot \Delta_\Omega \psi \right| \;\leq\; c_u \|\psi\|_{W^{1,p^*}(\Omega, Z^*)} \qquad \forall \psi \in \mathcal{C}^\infty_\nu(\Omega \times M).$$

Then $u \in W^{1,p}(\Omega, Z)$ and

$$\|u\|_{W^{1,p}(\Omega,Z)} \leq C\big(c_u + \|u\|_{L^p(\Omega,Z)}\big).$$

If moreover $\int_\Omega u = 0$ then in fact $\quad \|u\|_{W^{1,p}(\Omega,Z)} \leq C c_u$.

The key to the proof of (i) and (ii) is the fact that the functions f and g can be approximated not only by smooth functions with values in the Banach space $L^p(M)$, but by smooth functions on $\Omega \times M$.

Lemma 5.8 *Let Ω be a compact manifold (possibly with boundary), let M be a closed manifold, let $1 < p, q < \infty$, and $k, \ell \in \mathbb{N}_0$. Then $\mathcal{C}^\infty(\Omega \times M)$ is dense in $W^{k,q}(\Omega, W^{\ell,p}(M))$.*

Remark 5.9

(i) A function $u \in W^{k,q}(\Omega, W^{\ell,p}(M))$ with zero boundary values $u|_{\partial\Omega} = 0$ can be approximated by $u^\nu \in \mathcal{C}^\infty(\Omega \times M)$ with $u^\nu|_{\partial\Omega\times M} = 0$.

This can be seen by first approximating in $\mathcal{C}^\infty(\Omega, W^{\ell,p}(M))$ with zero boundary values and then mollifying on M as in lemma 5.8. In case $k = 0$ the boundary condition is meaningless, but the approximation with zero boundary values can be done elementary by cutting off in small neighbourhoods of the boundary. For $k \geq 1$ consider a local chart of Ω in $[0,1]\times\mathbb{R}^n$ such that $\{t = 0\}$ corresponds to the boundary, where t denotes the $[0,1]$-coordinate. Let $f \in W^{k,q}([0,1]\times\mathbb{R}^n, Z)$ for any vector space Z with $f|_{t=0} = 0$ and compact support. Let σ_ε be mollifiers on $\mathbb{R}^n$ as in the proof of lemma 5.8 below, then $f_\varepsilon(t,\cdot) := \sigma_\varepsilon * f(t,\cdot)$ defines $f_\varepsilon \in \mathcal{C}^\infty(\mathbb{R}^n, W^{k,q}([0,1], Z))$ for all $\varepsilon > 0$. One checks that $\|f_\varepsilon - f\|_{W^{k,q}([0,1]\times\mathbb{R}^n)} \to 0$ as $\varepsilon \to 0$. We choose the σ_ε with compact support, then the f_ε are also compactly supported and hence have finite $W^{k,q}([0,1], W^{\ell,q}(\mathbb{R}^n))$-norm for any $\ell \in \mathbb{N}$. Moreover, note that still $f_\varepsilon|_{t=0} = 0$. In order to approximate f_ε with zero boundary values one chooses $\ell = k$, then lemma 5.8 gives a smooth approximation $g^\nu \to \partial_t f_\varepsilon$ in the $W^{k-1,q}([0,1], W^{k,q}(\mathbb{R}^n))$-norm. Now $f_\varepsilon^\nu(t,x) := \int_0^t g^\nu(\tau,x)\mathrm{d}\tau$ defines functions in $\mathcal{C}^\infty([0,1]\times\mathbb{R}^n, Z)$ that vanish at $t = 0$ and approximate f_ε in the $W^{k,q}([0,1], W^{k,q}(\mathbb{R}^n))$-norm, which is even stronger than the $W^{k,q}$-norm on $[0,1]\times\mathbb{R}^n$.

(ii) If $\ell p > \dim M$ and $z \in M$, then a function $u \in W^{k,q}(\Omega, W^{\ell,p}(M))$ with $u(\cdot,z) = 0 \in W^{k,q}(\Omega)$ can be approximated by $u^\nu \in \mathcal{C}^\infty(\Omega \times M)$ with $u^\nu(\cdot,z) \equiv 0$.

Indeed choose an approximation by $u^\nu \in \mathcal{C}^\infty(\Omega\times M)$, then $u^\nu(\cdot,z) \to 0$ in $W^{k,q}(\Omega)$ since the evaluation at z is a continuous map $W^{\ell,p}(M) \to \mathbb{R}$. Now $u^\nu - u^\nu(\cdot,z) \in \mathcal{C}^\infty(\Omega\times M)$ still converges to u in $W^{k,q}(\Omega, W^{\ell,p}(M))$ but it vanishes at z.

Proof of lemma 5.8:
By definition $\mathcal{C}^\infty(\Omega, W^{\ell,p}(M))$ is dense in $W^{k,q}(\Omega, W^{\ell,p}(M))$, so it suffices to fix $g \in \mathcal{C}^\infty(\Omega, W^{\ell,p}(M))$ and show that in every $W^{k,q}(\Omega, W^{\ell,p}(M))$-neighbourhood of g one can find a $\tilde{g} \in \mathcal{C}^\infty(\Omega \times M)$. Firstly, we prove this in the case $k = 0$ for closed manifolds M as well as in the following case (that will be needed for the proof in the case $k \geq 1$): $M = \mathbb{R}^n$, g is supported in $\Omega \times V$ and $\tilde{g}$ is required to have support in $\Omega \times U$ for some open bounded domains $V, U \subset \mathbb{R}^n$ such that $\overline{V} \subset U$.

Fix $\delta > 0$. Since Ω is compact one finds a finite covering $\Omega = \bigcup_{i=1}^N U_i$ by neighbourhoods U_i of $x_i \in \Omega$ such that

$$\|g(x) - g(x_i)\|_{W^{\ell,p}(M)} \leq \tfrac{\delta}{2} \qquad \forall x \in U_i.$$

Next, choose $g_i \in \mathcal{C}^\infty(M)$ such that $\|g_i - g(x_i)\|_{W^{\ell,p}(M)} \leq \frac{\delta}{2}$. In the case $M = \mathbb{R}^n$ one has $\operatorname{supp} g(x_i) \subset V$ and hence can choose g_i such that it is supported in U (e.g. using mollifiers with compact support). Then choose a partition of unity $\sum_{i=1}^N \phi_i = 1$ by $\phi_i \in \mathcal{C}^\infty(\Omega, [0,1])$ with $\operatorname{supp} \phi_i \subset U_i$. Now one can define $\tilde{g} \in \mathcal{C}^\infty(\Omega \times M)$ by

$$\tilde{g}(x,z) := \sum_{i=1}^N \phi_i(x) g_i(z) \qquad \forall x \in \Omega, z \in M.$$

In the case $M = \mathbb{R}^n$ this satisfies $\operatorname{supp} \tilde{g} \subset \Omega \times U$ as required. Moreover,

$$\begin{aligned} \|\tilde{g} - g\|^q_{L^q(\Omega, W^{\ell,p}(M))} &= \int_\Omega \left\| \textstyle\sum_{i=1}^N \phi_i (g_i - g) \right\|^q_{W^{\ell,p}(M)} \\ &\leq \int_\Omega \left(\textstyle\sum_{i=1}^N \phi_i \cdot \sup_{x \in U_i} \|g_i - g(x)\|_{W^{\ell,p}(M)} \right)^q \\ &\leq \int_\Omega \delta^q \;=\; \delta^q \operatorname{Vol} \Omega. \end{aligned}$$

Thus we have proven the lemma in the case $k = 0$. For $k \geq 1$ this method does not work since one picks up derivatives of the cutoff functions ϕ_i. Instead, one has to use mollifiers and the result for $k = 0$ on $M = \mathbb{R}^n$.

So we assume $k \geq 1$, fix $g \in \mathcal{C}^\infty(\Omega, W^{\ell,p}(M))$ and pick some $\delta > 0$. Let $M = \bigcup_{i=1}^N \Phi_i(U_i)$ be an atlas with bounded open domains $U_i \subset \mathbb{R}^n$ and charts $\Phi_i : U_i \to M$. Let $V_i \subset \overline{V}_i \subset U_i$ be open sets such that still $M = \bigcup_{i=1}^N \Phi_i(V_i)$. Then there exists a partition of unity $\sum_{i=1}^N \psi_i \circ \Phi_i^{-1} = 1$ by $\psi_i \in \mathcal{C}^\infty(\mathbb{R}^n, [0,1])$ such that $\operatorname{supp} \psi_i \subset V_i$. Now $g = \sum_{i=1}^N g_i \circ (\mathrm{id}_\Omega \times \Phi_i^{-1})$ with

$$g_i(x,y) = \psi_i(y) \cdot g(x, \Phi_i(y)) \qquad \forall x \in \Omega, y \in U_i.$$

Here $g_i \in \mathcal{C}^\infty(\Omega, W^{\ell,p}(\mathbb{R}^n))$ is extended by 0 outside of $\operatorname{supp} g_i \subset \Omega \times V_i$, and it suffices to prove that each of these functions can be approximated in $W^{k,q}(\Omega, W^{\ell,p}(\mathbb{R}^n))$ by $\tilde{g}_i \in \mathcal{C}^\infty(\Omega \times \mathbb{R}^n)$ with $\operatorname{supp} \tilde{g}_i \subset \Omega \times U_i$. So drop the subscript i and consider $g \in \mathcal{C}^\infty(\Omega, W^{\ell,p}(\mathbb{R}^n))$ that is supported in $\Omega \times V$, where $V, U \subset \mathbb{R}^n$ are open bounded domains such that $\overline{V} \subset U$.

Let $\sigma_\varepsilon(y) = \varepsilon^{-n}\sigma(y/\varepsilon)$ be a family of compactly supported mollifiers for $\varepsilon > 0$, i.e. $\sigma \in \mathcal{C}^\infty(\mathbb{R}^n, [0,\infty))$ such that $\operatorname{supp} \sigma \subset B_1(0)$ and $\int \sigma = 1$. Then for all $\varepsilon > 0$ define $\tilde{g}_\varepsilon \in \mathcal{C}^\infty(\Omega \times \mathbb{R}^n)$ by

$$\tilde{g}_\varepsilon(x,y) := [\sigma_\varepsilon * g(x,\cdot)](y) \qquad \forall x \in \Omega, y \in \mathbb{R}^n.$$

Firstly, $\operatorname{supp} \sigma_\varepsilon \subset B_\varepsilon(0)$, so for sufficiently small $\varepsilon > 0$ the support of $\tilde{g}_\varepsilon$ lies within $\Omega \times U$. Secondly, we abbreviate for $j \le k$, $m \le \ell$

$$f_{j,m} \;:=\; \nabla_\Omega^j \nabla_{\mathbb{R}^n}^m g \;\in\; \mathcal{C}^\infty(\Omega, L^p(\mathbb{R}^n)),$$

which are supported in $\Omega \times V$. Then

$$\begin{aligned}\|\tilde{g}_\varepsilon - g\|^q_{W^{k,q}(\Omega, W^{\ell,p}(\mathbb{R}^n))} &= \sum_{j\le k} \int_\Omega \bigl\| \nabla_\Omega^j \bigl(\sigma_\varepsilon * g(x,\cdot) - g(x,\cdot)\bigr)\bigr\|^q_{W^{\ell,p}(\mathbb{R}^n)} \\ &\le (\ell+1)^{\frac{q}{p}} \sum_{j\le k}\sum_{m\le \ell} \int_\Omega \bigl\| \sigma_\varepsilon * f_{j,m}(x,\cdot) - f_{j,m}(x,\cdot)\bigr\|^q_{L^p(\mathbb{R}^n)}.\end{aligned}$$

Now use the result for $k=0$ on $M = \mathbb{R}^n$ (with values in a vector bundle) to find $\tilde{f}_{j,m} \in \mathcal{C}^\infty(\Omega \times \mathbb{R}^n)$ supported in $\Omega \times U$ such that

$$\|\tilde{f}_{j,m} - f_{j,m}\|_{L^q(\Omega, L^p(\mathbb{R}^n))} \le \delta.$$

Then for all $x \in \Omega$ and sufficiently small $\varepsilon > 0$ the functions $\sigma_\varepsilon * \tilde{f}_{j,m}(x,\cdot)$ are supported in some fixed bounded domain $U' \subset \mathbb{R}^n$ containing U. Moreover, the $\tilde{f}_{j,m}$ are Lipschitz continuous, hence one finds a constant C (depending on the $\tilde{f}_{j,m}$, i.e. on g and δ) such that for all $x \in \Omega$

$$\begin{aligned}&\bigl\|\sigma_\varepsilon * \tilde{f}_{j,m}(x,\cdot) - \tilde{f}_{j,m}(x,\cdot)\bigr\|^p_{L^p(\mathbb{R}^n)} \\ &= \int_{U'} \Bigl| \int_{\mathbb{R}^n} \sigma_\varepsilon(y'-y)\bigl(\tilde{f}_{j,m}(x,y') - \tilde{f}_{j,m}(x,y)\bigr) \mathrm{d}^n y' \Bigr|^p \mathrm{d}^n y \\ &\le \int_{U'} \Bigl(\int_{\mathbb{R}^n} \sigma_\varepsilon(y'-y) \sup_{|y-y'|\le\varepsilon} |\tilde{f}_{j,m}(x,y') - \tilde{f}_{j,m}(x,y)| \,\mathrm{d}^n y' \Bigr)^p \mathrm{d}^n y \\ &\le \operatorname{Vol} U' (C\varepsilon)^p.\end{aligned}$$

Now use the fact that the convolution with σ_ε is continuous with respect to the L^p-norm, $\|\sigma_\varepsilon * f\|_p \le \|f\|_p$ (see e.g. [Ad, Lemma 2.18]) to estimate

$$
\begin{aligned}
&\int_\Omega \big\|\sigma_\varepsilon * f_{j,m}(x,\cdot) - f_{j,m}(x,\cdot)\big\|_{L^p(\mathbb{R}^n)}^q \\
&\le \int_\Omega \Big(\big\|\sigma_\varepsilon * \big(f_{j,m}(x,\cdot) - \tilde f_{j,m}(x,\cdot)\big)\big\|_{L^p(\mathbb{R}^n)} + \big\|f_{j,m}(x,\cdot) - \tilde f_{j,m}(x,\cdot)\big\|_{L^p(\mathbb{R}^n)} \\
&\qquad + \big\|\sigma_\varepsilon * \tilde f_{j,m}(x,\cdot) - \tilde f_{j,m}(x,\cdot)\big\|_{L^p(\mathbb{R}^n)}\Big)^q \\
&\le 2\cdot 3^q \big\|f_{j,m} - \tilde f_{j,m}\big\|_{L^q(\Omega,L^p(\mathbb{R}^n))}^q + 3^q \operatorname{Vol}\Omega\,(\operatorname{Vol}U)^{\frac{q}{p}}(C\varepsilon)^q \quad \le\; 3\cdot 3^q\delta^q.
\end{aligned}
$$

Here we have chosen $0 < \varepsilon \le C^{-1}(\operatorname{Vol}\Omega)^{-\frac{1}{q}}(\operatorname{Vol}U)^{-\frac{1}{p}}\,\delta$. Thus we obtain

$$\|\tilde g_\varepsilon - g\|_{W^{k,q}(\Omega,W^{\ell,p}(\mathbb{R}^n))} \le 3(\ell+1)^{\frac{1}{p}}(3(k+1)(\ell+1))^{\frac{1}{q}}\,\delta.$$

This proves the lemma. □

In the case $q = p$ this lemma provides the continuous inclusion

$$W^{k,p}(\Omega, W^{\ell,p}(M)) \subset W^{\ell,p}(M, W^{k,p}(\Omega))$$

since the norms on these spaces are identical. [1] Moreover, for $p = q$ and $k = \ell = 0$ the lemma identifies $L^p(\Omega, L^p(M)) = L^p(\Omega\times\Sigma)$ as the closure of $\mathcal{C}^\infty(\Omega\times M)$ under the L^p-norm.

Proof of lemma 5.7 (i) and (ii) :
We first give the proof of the regularity for the inhomogenous Neumann problem (ii) in full detail; (i) is proven in complete analogy – using the regularity theory for the Laplace equation on $\mathbb{R}$-valued functions with Dirichlet boundary condition instead of the Neumann condition.

So let $f \in W^{k-1,p}(\Omega, Z)$, $g \in W^{k,p}(\Omega, Z)$, and choose approximating sequences $f^i, g^i \in \mathcal{C}^\infty(\Omega\times M)$ given by lemma 5.8. Note that testing the weak equation with $\psi \equiv \alpha$ for all $\alpha \in Z^*$ yields the identity $\int_\Omega f + \int_{\partial\Omega} g = 0$. Thus $h^i := \int_\Omega f^i + \int_{\partial\Omega} g^i \to 0$ in Z as $i \to \infty$, so one can replace the f^i by $f^i - h^i/\operatorname{Vol}\Omega \in \mathcal{C}^\infty(\Omega, Z)$ to achieve

$$\int_\Omega f^i(\cdot,y) + \int_{\partial\Omega} g^i(\cdot,y) = 0 \qquad \forall y \in M, i \in \mathbb{N}.$$

[1] The spaces are actually equal. The proof requires an extension of the approximation argument to manifolds with boundary. We do not carry this out here because we will only need this one inclusion.

Now for each $y \in M$ there exist unique solutions $u^i(\cdot,y) \in \mathcal{C}^\infty(\Omega)$ of

$$\begin{cases} \Delta u^i(\cdot,y) = f^i(\cdot,y), \\ \frac{\partial}{\partial\nu} u^i(\cdot,y)\big|_{\partial\Omega} = g^i(\cdot,y)\big|_{\partial\Omega}, \\ \int_\Omega u^i(\cdot,y) = 0. \end{cases}$$

For each of these Laplace equations with Neumann boundary conditions one obtains an L^p-estimate for the solution, see proposition 3.5 and e.g. [We, Theorem 3.1] for the existence. The constant can be chosen independently of $y \in M$ since it varies continuously with y and M is compact. Then integration of those estimates yields (with different constants C)

$$\begin{aligned} \|u^i\|^p_{W^{k+1,p}(\Omega,Z)} &= \int_M \|u^i\|^p_{W^{k+1,p}(\Omega)} \\ &\leq \int_M C\bigl(\|f^i\|_{W^{k-1,p}(\Omega)} + \|g^i\|_{W^{k,p}(\Omega)}\bigr)^p \\ &\leq C\bigl(\|f^i\|_{W^{k-1,p}(\Omega,Z)} + \|g^i\|_{W^{k,p}(\Omega,Z)}\bigr)^p. \end{aligned}$$

Here one uses the crucial fact that $L^p(\Omega, L^p(M)) \subset L^p(M, L^p(\Omega))$ with identical norms. (Note that this is not the case if the integrability indices over Ω and M are different.) Similarly, one obtains for all $i,j \in \mathbb{N}$

$$\|u^i - u^j\|_{W^{k+1,p}(\Omega,Z)} \leq C\bigl(\|f^i - f^j\|_{W^{k-1,p}(\Omega,Z)} + \|g^i - g^j\|_{W^{k,p}(\Omega,Z)}\bigr).$$

So u^i is a Cauchy sequence and hence converges to some $\tilde{u} \in W^{k+1,p}(\Omega, Z)$. Now suppose that $u \in W^{k,p}(\Omega,Z)$ solves the weak Neumann equation for f and g, then we claim that in fact $u = \tilde{u} + c \in W^{k+1,p}(\Omega,Z)$, where $c \in Z$ is given by

$$c(y) := \frac{1}{\operatorname{Vol}\Omega} \int_\Omega \bigl(u(\cdot,y) - \tilde{u}(\cdot,y)\bigr) \qquad \forall y \in M.$$

In order to see that indeed $c \in L^p(M) = Z$ and that for some constant C one has $\|c\|_{L^p(M)} \leq C(\|u\|_{L^p(\Omega,Z)} + \|\tilde{u}\|_{L^p(\Omega,Z)})$ note that lemma 5.8 yields the continuous inclusion $W^{k,p}(\Omega, L^p(M)) \subset L^p(M, W^{k,p}(\Omega)) \subset L^p(M, L^1(\Omega))$. To establish the identity $u = \tilde{u} + c$, we first note that for all $\phi \in \mathcal{C}^\infty(M) \subset Z^*$

$$\int_\Omega \langle\, \tilde{u} + c - u \,,\, \phi \,\rangle \;=\; \int_M \phi \cdot \Bigl(\operatorname{Vol}\Omega \cdot c - \int_\Omega (u - \tilde{u})\Bigr) \;=\; 0.$$

Next, for any $\phi \in \mathcal{C}^\infty(\Omega \times M)$ let

$$\phi_0 := \frac{1}{\operatorname{Vol}\Omega} \int_\Omega \phi \quad \in\; \mathcal{C}^\infty(M).$$

Then one finds $\psi\in\mathcal{C}^\infty_\nu(\Omega\times M)$ such that $\phi=\Delta_\Omega\psi+\phi_0$. (There exist unique solutions $\psi(\cdot,y)$ of the Neumann problem for $\phi(\cdot,y)-\phi_0(y)$, and these depend smoothly on $y\in M$.) So we find that for all $\phi\in\mathcal{C}^\infty(\Omega\times M)$, abbreviating $\Delta_\Omega=\Delta$

$$\begin{aligned}\int_\Omega\langle\, u-\tilde{u}-c\,,\,\phi\,\rangle &= \int_\Omega\langle\, u\,,\,\Delta\psi\,\rangle-\int_\Omega\langle\,\tilde{u}+c\,,\,\Delta\psi\,\rangle+\int_\Omega\langle\, u-\tilde{u}-c\,,\,\phi_0\,\rangle\\ &= \int_\Omega\langle\, f\,,\,\psi\,\rangle+\int_{\partial\Omega}\langle\, g\,,\,\psi\,\rangle-\lim_{i\to\infty}\int_\Omega\langle\, u^i\,,\,\Delta\psi\,\rangle\\ &= \lim_{i\to\infty}\left(\int_\Omega\langle\, f-\Delta u^i\,,\,\psi\,\rangle+\int_{\partial\Omega}\langle\, g-\tfrac{\partial u^i}{\partial\nu}\,,\,\psi\,\rangle\right)\;=\;0.\end{aligned}$$

This proves $u=\tilde{u}+c\in W^{k+1,p}(\Omega,Z)$ and the estimate for u^i yields in the limit

$$\begin{aligned}\|u\|_{W^{k+1,p}(\Omega,Z)} &\leq \|\tilde{u}\|_{W^{k+1,p}(\Omega,Z)}+(\operatorname{Vol}\Omega)^{\frac{1}{p}}\|c\|_{L^p(M)}\\ &\leq C\bigl(\|f\|_{W^{k-1,p}(\Omega,Z)}+\|g\|_{W^{k,p}(\Omega,Z)}+\|u\|_{L^p(\Omega,Z)}\bigr).\end{aligned}$$

This finishes the proof of (ii), and analogously of (i). □

Proof of lemma 5.7 (iii) and (iv) :
Let $u\in L^p(\Omega,Z)$ be as supposed in (iii) or (iv), where $Z=L^p(M)$ and thus $Z^*=L^{p^*}(M)$. Then we have $u\in L^p(\Omega\times M)$ and the task is to prove that $\mathrm{d}_\Omega u$ also is of class L^p on $\Omega\times M$. So we have to consider $\int_{\Omega\times M}u\cdot\mathrm{d}^*_\Omega\tau$ for $\tau\in\mathcal{C}^\infty_\delta(\Omega\times M,\mathrm{T}^*\Omega)$ (which are dense in $L^{p^*}(\Omega\times M,\mathrm{T}^*\Omega)$). In the case (iii) one finds for any such smooth family τ of 1-forms on Ω a smooth function $\psi\in\mathcal{C}^\infty_\delta(\Omega\times M)$ such that $\mathrm{d}^*_\Omega\tau=\Delta_\Omega\psi$. Then there is a constant C such that for all $y\in M$ (see proposition 3.5)

$$\|\psi(\cdot,y)\|_{W^{1,p^*}}\;\leq\;C\,\|\Delta_\Omega\psi(\cdot,y)\|_{(W^{1,p})^*}\;\leq\;C\,\|\tau(\cdot,y)\|_{p^*}.$$

In the case (iv) one similarly finds $\psi\in\mathcal{C}^\infty_\nu(\Omega\times M)$ such that $\mathrm{d}^*_\Omega\tau=\Delta_\Omega\psi$ and $\|\psi(\cdot,y)\|_{W^{1,p^*}}\leq C\,\|\tau(\cdot,y)\|_{p^*}$ for all $y\in M$ and some constant C. (Note that $\int_\Omega\mathrm{d}^*_\Omega\tau\equiv 0$ since τ vanishes on $\partial\Omega\times M$ and we have used e.g. [We, Theorems 2.2,2.3'].) In both cases we can thus estimate for all $\tau\in\mathcal{C}^\infty_\delta(\Omega\times M,\mathrm{T}^*\Omega)$

using the assumption

$$\left|\int_{\Omega\times M} u\cdot \mathrm{d}_\Omega^*\tau\right| = \left|\int_{\Omega\times M} u\cdot \Delta_\Omega\psi\right| \le c_u\left(\int_M \|\psi\|_{W^{1,p^*}(\Omega)}^{p^*}\right)^{\frac{1}{p^*}}$$
$$\le Cc_u\left(\int_M \|\tau\|_{L^{p^*}(\Omega)}^{p^*}\right)^{\frac{1}{p^*}} \le Cc_u\|\tau\|_{L^{p^*}(\Omega\times M)}.$$

Now in both cases the Riesz representation theorem (e.g. [Ad, Theorem 2.33]) asserts that $\int_{\Omega\times M} u\cdot \mathrm{d}_\Omega^*\tau = \int_{\Omega\times M} f\cdot\tau$ for all τ with some $f\in L^p(\Omega\times M)$. This proves the L^p-regularity of $\mathrm{d}_\Omega u$ and yields the estimate

$$\|\mathrm{d}_\Omega u\|_{L^p(\Omega\times M)} \le Cc_u.$$

In the case (iii), one can moreover deduce $u|_{\partial\Omega}=0$. Indeed, partial integration in the weak equation gives for all $\psi\in\mathcal{C}^\infty_\delta(\Omega\times M)$

$$\left|\int_{\partial\Omega\times M} u\cdot\tfrac{\partial\psi}{\partial\nu}\right| = \left|\int_{\Omega\times M} u\cdot\Delta_\Omega\psi \;-\int_{\Omega\times M}\langle\, \mathrm{d}_\Omega u\,,\, \mathrm{d}_\Omega\psi\,\rangle\right|$$
$$\le (c_u + \|\mathrm{d}_\Omega u\|_{L^p(\Omega\times M)})\|\psi\|_{W^{1,p^*}(\Omega\times M)}.$$

For any given $g\in\mathcal{C}^\infty(\partial\Omega\times M)$ one now finds $\psi\in\mathcal{C}^\infty(\Omega\times M)$ with $\psi|_{\partial\Omega}=0$ and $\frac{\partial\psi}{\partial\nu}=g$, and these can be chosen such that $\|\psi\|_{W^{1,p^*}}$ becomes arbitrarily small. Then one obtains $\int_{\partial\Omega\times M} u\,g = 0$ and thus $u|_{\partial\Omega}=0$. Thus in the case (iii) one finds a constant C' such that

$$\|u\|_{W^{1,p}(\Omega,Z)}^p = \int_M \|u\|_{W^{1,p}(\Omega)}^p \le C'\int_M \|\mathrm{d}_\Omega u\|_{L^p(\Omega)}^p = C'\|\mathrm{d}_\Omega u\|_{L^p(\Omega\times M)}^p,$$

which finishes the proof of (iii).

In the case (iv) with $\int_\Omega u = 0$ one also has a constant C' such that $\|u(\cdot,y)\|_{W^{1,p}(\Omega)} \le C'\|\mathrm{d}_\Omega u(\cdot,y)\|_{L^p(\Omega)}$ for all $y\in M$ and thus

$$\|u\|_{W^{1,p}(\Omega,Z)} \le C'\|\mathrm{d}_\Omega u\|_{L^p(\Omega\times M)} \le C'Cc_u.$$

In the general case (iv) one similarly has

$$\|u\|_{W^{1,p}(\Omega,Z)}^p = \|\mathrm{d}_\Omega u\|_{L^p(\Omega\times M)}^p + \|u\|_{L^p(\Omega\times M)}^p \le \big(Cc_u + \|u\|_{L^p(\Omega,Z)}\big)^p.$$

□

The proof of theorem 5.3 will moreover use the following quantitative version of the implicit function theorem. This is proven e.g. in [MS2, Proposition A.3.4] by a Newton-Picard method. (Here we only need the special case $x_0 = x_1 = 0$.)

Proposition 5.10 *Let X and Y be Banach spaces and let $U \subset Y$ be a neighbourhood of 0. Suppose that $f : \mathcal{U} \to X$ is continuously differentiable map such that $\mathrm{d}_0 f : Y \to X$ is bijective. Then choose constants $c \geq \|(\mathrm{d}_0 f)^{-1}\|$ and $\delta > 0$ such that $B_\delta(0) \subset \mathcal{U}$ and*

$$\|\mathrm{d}_y f - \mathrm{d}_0 f\| \leq \tfrac{1}{2c} \qquad \forall y \in B_\delta(0).$$

Now if $\|f(0)\| \leq \frac{\delta}{4c}$ then there exists a unique solution $y \in B_\delta(0)$ of $f(y) = 0$. Moreover, this solution satisfies

$$\|y\| \leq 2c\|f(0)\|.$$

Proof of theorem 5.3 :
For every $(s_0, 0) \in \Omega$ one finds a Banach manifold chart $\phi : V \to \mathcal{L}$ from a neighbourhood $V \subset Y$ of 0 to a neighbourhood of $\phi(0) = u_0(s_0, 0) =: x_0$. Choose a complex structure $J \in \operatorname{End} X$, then one obtains a Banach submanifold chart of $\mathcal{L} \subset X$ from a neighbourhood $\mathcal{W} \subset Y \times Y$ of zero to a ball $B_\varepsilon(x_0) \subset X$ around x_0,

$$\Theta : \begin{array}{ccc} \mathcal{W} & \xrightarrow{\sim} & B_\varepsilon(x_0) \\ (v_1, v_2) & \longmapsto & \phi(v_1) + J\mathrm{d}_{v_1}\phi(v_2). \end{array}$$

To see that this is indeed a diffeomorphism for sufficiently small $\mathcal{W}$ and $\varepsilon > 0$ note that $D := \mathrm{d}_{(0,0)}\Theta = \mathrm{d}_0\phi \oplus J\mathrm{d}_0\phi$. Here $\mathrm{d}_0\phi : Y \to \mathrm{T}_{x_0}\mathcal{L}$ is an isomorphism and so is the map $\mathrm{T}_{x_0}\mathcal{L} \times \mathrm{T}_{x_0}\mathcal{L} \to X$ given by the splitting $X = \mathrm{T}_{x_0}\mathcal{L} \oplus J\,\mathrm{T}_{x_0}\mathcal{L}$.

Now let the complex structure J vary in a sufficiently small neighbourhood of $J_0(s_0, 0) \in \operatorname{End} X$ such that $\|D^{-1}\| \leq c$ with a uniform (i.e. J-independent) constant c. Since ϕ is smooth one then also finds a uniform constant $\delta > 0$ such that $\|\mathrm{d}_v\Theta - D\| \leq \frac{1}{2c}$ for all $v = (v_1, v_2) \in B_\delta(0)$ and $B_\delta(0) \subset \mathcal{W}$. So one can apply the quantitative implicit function proposition 5.10 to the maps $f = \Theta - x$ for varying J and $x \in X$ if $\|x - x_0\| = \|f(0)\|$ is sufficiently small. This yields uniform constants $\varepsilon, \delta > 0$ and C such that $\Theta^{-1} : B_\varepsilon(x_0) \to B_\delta(0,0) \subset Y \times Y$ is uniquely defined for all complex structures J in the neighbourhood of $J_0(s_0, 0)$, and moreover

$$\|\Theta^{-1}(x)\|_{Z \times Z} \leq C\|x - x_0\|_X \qquad \forall x \in B_\varepsilon(x_0). \tag{5.2}$$

(Recall that Y is a closed subspace of the Banach space Z, so the norm on Y is induced by the norm on Z.) In particular, the uniform estimate (5.2) holds for all $J = J_{s,t}$ if $(s,t) \in U$ for a neighbourhood $U \subset \Omega$ of $(s_0, 0)$ and provided that $J \in \mathcal{C}^{k+1}(\Omega, \mathrm{End}\, X)$ satisfies the assumption $\|J - J_0\|_{\mathcal{C}^{k+1}} \leq \delta_2$ for sufficiently small $\delta_2 > 0$. Thus one obtains a $\mathcal{C}^{k+1}$-family of chart maps for $(s,t) \in \overline{U}$,

$$\Theta_{s,t} : Y \times Y \supset \mathcal{W}_{s,t} \xrightarrow{\sim} B_\varepsilon(x_0).$$

Next, choose U even smaller such that $u_0(s,t) \in B_{\frac{\varepsilon}{2}}(x_0)$ for all $(s,t) \in U$ and let $\delta_1 = \frac{\varepsilon}{2}$. Then every $u \in W^{k,q}(\Omega, X)$ that satisfies $\|u - u_0\|_{L^\infty(\Omega,X)} \leq \delta_1$ can be expressed in local coordinates,

$$u(s,t) = \Theta_{s,t}(v(s,t)) \qquad \forall (s,t) \in U,$$

where $v \in W^{k,q}(U, Z \times Z)$. This follows from the fact that the composition of the $\mathcal{C}^{k+1}$-map Θ^{-1} with a $W^{k,q}$-map u is again $W^{k,q}$-regular if $kq > 2$ (see e.g. [We, Lemma B.8]). Moreover, v actually takes values in $\mathcal{W} \subset Y \times Y$. Integration of (5.2) together with the fact that all derivatives of Θ^{-1} up to order k are bounded yields the estimate

$$\|v\|_{W^{k,q}(U,Z\times Z)} \;\leq\; C\|u - u_0\|_{W^{k,q}(U,X)} \;\leq\; C\delta_1. \tag{5.3}$$

Here and in the following C denotes any constant that is independent of the specific choices of J and u in the fixed neighbourhoods of J_0 and u_0.

In these coordinates, the boundary value problem (5.1) now becomes

$$\begin{cases} \partial_s v + I\partial_t v = f, \\ v_2(s,0) = 0 \quad \forall s \in \mathbb{R}. \end{cases} \tag{5.4}$$

with $v = (v_1, v_2)$ and

$$\begin{aligned} f &= (\mathrm{d}_v\Theta)^{-1}\big(G - \partial_s\Theta(v) - J\partial_t\Theta(v)\big) &&\in\; W^{k,q}(U, Y \times Y), \\ I &= (\mathrm{d}_v\Theta)^{-1} J \mathrm{d}_v\Theta &&\in\; W^{k,q}(U, \mathrm{End}(Y \times Y)). \end{aligned}$$

Note the following difficulty: The complex structure I now explicitly depends on the solution v of the equation (5.4) and thus is only $W^{k,q}$-regular. This cannot be avoided when straightening out the Lagrangian boundary condition. However, one obtains one more simplification of the boundary value problem: Θ was constructed such that one obtains the standard complex structure along $\mathcal{L}$. Indeed, for all $(s,0) \in U$ using that $J^2 = -\mathbb{1}$

$$\begin{aligned} I(s,0) &= (\mathrm{d}_{(v_1,0)}\Theta)^{-1} J \mathrm{d}_{(v_1,0)}\Theta = \big(\mathrm{d}_{v_1}\phi \oplus J\mathrm{d}_{v_1}\phi\big)^{-1} J\big(\mathrm{d}_{v_1}\phi \oplus J\mathrm{d}_{v_1}\phi\big) \\ &= \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} =: I_0. \end{aligned}$$

Moreover, one has the following estimates on U:

$$\begin{aligned} \|I\|_{W^{k,q}} &\leq C, \\ \|f\|_{W^{k,q}} &\leq C\big(\|G\|_{W^{k,q}} + \|u - u_0\|_{W^{k,q}}\big). \end{aligned}$$

So for every boundary point $(s_0,0) \in \Omega \cap \partial\mathbb{H}$ we have rewritten the boundary value problem (5.1) over some neighbourhood $U \subset \Omega$. Now for the compact set $K \subset \Omega$ one finds a covering $K \subset V \cup \bigcup_{i=1}^N U_i$ by finitely many such neighbourhoods U_i at the boundary and a compact domain $V \subset \Omega \setminus \partial\Omega$ away from the boundary. Note that the U_i can be replaced by interior domains $\tilde{U}_i$ (that intersect ∂U_i only on $\partial\mathbb{H}$) that together with V still cover K. We will establish the regularity and estimate for u on all domains $\tilde{U}_i$ near the boundary and on the remaining domain V separately. So firstly consider a domain U_i near the boundary and drop the subscript i. After possibly replacing U by a slightly smaller domain one can assume that U is a manifold with smooth boundary and still $\tilde{U} \cap \partial U \subset \partial\mathbb{H}$. The task is now to prove the regularity and estimate for $u = \Theta \circ v$ on $\tilde{U}$ from (5.4).

Since $\Theta_{s,t} : Y \times Y \to X$ are smooth maps depending smoothly on $(s,t) \in U$, it suffices to prove that $v \in W^{k+1,p}(\tilde{U}, Z \times Z)$ with the according estimate. (One already knows that v takes values – almost everywhere – in $Y \times Y$, so one automatically also obtains $v \in W^{k+1,p}(\tilde{U}, Y \times Y)$.) For that purpose fix a cutoff function $h \in \mathcal{C}^\infty(\mathbb{H}, [0,1])$ with $h \equiv 1$ on $\tilde{U}$ and $h \equiv 0$ on $\mathbb{H} \setminus U$. Moreover, this function can be chosen such that $\partial_t h|_{t=0} = 0$. Note that $h \equiv 0$ on $\partial U \setminus \partial\mathbb{H}$, so hv_2 satisfies the Dirichlet boundary condition on ∂U. Indeed, we will see that $hv_2 \in W^{k,p}(U, Z)$ solves a weak Dirichlet problem.

In the following Y^* denotes the dual space of Y and $\langle \cdot, \cdot \rangle$ denotes the pairing between Y and Y^* as well as the pairing between $Y \times Y$ and $Y^* \times Y^*$. Let $I^* \in W^{k,q}(\Omega, \mathrm{End}(Y^* \times Y^*))$ be the pointwise dual operator of I. Then one has $\Delta = -(\partial_s + \partial_t I^*)(\partial_s - I^*\partial_t) + (\partial_t I^*)\partial_s - (\partial_s I^*)\partial_t$ and thus for all $\phi \in \mathcal{C}^\infty(\Omega, Y^* \times Y^*)$

$$\begin{aligned} h\Delta\phi = &-(\partial_s + \partial_t I^*)(\partial_s - I^*\partial_t)(h\phi) - (\Delta h)\phi + 2(\partial_s h)\partial_s\phi + 2(\partial_t h)\partial_t\phi \\ &+ (\partial_t I^*)\partial_s(h\phi) - (\partial_s I^*)\partial_t(h\phi). \end{aligned}$$

Hence partial integration (using smooth approximations of v, f, and I) yields due to (5.4)

$$\begin{aligned} &\int_U \langle\, hv\,,\, \Delta\phi \,\rangle \\ &= \int_U \langle\, \partial_s v + I\partial_t v\,,\, (\partial_s - I^*\partial_t)(h\phi) \,\rangle \\ &\quad - \int_U \langle\, (\Delta h)v + 2(\partial_s h)\partial_s v + 2(\partial_t h)\partial_t v + h(\partial_t I)\partial_s v - h(\partial_s I)\partial_t v\,,\, \phi \,\rangle \\ &\quad + \int_{\partial U\cap\partial\mathbb{H}} \langle\, Iv\,,\, (\partial_s - I^*\partial_t)(h\phi) \,\rangle + \langle\, h(\partial_s I)v - 2(\partial_t h)v\,,\, \phi \,\rangle \\ &= \int_U \langle\, h(-\partial_s f + I\partial_t f + (\partial_t I)f - (\partial_t I)\partial_s v + (\partial_s I)\partial_t v) \\ &\qquad\qquad - (\Delta h)v - 2(\partial_s h)\partial_s v - 2(\partial_t h)\partial_t v\,,\, \phi \,\rangle \\ &\quad + \int_{\partial U\cap\partial\mathbb{H}} \langle\, h\cdot If\,,\, \phi \,\rangle + \int_{\partial U\cap\partial\mathbb{H}} \langle\, v\,,\, \partial_t(h\phi) \,\rangle + \langle\, Iv\,,\, \partial_s(h\phi) \,\rangle \\ &= \int_U \langle\, F\,,\, \phi \,\rangle + \int_{\partial U} \langle\, H\,,\, \phi \,\rangle + \int_{\partial U\cap\partial\mathbb{H}} \langle\, v_1\,,\, \partial_t(h\phi_1) + \partial_s(h\phi_2) \,\rangle. \end{aligned} \tag{5.5}$$

Here we used the notation $\phi = (\phi_1, \phi_2)$, the boundary condition $v_2|_{t=0} = 0$, and the fact that $I|_{t=0} \equiv I_0$. One reads off $F = (F_1, F_2) \in W^{k-1,p}(U, Y \times Y)$, $H = (H_1, H_2) \in W^{k,p}(U, Y \times Y)$, and that for some finite constants C

$$\begin{aligned} \|F\|_{W^{k-1,p}} + \|H\|_{W^{k,p}} &\le C\big(\|f\|_{W^{k,q}} + \|I\|_{W^{k,q}}\|f\|_{W^{k,q}} + \|I\|_{W^{k,q}}\|v\|_{W^{k,q}}\big) \\ &\le C\big(\|G\|_{W^{k,q}} + \|u - u_0\|_{W^{k,q}}\big). \end{aligned}$$

We point out that the crucial terms here are $(\partial_s I)\partial_t v$ and $(\partial_t I)\partial_s v$. In the case $k \ge 3$ the estimate holds with $q = p$ due to the Sobolev embedding $W^{k-1,p} \cdot W^{k-1,p} \hookrightarrow W^{k-1,p}$. In the case $k = 1$ one only has $L^{2p} \cdot L^{2p} \hookrightarrow L^p$ and hence one needs $q = 2p$ in the above estimate. In the case $k = 2$ the Sobolev embedding $W^{1,p} \cdot W^{1,p} \hookrightarrow W^{1,p}$ requires $p > 2$. Let us make this assumption for the moment and finish the proof of this theorem under the additional assumption $p > 2$ in case $k = 2$. Then after that we will show how an iteration of the $k = 1$-case of the theorem will give the required $W^{1,p}$-regularity of $(\partial_s I)\partial_t v$ and $(\partial_t I)\partial_s v$ also in case $1 < p \leq 2$. (Note that this would follow from $W^{2,p'}$-regularity of u for any $p' > 2$.)

Now in order to obtain a weak Laplace equation for v_2 we test the weak equation (5.5) with $\phi = (\phi_1, \phi_2) = (0, \pi \circ \psi)$ for $\psi \in \mathcal{C}^\infty_\delta(U, Z^*)$ and where $\pi : Z^* \to Y^*$ is the canonical embedding. In that case, both boundary terms vanish and one obtains for all $\psi \in \mathcal{C}^\infty_\delta(U, Z^*)$

$$\int_U \langle\, hv_2\,,\, \Delta\psi\,\rangle = \int_U \langle\, F_2\,,\, \psi\,\rangle.$$

By lemma 5.7 (i) this weak equation for $hv_2 \in W^{k,p}(U, Z)$ now implies that $hv_2 \in W^{k+1,p}(U, Z)$ and thus $v_2 \in W^{k+1,p}(\tilde{U}, Z)$. Moreover, one obtains the estimate

$$\begin{aligned}\|v_2\|_{W^{k+1,p}(\tilde{U},Z)} \leq \|hv_2\|_{W^{k+1,p}(U,Z)} &\leq C\|F_2\|_{W^{k-1,p}(U,Z)} \\ &\leq C\big(\|G\|_{W^{k,q}(\Omega,X)} + \|u - u_0\|_{W^{k,q}(\Omega,X)}\big).\end{aligned}$$

To obtain a weak Laplace equation for v_1 we have to test the weak equation (5.5) with $\phi = (\phi_1, \phi_2) = (\pi \circ \psi, 0)$, where $\psi \in \mathcal{C}^\infty(U, Z^*)$ such that $\partial_t\psi|_{t=0} = 0$ in order to make the second boundary term vanish. One obtains for all $\psi \in \mathcal{C}^\infty_\nu(U, Z^*)$

$$\int_\Omega \langle\, hv_1\,,\, \Delta\psi\,\rangle = \int_\Omega \langle\, F_1\,,\, \psi\,\rangle + \int_{\partial\Omega} \langle\, H_1\,,\, \psi\,\rangle.$$

So we have established a weak Laplace equation with Neumann boundary condition for hv_1. Now lemma 5.7 (ii) implies that $hv_1 \in W^{k+1,p}(U, Z)$, hence $v_1 \in W^{k+1,p}(\tilde{U}, Z)$. Moreover, one obtains the estimate

$$\begin{aligned}\|v_1\|_{W^{k+1,p}(\tilde{U},Z)} &\leq \|hv\|_{W^{k+1,p}(U,Z)} \\ &\leq C\big(\|F_1\|_{W^{k-1,p}(U,Z)} + \|H_1\|_{W^{k,p}(U,Z)} + \|hv_1\|_{W^{k,p}(U,Z)}\big) \\ &\leq C\big(\|G\|_{W^{k,q}(\Omega,X)} + \|u - u_0\|_{W^{k,q}(\Omega,X)}\big).\end{aligned}$$

This now provides the regularity and the estimate for $u = \Theta \circ v$ on $\tilde{U}$ as follows. We have established that $v : \tilde{U} \to Z \times Z$ is a $W^{k+1,p}$-map that takes values in $\mathcal{W} \subset Y \times Y$. All derivatives of $\Theta : \Omega \times \mathcal{W} \to X$ are uniformly bounded on Ω. Hence $u \in W^{k+1,p}(\tilde{U}, X)$ and

$$\begin{aligned}\|u - u_0\|_{W^{k+1,p}(\tilde{U},X)} &\leq C\bigl(1 + \|v\|_{W^{k+1,p}(\tilde{U},X)}\bigr) \\ &\leq C\bigl(1 + \|G\|_{W^{k,q}(\Omega,X)} + \|u - u_0\|_{W^{k,q}(\Omega,X)}\bigr).\end{aligned}$$

For the regularity of u on the domain $V \subset \Omega \setminus \partial\Omega$ away from the boundary one does not need any special coordinates. As for U, one replaces Ω by a possibly smaller domain with smooth boundary. Moreover, one chooses a cutoff function $h \in \mathcal{C}^\infty(\mathbb{H}, [0,1])$ such that $h|_V \equiv 1$ and that vanishes outside of $\Omega \subset \mathbb{H}$ and in a neighbourhood of $\partial\Omega$. Then in the same way as for (5.5) one obtains a weak Dirichlet equation. For all $\phi \in \mathcal{C}^\infty_\delta(\Omega, X^*)$

$$\begin{aligned}\int_\Omega \langle\, hu \,,\, \Delta\phi \,\rangle = \int_\Omega \langle\, h\bigl(-\partial_s G + J\partial_t G + (\partial_t J)G - (\partial_t J)\partial_s u + (\partial_s J)\partial_t u\bigr) \\ - (\Delta h)u - 2(\partial_s h)\partial_s u - 2(\partial_t h)\partial_t u \,,\, \phi \,\rangle.\end{aligned}$$

Note that $X \cong Y \times Y \subset Z \times Z$ also is bounded isomorphic to a closed subspace of an L^p-space. So by lemma 5.7 this weak equation implies that $hu \in W^{k+1,p}(\Omega, X)$, and thus $u \in W^{k+1,p}(V, X)$ with the estimate

$$\|u\|_{W^{k+1,p}(V,X)} \leq C\bigl(\|G\|_{W^{k,q}(\Omega,X)} + \|u - u_0\|_{W^{k,q}(\Omega,X)}\bigr).$$

Thus we have proven the regularity and estimates of u on all parts of the finite covering $K \subset V \cup \bigcup_{i=1}^N U_i$. This finishes the proof of the theorem under the additional assumption $p > 2$ in case $k = 2$.

Finally, let $u \in W^{2,p}(\Omega, X)$ and $G \in W^{2,p}(\Omega, X)$ be as supposed for $1 < p \leq 2$. Then the task is to establish $W^{2,p'}$-regularity and -estimates for some $p' > 2$. This follows from the following iteration.

One starts with $W^{2,p_0}(\Omega_0)$-regularity and -estimates for $p_0 = p \in (1,2)$ on $\Omega_0 = \Omega$. (In case $p = 2$ one chooses a smaller value for p.) Now as long as $p_i \leq \frac{4p}{2+p} < 2$ and $\Omega_i \subset \mathbb{H}$ is compact one has the Sobolev embeddings $W^{2,p}(\Omega) \hookrightarrow W^{1,\frac{p_i}{2-p_i}}(\Omega_i)$ and $W^{2,p_i}(\Omega_i) \hookrightarrow L^{\frac{2p_i}{2-p_i}}(\Omega_i)$. So choose a compact submanifold $\Omega_{i+1} \subset \mathbb{H}$ such that $K \subset \operatorname{int}\Omega_{i+1}$ and $\Omega_{i+1} \subset \operatorname{int}\Omega_i$. Then the theorem in case $k = 1$ with p replaced by $\frac{p_i}{2-p_i}$ gives regularity and estimates in $W^{2,p_{i+1}}(\Omega_{i+1})$ for $p_{i+1} = \frac{p_i}{2-p_i}$. One sees that the sequence (p_i) grows at a rate of at least $\frac{1}{2-p} > 1$ until it reaches $p_N \geq \frac{4p}{2+p}$ after finitely many steps. A further step of the iteration with $p_N = \frac{4p}{2+p}$ then gives $W^{2,p_{N+1}}$-regularity and -estimates with $p_{N+1} = \frac{4p}{2-p} > 2$ on $\Omega_{N+1} := K$. This is exactly the regularity for u that still was to be established. □

Proof of theorem 5.4 :
The Banach manifold charts near the path $x : \mathbb{R} \to \mathcal{L}$ give rise to a smooth path of isomorphisms $\phi_s : Y \xrightarrow{\sim} \mathrm{T}_{x(s)}\mathcal{L}$ for all $s \in \mathbb{R}$. Together with the family of complex structures $J \in \mathcal{C}^\infty(\overline{\Omega}, \mathrm{End}\, X)$ these give rise to a smooth family $\Theta \in \mathcal{C}^\infty(\overline{\Omega}, \mathrm{Hom}(Y \times Y, X))$ of bounded isomorphisms

$$\Theta_{s,t} : \begin{array}{ccc} Y \times Y & \xrightarrow{\sim} & X \\ (z_1, z_2) & \longmapsto & \phi_s(z_1) + J_{s,t}\phi_s(z_2). \end{array}$$

The inverses of the dual operators of $\Theta_{s,t}$ give a smooth family of bounded isomorphisms $\Theta' \in \mathcal{C}^\infty(\overline{\Omega}, \mathrm{Hom}(Y^* \times Y^*, X^*))$,

$$\Theta'_{s,t} := (\Theta^*_{s,t})^{-1} : \; Y^* \times Y^* \;\xrightarrow{\sim}\; X^*.$$

One checks that for all $(s,t) \in \overline{\Omega}$

$$\Theta^{-1}_{s,t} J_{s,t} \Theta_{s,t} \;=\; \left(\begin{smallmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{smallmatrix}\right) \;=:\; I_0 \;\in\; \mathrm{End}(Y \times Y).$$

Next, after possibly replacing Ω by a slightly smaller domain that still contains K in its interior, one can assume that Ω is a manifold with smooth boundary. Then fix a cutoff function $h \in \mathcal{C}^\infty(\mathbb{H}, [0,1])$ such that $h|_K \equiv 1$ and $\mathrm{supp}\, h \subset \Omega$, i.e. $h \equiv 0$ near $\partial\Omega \setminus \partial\mathbb{H}$. Now let $u \in L^p(\Omega, X)$ be given as in the theorem and express it in the above coordinates as $u = \Theta \circ v$, where $v \in L^p(\Omega, Y \times Y)$. We will show that v satisfies a weak Laplace equation. For all $\phi \in \mathcal{C}^\infty(\Omega, Y^* \times Y^*)$ we introduce $\psi := \Theta'((\partial_s + I_0\partial_t)\phi) \in \mathcal{C}^\infty(\Omega, X^*)$. Then

$$\begin{aligned} \partial_s(h\psi) + \partial_t(J^* h\psi) &= h\Theta'((\partial_s + I_0^*\partial_t)(\partial_s + I_0\partial_t)\phi) \\ &\quad + (\partial_s h)\psi + \partial_t(hJ^*)\psi + h(\partial_s\Theta' + J^*\partial_t\Theta'){\Theta'}^{-1}(\psi). \end{aligned}$$

So if $\psi(s,0) \in (J(s,0)\mathrm{T}_{x(s)}\mathcal{L})^\perp$ for all $(s,0) \in \partial\Omega \cap \partial\mathbb{H}$, then $h\psi$ is an admissible test function in the given weak estimate for u in the theorem and we obtain, denoting all constants by C and using $\Theta^*\Theta' = \mathrm{id}$,

$$\begin{aligned} \left| \int_\Omega \langle\, hv \,,\, \Delta\phi \,\rangle \right| &= \left| \int_\Omega \langle\, \Theta(v) \,,\, h\Theta'\big((-\partial_s + I_0\partial_t)(\partial_s + I_0\partial_t)\phi\big) \,\rangle \right| \\ &= \left| \int_\Omega \langle\, u \,,\, \partial_s(h\psi) + \partial_t(J^* h\psi) \,\rangle \right| \\ &\quad + \left| \int_\Omega \langle\, u \,,\, (\partial_s h)\psi + (\partial_t h J^*)\psi + h(\partial_s\Theta' + J^*\partial_t\Theta'){\Theta'}^{-1}(\psi) \,\rangle \right| \\ &\le \big(c_u + C\|u\|_{L^p(\Omega,X)}\big)\|\psi\|_{L^{p^*}(\Omega,X^*)} \\ &\le C\big(c_u + \|u\|_{L^p(\Omega,X)}\big)\|\phi\|_{W^{1,p^*}(\Omega,Y^*\times Y^*)}. \end{aligned}$$

Here we used the fact that J^* and Θ' as well as their first derivatives and inverses are bounded linear operators between $Y^*\times Y^*$ and X^*. This inequality then holds for all $\phi=(\phi_1,\phi_2)$ with $\phi_1\in\mathcal{C}^\infty_\nu(\Omega,Y^*)$ and $\phi_2\in\mathcal{C}^\infty_\delta(\Omega,Y^*)$ since in that case ψ is admissible. Indeed, $\psi|_{t=0}=\Theta'(\partial_s\phi_1-\partial_t\phi_2,0)\in(J\mathrm{T}_x\mathcal{L})^\perp$ due to $\Theta'(Y^*\times\{0\})=\Theta'(I_0(Y\times\{0\}))^\perp=(J\mathrm{T}_x\mathcal{L})^\perp$.

Recall that $Y\subset Z$ is a closed subset of the Banach space Z with the induced norm. So one also has $v\in L^p(\Omega,Z\times Z)$. Let $\pi:Z^*\to Y^*$ be the natural embedding, then above inequality holds with $\phi=(\pi\circ\psi_1,\pi\circ\psi_2)$ for all $\psi_1\in\mathcal{C}^\infty_\nu(\Omega,Z^*)$ and $\psi_2\in\mathcal{C}^\infty_\delta(\Omega,Z^*)$. Since $\|\pi\circ\psi_i\|_{Y^*}\leq\|\psi_i\|_{Z^*}$ one then obtains for all such $\Psi=(\psi_1,\psi_2)\in\mathcal{C}^\infty(\Omega,Z^*\times Z^*)$

$$\left|\int_\Omega\langle\, hv\,,\,\Delta\Psi\,\rangle\right|\leq C\bigl(c_u+\|u\|_{L^p(\Omega,X)}\bigr)\|\Psi\|_{W^{1,p^*}(\Omega,Z^*\times Z^*)}.$$

Now lemma 5.7 (iii) and (iv) asserts the $W^{1,p}$-regularity of hv and hence one obtains $v\in W^{1,p}(\Omega,Z\times Z)$ with the estimate

$$\|v\|_{W^{1,p}(K,Z\times Z)}\;\leq\;\|hv\|_{W^{1,p}(\Omega,Z\times Z)}\;\leq\;C\bigl(c_u+\|u\|_{L^p(\Omega,X)}+\|v\|_{L^p(\Omega,Z\times Z)}\bigr).$$

For the first factor of $Z\times Z$, this follows from lemma 5.7 (iv), in the second factor one uses (iii). Since it was already known that v takes values in $Y\times Y$ (almost everywhere), one in fact has $v\in W^{1,p}(\Omega,Y\times Y)$ with the same estimate as above. Finally, recall that $u=\Theta\circ v$ and use the fact that all derivatives of Θ and Θ^{-1} are bounded to obtain $u\in W^{1,p}(K,X)$ with the claimed estimate (using again [We, Lemma B.8])

$$\|u\|_{W^{1,p}(K,X)}\;\leq\;C\|v\|_{W^{1,p}(K,Z\times Z)}\;\leq\;C\bigl(c_u+\|u\|_{L^p(\Omega,X)}\bigr).$$

□

Chapter 6

Regularity and compactness

This chapter is devoted to the proofs of theorems A and B, restated below as theorems 6.1 and 6.2. Let (X, τ) be a 4-manifold with boundary space-time splitting. So X is oriented and

$$X = \bigcup_{k \in \mathbb{N}} X_k,$$

where all X_k are compact submanifolds and deformation retracts of X such that $X_k \subset \operatorname{int} X_{k+1}$ for all $k \in \mathbb{N}$. Here the interior of a submanifold $X' \subset X$ is to be understood with respect to the relative topology, i.e. we define $\operatorname{int} X' = X \setminus \operatorname{cl}(X \setminus X')$. Moreover,

$$\partial X = \bigcup_{i=1}^{n} \tau_i(\mathcal{S}_i \times \Sigma_i),$$

where each Σ_i is a Riemann surface, each $\mathcal{S}_i$ is either an open interval in $\mathbb{R}$ or is equal to $S^1 = \mathbb{R}/\mathbb{Z}$, and the embeddings $\tau_i : \mathcal{S}_i \times \Sigma_i \to X$ have disjoint images. We then consider the trivial G-bundle over X, where G is a compact Lie group with Lie algebra $\mathfrak{g}$. For $i = 1, \ldots, n$ let $\mathcal{L}_i \subset \mathcal{A}^{0,p}(\Sigma_i)$ be a Lagrangian submanifold as described in chapter 4, i.e. suppose that

$$\mathcal{L}_i \subset \mathcal{A}^{0,p}_{\text{flat}}(\Sigma_i) \qquad \text{and} \qquad \mathcal{G}^{1,p}(\Sigma_i)^* \mathcal{L}_i \subset \mathcal{L}_i.$$

Furthermore, let X be equipped with a metric g that is compatible with the space-time splitting τ. This means that for each $i = 1, \ldots, n$ the map

$\mathcal{S}_i \times [0,\infty) \times \Sigma_i \to X$, $(s,t,z) \mapsto \gamma_{(s,z)}(t)$ given by the normal geodesics $\gamma_{(s,z)}$ starting at $\gamma_{(s,z)}(0) = \tau_i(s,z)$ restricts to an embedding $\bar\tau_i : \mathcal{U}_i \times \Sigma_i \hookrightarrow X$ for some neighbourhood $\mathcal{U}_i \subset \mathcal{S}_i \times [0,\infty)$ of $\mathcal{S}_i \times \{0\}$. Now consider the following boundary value problem for connections $A \in \mathcal{A}^{1,p}_{\mathrm{loc}}(X)$,

$$\begin{cases} *F_A + F_A = 0, \\ \tau_i^* A|_{\{s\}\times\Sigma_i} \in \mathcal{L}_i \quad \forall s \in \mathcal{S}_i,\ i = 1,\dots,n. \end{cases} \tag{6.1}$$

The anti-self-duality equation is welldefined for $A \in \mathcal{A}^{1,p}_{\mathrm{loc}}(X)$ with any $p \geq 1$, but in order to be able to state the boundary condition correctly we have to assume $p > 2$. Then the trace theorem for Sobolev spaces (e.g. [Ad, Theorem 6.2]) ensures that $\tau_i^* A|_{\{s\}\times\Sigma_i} \in \mathcal{A}^{0,p}(\Sigma_i)$ for all $s \in \mathcal{S}_i$.

The aim of this chapter is to prove the following two theorems.

Theorem 6.1 *Let $p > 2$. Then for every solution $A \in \mathcal{A}^{1,p}_{\mathrm{loc}}(X)$ of (6.1) there exists a gauge transformation $u \in \mathcal{G}^{2,p}_{\mathrm{loc}}(X)$ such that $u^*A \in \mathcal{A}(X)$ is a smooth solution.*

Theorem 6.2 *Let $p > 2$ and let g^ν be a sequence of metrics compatible with τ that uniformly converges with all derivatives to a smooth metric. Suppose that $A^\nu \in \mathcal{A}^{1,p}_{\mathrm{loc}}(X)$ is a sequence of solutions of (6.1) with respect to the metrics g^ν such that for every compact subset $K \subset X$ there is a uniform bound on $\|F_{A^\nu}\|_{L^p(K)}$.*

Then there exists a subsequence (again denoted A^ν) and a sequence of gauge transformations $u^\nu \in \mathcal{G}^{2,p}_{\mathrm{loc}}(X)$ such that ${u^\nu}^* A^\nu$ converges uniformly with all derivatives on every compact set to a connection $A \in \mathcal{A}(X)$.

Both theorems are dealing with the noncompact base manifold X. However, we shall use an extension argument by Donaldson and Kronheimer [DK, Lemma 4.4.5] to reduce the problem to compact base manifolds. For the following special version of this argument a detailed proof can be found in [We, Propositions 8.6,10.8]. At this point, the assumption that the exhausting compact submanifolds X_k are deformation retracts of X comes in crucially. It ensures that every gauge transformation on X_k can be extended to X, which is a central point in the argument of Donaldson and Kronheimer that proves the following proposition.

Proposition 6.3 *Let the* 4*-manifold* $\tilde{M} = \bigcup_{k\in\mathbb{N}} M_k$ *be exhausted by compact submanifolds* $M_k \subset \operatorname{int} M_{k+1}$ *that are deformation retracts of* $\tilde{M}$*, and let* $p > 2$*.*

(i) Let $A \in \mathcal{A}^{1,p}_{\mathrm{loc}}(\tilde{M})$ *and suppose that for each* $k \in \mathbb{N}$ *there exists a gauge transformation* $u_k \in \mathcal{G}^{2,p}(M_k)$ *such that* $u_k^* A|_{M_k}$ *is smooth. Then there exists a gauge transformation* $u \in \mathcal{G}^{2,p}_{\mathrm{loc}}(\tilde{M})$ *such that* u^*A *is smooth.*

(ii) Let a sequence of connections $(A^\nu)_{\nu\in\mathbb{N}} \subset \mathcal{A}^{1,p}_{\mathrm{loc}}(\tilde{M})$ *be given and suppose that the following holds:*

For every $k \in \mathbb{N}$ *and every subsequence of* $(A^\nu)_{\nu\in\mathbb{N}}$ *there exist a further subsequence* $(\nu_{k,i})_{i\in\mathbb{N}}$ *and gauge transformations* $u^{k,i} \in \mathcal{G}^{2,p}(M_k)$ *such that*

$$\sup_{i\in\mathbb{N}} \left\| u^{k,i\,*} A^{\nu_{k,i}} \right\|_{W^{\ell,p}(M_k)} < \infty \qquad \forall \ell \in \mathbb{N}.$$

Then there exists a subsequence $(\nu_i)_{i\in\mathbb{N}}$ *and a sequence of gauge transformations* $u^i \in \mathcal{G}^{2,p}_{\mathrm{loc}}(\tilde{M})$ *such that*

$$\sup_{i\in\mathbb{N}} \left\| u^{i\,*} A^{\nu_i} \right\|_{W^{\ell,p}(M_k)} < \infty \qquad \forall k \in \mathbb{N}, \ell \in \mathbb{N}.$$

So in order to prove theorem 6.1 it suffices to find smoothing gauge transformations on the compact submanifolds X_k in view of proposition 6.3 (i). For that purpose we shall use the so-called local slice theorem. The following version is proven e.g. in [We, Theorem 9.1]. Note that we are dealing with trivial bundles, so we will be using the product connection as reference connection in the definition of the Sobolev norms of connections.

Proposition 6.4 (Local Slice Theorem)
Let M *be a compact* 4*-manifold, let* $p > 2$*, and let* $q > 4$ *be such that* $\frac{1}{q} > \frac{1}{p} - \frac{1}{4}$ *(or* $q = \infty$ *in case* $p > 4$*). Fix* $\hat{A} \in \mathcal{A}^{1,p}(M)$ *and let a constant* $c_0 > 0$ *be given. Then there exist constants* $\varepsilon > 0$ *and* C_{CG} *such that the following holds. For every* $A \in \mathcal{A}^{1,p}(M)$ *with*

$$\|A - \hat{A}\|_q \le \varepsilon \qquad \textit{and} \qquad \|A - \hat{A}\|_{W^{1,p}} \le c_0$$

there exists a gauge transformation $u \in \mathcal{G}^{2,p}(M)$ *such that*

$$\begin{cases} \mathrm{d}^*_{\hat{A}}(u^*A - \hat{A}) = 0, \\ *(u^*A - \hat{A})|_{\partial M} = 0, \end{cases} \qquad \textit{and} \qquad \begin{aligned} \|u^*A - \hat{A}\|_q &\le C_{CG}\|A - \hat{A}\|_q, \\ \|u^*A - \hat{A}\|_{W^{1,p}} &\le C_{CG}\|A - \hat{A}\|_{W^{1,p}}. \end{aligned}$$

Remark 6.5

(i) If the boundary value problem in proposition 6.4 is satisfied one says that u^*A is in Coulomb gauge relative to $\hat{A}$. This is equivalent to $v^*\hat{A}$ being in Coulomb gauge relative to A for $v = u^{-1}$, i.e. the boundary value problem can be replaced by

$$\begin{cases} \mathrm{d}_A^*(v^*\hat{A} - A) = 0, \\ *(v^*\hat{A} - A)|_{\partial M} = 0. \end{cases}$$

(ii) The assumptions in proposition 6.4 on p and q guarantee that one has a compact Sobolev embedding

$$W^{1,p}(M) \hookrightarrow L^q(M).$$

(iii) One can find uniform constants for varying metrics in the following sense. Fix a metric g on M. Then there exist constants $\varepsilon, \delta > 0$, and C_{CG} such that the assertion of proposition 6.4 holds for all metrics g' with $\|g - g'\|_{\mathcal{C}^1} \leq \delta$.

In the following we shall briefly outline the proof of theorem 6.1. Given a solution $A \in \mathcal{A}^{1,p}_{\mathrm{loc}}(X)$ of (6.1) one fixes $k \in \mathbb{N}$ and proves the assumption of proposition 6.3 (i) as follows. One finds some sufficiently large compact submanifold $M \subset X$ with $X_k \subset M$. Then one chooses a smooth connection $A_0 \in \mathcal{A}(M)$ sufficiently $W^{1,p}$-close to A and applies the local slice theorem with the reference connection $\hat{A} = A$ to find a gauge tranformation that puts A_0 into relative Coulomb gauge with respect to A. This is equivalent to finding a gauge transformation that puts A into relative Coulomb gauge with respect to A_0. We denote this gauge transformed connection again by $A \in \mathcal{A}^{1,p}(M)$. It satisfies the following boundary value problem:

$$\begin{cases} \mathrm{d}_{A_0}^*(A - A_0) = 0, \\ *F_A + F_A = 0, \\ *(A - A_0)|_{\partial M} = 0, \\ \tau_i^* A|_{\{s\}\times\Sigma_i} \in \mathcal{L}_i \quad \forall s \in \mathcal{S}_i,\ i = 1, \dots, n. \end{cases} \tag{6.2}$$

More precisely, the Lagrangian boundary condition only holds for those $s \in \mathcal{S}_i$ and $i \in \{1, \dots n\}$ for which $\tau_i(\{s\} \times \Sigma_i)$ is entirely contained in ∂M.

If M was chosen large enough, then the regularity theorem 6.8 below will assert the smoothness of $\tilde{A}$ on X_k.

The proof outline of the proof of theorem 6.2 goes along similar lines. We will use proposition 6.3 (ii) to reduce the problem to compact base manifolds. On these, we shall use the following weak Uhlenbeck compactness theorem (see [U1], [We, Theorem 8.1]) to find a subsequence of gauge equivalent connections that converges $W^{1,p}$-weakly.

Proposition 6.6 (Weak Uhlenbeck Compactness)
Let M be a compact 4-manifold and let $p > 2$. Suppose that the sequence of connections $A^\nu \in \mathcal{A}^{1,p}(M)$ is such that $\|F_{A^\nu}\|_p$ is uniformly bounded. Then there exists a subsequence (again denoted $(A^\nu)_{\nu\in\mathbb{N}}$) and a sequence $u^\nu \in \mathcal{G}^{2,p}(M)$ of gauge transformations such that ${u^\nu}^* A^\nu$ weakly converges in $\mathcal{A}^{1,p}(M)$.

The limit A_0 of the convergent subsequence then serves as reference connection $\hat{A}$ in the local slice theorem, proposition 6.4, and this way one obtains a $W^{1,p}$-bounded sequence of connections $\tilde{A}^\nu$ that solve the boundary value problem (6.2). This makes crucial use of the compact Sobolev embedding $W^{1,p} \hookrightarrow L^q$ on compact 4-manifolds (with q from the local slice theorem). The estimates in the subsequent theorem 6.8 then provide the higher $W^{k,p}$-bounds on the connections that will imply the compactness. One difficulty in the proof of this regularity theorem is that due to the global nature of the boundary conditions one has to consider the Σ-components of the connections near the boundary as maps into the Banach space $\mathcal{A}^{0,p}(\Sigma)$ that solve a Cauchy-Riemann equation with Lagrangian boundary conditions. In order to prove a regularity result for such maps one has to straighten out the Lagrangian submanifold by using coordinates in $\mathcal{A}^{0,p}(\Sigma)$. This was done in theorem 5.3 above. Thus on domains $\mathcal{U} \times \Sigma$ at the boundary a crucial assumption is that the Σ-components of the connections all lie in one such coordinate chart, that is one needs the connections to converge strongly in the $L^\infty(\mathcal{U}, L^p(\Sigma))$-norm. In the case $p > 4$ this is ensured by the compact embedding $W^{1,p} \hookrightarrow L^\infty$ on $\mathcal{U} \times \Sigma$. To treat the case $2 < p \leq 4$ we shall make use of the following special Sobolev embedding. The proof mainly uses techniques from [Ad].

Lemma 6.7 *Let M, N be compact manifolds and let $p > m = \dim M$ and $p > n = \dim N$. Then the following embedding is compact,*

$$W^{1,p}(M \times N) \hookrightarrow L^\infty(M, L^p(N)).$$

Proof of lemma 6.7:
Since M is compact it suffices to prove the embedding in (finitely many) coordinate charts. These can be chosen as either balls $B_2 \subset \mathbb{R}^m$ in the interior or half balls $D_2 = B_2 \cap \mathbb{H}^m$ in the half space $\mathbb{H}^m = \{x \in \mathbb{R}^m \,\big|\, x_1 \geq 0\}$ at the boundary of M. We can choose both of radius 2 but cover M by balls and half balls of radius 1. So it suffices to consider a bounded set $\mathcal{K} \subset W^{1,p}(B_2 \times N)$ and prove that it restricts to a precompact set in $L^\infty(B_1, L^p(N))$, and similarly with the half balls. Here we use the Euclidean metric on $\mathbb{R}^m$, which is equivalent to the metric induced from M.

For a bounded subset $\mathcal{K} \subset W^{1,p}(D_2 \times N)$ over the half ball define the subset $\mathcal{K}' \subset W^{1,p}(B_2 \times N)$ by extending all $u \in \mathcal{K}$ to $B_2 \setminus \mathbb{H}^m$ by $u(x_1, x_2, \ldots, x_m) := u(-x_1, x_2, \ldots, x_m)$ for $x_1 \leq 0$. The thus extended function is still $W^{1,p}$-regular with twice the norm of u. So $\mathcal{K}'$ also is a bounded subset, and if this restricts to a precompact set in $L^\infty(B_1, L^p(N))$, then also $\mathcal{K} \subset L^\infty(D_1, L^p(N))$ is compact. Hence it suffices to consider the interior case of the full ball.

The claimed embedding is continuos by the standard Sobolev estimates – check for example in [Ad] that the estimates generalize directly to functions with values in a Banach space. In fact, one obtains an embedding

$$W^{1,p}(B_2 \times N) \subset W^{1,p}(B_2, L^p(N)) \hookrightarrow \mathcal{C}^{0,\lambda}(B_2, L^p(N))$$

into some Hölder space with $\lambda = 1 - \frac{m}{p} > 0$. One can also use this Sobolev estimate for $W^{1,p}(N)$ with $\lambda' = 1 - \frac{n}{p} > 0$ combined with the inclusion $L^p \hookrightarrow L^1$ on B_2 to obtain a continuous embedding

$$W^{1,p}(B_2 \times N) \subset L^p(B_2, W^{1,p}(N)) \hookrightarrow L^p(B_2, \mathcal{C}^{0,\lambda'}(N)) \subset L^1(B_2, \mathcal{C}^{0,\lambda'}(N)).$$

Now consider a bounded subset $\mathcal{K} \subset W^{1,p}(B_2 \times N)$. The first embedding ensures that the functions $u \in \mathcal{K}$, $u : B_2 \to L^p(N)$ are equicontinuous. For some constant C

$$\|u(x) - u(y)\|_{L^p(N)} \leq C|x - y|^\lambda \qquad \forall x, y \in B_2, u \in \mathcal{K}. \tag{6.3}$$

The second embedding asserts that for some constant C'

$$\int_{B_2} \|u\|_{\mathcal{C}^{0,\lambda'}(N)} \leq C' \qquad \forall u \in \mathcal{K}. \tag{6.4}$$

In order to prove that $\mathcal{K} \subset L^\infty(B_1, L^p(N))$ is precompact we now fix any $\varepsilon > 0$ and show that $\mathcal{K}$ can be covered by finitely many ε-balls.

Let $J \in \mathcal{C}^\infty(\mathbb{R}^m, [0,\infty))$ be such that $\operatorname{supp} J \subset B_1$ and $\int J = 1$. Then $J_\delta(x) := \delta^{-m} J(x/\delta)$ are mollifiers for $\delta > 0$ with $\operatorname{supp} J_\delta \subset B_\delta$ and $\int J_\delta = 1$. Let $\delta \leq 1$, then $J_\delta * u|_{B_1} \in \mathcal{C}^\infty(B_1, L^p(N))$ is welldefined. Moreover, choose $\delta > 0$ sufficiently small such that for all $u \in \mathcal{K}$

$$\begin{aligned}\big\|J_\delta * u - u\big\|_{L^\infty(B_1,L^p(N))} &= \sup_{x\in B_1}\Big\|\int_{B_\delta} J_\delta(y)\,(u(x-y)-u(x))\,\mathrm{d}^m y\Big\|_{L^p(N)} \\ &\leq \sup_{x\in B_1}\int_{B_\delta} J_\delta(y)\, C|y|^\lambda\,\mathrm{d}^m y \\ &\leq C\delta^\lambda \;\leq\; \tfrac14\varepsilon.\end{aligned}$$

Now it suffices to prove the precompactness of $\mathcal{K}_\delta := \{J_\delta * u \,\big|\, u \in \mathcal{K}\}$, then this set can be covered by $\frac12\varepsilon$-balls around $J_\delta * u_i$ with $u_i \in \mathcal{K}$ for $i = 1, \dots, I$ [1] and above estimate shows that $\mathcal{K}$ is covered by the ε-balls around the u_i. Indeed, for each $u \in \mathcal{K}$ one has $\|J_\delta * u - J_\delta * u_i\|_{L^\infty(B_1,L^p(N))} \leq \frac{\varepsilon}{2}$ for some $i = 1, \dots, I$ and thus

$$\|u - u_i\| \;\leq\; \|u - J_\delta * u\| + \|J_\delta * u - J_\delta * u_i\| + \|J_\delta * u_i - u_i\| \;\leq\; \varepsilon.$$

The precompactness of $\mathcal{K}_\delta \subset L^\infty(B_1, L^p(N))$ will follow from the Arzéla-Ascoli theorem (see e.g. [L, IX §4]). Firstly, the smoothened functions $J_\delta * u$ are still equicontinuous on B_1. For all $u \in \mathcal{K}$ and $x, y \in B_1$ use (6.3) to obtain

$$\begin{aligned}\|(J_\delta * u)(x) - (J_\delta * u)(y)\|_{L^p(N)} &\leq \int_{B_\delta} J_\delta(z)\,\|u(x-z) - u(y-z)\|_{L^p(N)}\,\mathrm{d}^m z \\ &\leq \int_{B_\delta} J_\delta(z)\, C|x-y|^\lambda\,\mathrm{d}^m z \;=\; C|x-y|^\lambda.\end{aligned}$$

Secondly, the L^∞-norm of the smoothened functions is bounded by the L^1-norm of the original ones, so for fixed $\delta > 0$ one obtains a uniform bound from (6.4) : For all $u \in \mathcal{K}$ and $x \in B_1$

$$\begin{aligned}\|(J_\delta * u)(x)\|_{\mathcal{C}^{0,\lambda'}(N)} &\leq \int_{B_2} J_\delta(x-y)\,\|u(y)\|_{\mathcal{C}^{0,\lambda'}(N)}\,\mathrm{d}^m y \\ &\leq C'\|J_\delta\|_\infty.\end{aligned}$$

[1] If a subset $K \subset (X, \mathrm{d})$ of a metric space is precompact, then for fixed $\varepsilon > 0$ one firstly finds $v_1, \dots, v_I \in X$ such that for each $x \in K$ one has $\mathrm{d}(x, v_i) \leq \varepsilon$ for some v_i. For each v_i choose one such $x_i \in K$, or simply drop v_i if this does not exist. Then K is covered by 2ε-balls around the x_i: For each $x \in K$ one has $\mathrm{d}(x, x_i) \leq \mathrm{d}(x, v_i) + \mathrm{d}(v_i, x_i)$ for some $i = 1, \dots, I$.

Now the embedding $\mathcal{C}^{0,\lambda'}(N) \hookrightarrow L^p(N)$ is a standard compact Sobolev embedding, so this shows that the subset $\{(J_\delta * u)(x) \mid u \in \mathcal{K}\} \subset L^p(N)$ is precompact for all $x \in B_1$. Thus the Arzéla-Ascoli theorem asserts that $\mathcal{K}_\delta \subset L^\infty(B_1, L^p(N))$ is compact, and this finishes the proof of the lemma. □

In the proof of theorem 6.2, the weak Uhlenbeck compactness together with the local slice theorem and this lemma will put us in the position to apply the following crucial regularity theorem that also is the crucial point in the proof of theorem 6.1. Here (X, τ) is a 4-manifold with a boundary space-time splitting as described in definition 1.1 and in the beginning of this chapter.

Theorem 6.8 *For every compact subset $K \subset X$ there exists a compact submanifold $M \subset X$ such that $K \subset M$ and the following holds for all $p > 2$.*

(i) Suppose that $A \in \mathcal{A}^{1,p}(M)$ solves the boundary value problem (6.2). Then $A|_K \in \mathcal{A}(K)$ is smooth.

(ii) Fix a metric g_0 that is compatible with τ and a smooth connection $A_0 \in \mathcal{A}(M)$ such that $\tau_i^* A_0|_{\{s\}\times\Sigma_i} \in \mathcal{L}_i$ for all $s \in \mathcal{S}_i$ and $i = 1, \dots, n$. Moreover, fix a compact neighbourhood $\mathcal{V} = \bigcup_{i=1}^n \bar\tau_{0,i}(\mathcal{U}_i \times \Sigma_i)$ of $K \cap \partial X$. (Here $\bar\tau_{0,i}$ denotes the extension of τ_i given by the geodesics of g_0.) Then for every given constant C_1 there exist constants $\delta > 0$, $\delta_k > 0$, and C_k for all $k \geq 2$ such that the following holds:

Fix $k \geq 2$ and let g be a metric that is compatible with τ and satisfies $\|g - g_0\|_{\mathcal{C}^{k+2}(M)} \leq \delta_k$. Suppose that $A \in \mathcal{A}^{1,p}(M)$ solves the boundary value problem (6.2) with respect to the metric g and satisfies

$$\|A - A_0\|_{W^{1,p}(M)} \leq C_1,$$
$$\|\bar\tau_{0,i}^*(A - A_0)|_{\Sigma_i}\|_{L^\infty(\mathcal{U}_i, \mathcal{A}^{0,p}(\Sigma_i))} \leq \delta \quad \forall i = 1, \dots, n.$$

Then $A|_K \in \mathcal{A}(K)$ is smooth by (i) and

$$\|A - A_0\|_{W^{k,p}(K)} \leq C_k.$$

We first give some preliminary results for the proof of theorem 6.8. The interior regularity as well as the regularity of the $\mathcal{U}_i$-components on a neighbourhood $\mathcal{U}_i \times \Sigma_i$ of a boundary component $\mathcal{S}_i \times \Sigma_i$ will be a consequence of the following regularity result for Yang-Mills connections. The proof is similar to lemma 3.3 and can be found in detail in [We, Proposition 10.5]. Here M is a compact Riemannian manifold with boundary ∂M and outer unit normal ν. One then deals with two different spaces of test functions, $\mathcal{C}^\infty_\delta(M,\mathfrak{g})$ and $\mathcal{C}^\infty_\nu(M,\mathfrak{g})$ as in lemma 3.3.

Proposition 6.9 *Let (M,g) be a compact Riemannian 4-manifold. Fix a smooth reference connection $A_0 \in \mathcal{A}(M)$. Let $X \in \Gamma(\mathrm{T}M)$ be a smooth vector field that is either perpendicular to the boundary, i.e. $X|_{\partial M} = h \cdot \nu$ for some $h \in \mathcal{C}^\infty(\partial M)$, or is tangential, i.e. $X|_{\partial M} \in \Gamma(\mathrm{T}\partial M)$. In the first case let $\mathcal{T} = \mathcal{C}^\infty_\delta(M,\mathfrak{g})$, in the latter case let $\mathcal{T} = \mathcal{C}^\infty_\nu(M,\mathfrak{g})$. Moreover, let $N \subset \partial M$ be an open subset such that X vanishes in a neighbourhood of $\partial M \setminus N \subset M$. Let $1 < p < \infty$ and $k \in \mathbb{N}$ be such that either $kp > 4$ or $k = 1$ and $2 < p < 4$. In the first case let $q := p$, in the latter case let $q := \frac{4p}{8-p}$. Then there exists a constant C such that the following holds.*

Let $A = A_0 + \alpha \in \mathcal{A}^{k,p}(M)$ be a connection. Suppose that it satisfies

$$\begin{cases} \mathrm{d}^*_{A_0}\alpha = 0, \\ *\alpha|_{\partial M} = 0 \quad \text{on } N \subset \partial M, \end{cases} \tag{6.5}$$

and that for all 1-forms $\beta = \phi \cdot \iota_X g$ with $\phi \in \mathcal{T}$

$$\int_M \langle F_A \,,\, \mathrm{d}_A\beta \rangle = 0. \tag{6.6}$$

Then $\alpha(X) \in W^{k+1,q}(M,\mathfrak{g})$ and

$$\|\alpha(X)\|_{W^{k+1,q}} \le C\left(1 + \|\alpha\|_{W^{k,p}} + \|\alpha\|^3_{W^{k,p}}\right).$$

Moreover, the constant C can be chosen such that it depends continuously on the metric g and the vector field X with respect to the $\mathcal{C}^{k+1}$-topology.

Remark 6.10 In the case $k = 1$ and $2 < p < 4$ the iteration of proposition 6.9 also allows to obtain $W^{2,p}$-regularity and -estimates from initial $W^{1,p}$-regularity and -estimates.

Indeed, the Sobolev embedding $W^{2,q} \hookrightarrow W^{1,p'}$ holds with $p' = \frac{4q}{4-q}$ since $q < 4$. Now as long as $p' < 4$ one can iterate the proposition and Sobolev embedding to obtain regularity and estimates in W^{1,p_i} with $p_0 = p$ and

$$p_{i+1} \;=\; \frac{4q_i}{4-q_i} \;=\; \frac{2p_i}{4-p_i} \;\ge\; \theta p_i \;>\; p_i.$$

Since $\theta := \frac{2}{4-p} > 1$ this sequence terminates after finitely many steps at some $p_N \geq 4$. Now in case $p_N > 4$ the proposition even yields W^{2,p_N}-regularity and -estimates. In case $p_N = 4$ one only uses W^{1,p_N} for some smaller $p'_N > \frac{8}{3}$ in order to conclude $W^{2,p'_{N+1}}$-regularity and -estimates for $p'_{N+1} > 4$.

Similarly, in case $k = 1$ and $p = 4$ one only needs two steps to reach $W^{2,p'}$ for some $p' > 4$.

The above proposition and remark can be used on all components of the connections in theorem 6.8 except for the Σ-components in small neighbourhoods $\mathcal{U} \times \Sigma$ of boundary components $\mathcal{S} \times \Sigma$. For the regularity of their higher derivatives in Σ-direction we shall use the following lemma. The crucial regularity of the derivatives in direction of $\mathcal{U}$ of the Σ-components will then follow from chapter 5.

Lemma 6.11 *Let $k \in \mathbb{N}_0$ and $1 < p < \infty$. Let Ω be a compact manifold, let Σ be a Riemann surface, and equip $\Omega \times \Sigma$ with a product metric $g_\Omega \oplus g$, where $g = (g_x)_{x\in\Omega}$ is a smooth family of metrics on Σ. Then there exists a constant C such that the following holds:*

Suppose that $\alpha \in W^{k,p}(\Omega \times \Sigma, \mathrm{T}^*\Sigma)$ such that both $\mathrm{d}_\Sigma\alpha$ and $\mathrm{d}_\Sigma^*\alpha$ are of class $W^{k,p}$ on $\Omega \times \Sigma$. Then $\nabla_\Sigma\alpha$ also is of class $W^{k,p}$ and one has the following estimate on $\Omega \times \Sigma$

$$\|\nabla_\Sigma\alpha\|_{W^{k,p}} \leq C\big(\|\mathrm{d}_\Sigma\alpha\|_{W^{k,p}} + \|\mathrm{d}_\Sigma^*\alpha\|_{W^{k,p}} + \|\alpha\|_{W^{k,p}}\big).$$

Here ∇_Σ denotes the family of Levi-Civita connections on Σ that is given by the family of metrics g. Moreover, for every fixed family of metrics g one finds a $\mathcal{C}^k$-neighbourhood of metrics for which this estimate holds with a uniform constant C.

Proof of lemma 6.11:
We first prove this for $k = 0$, i.e. suppose that $\alpha \in L^p(\Omega \times \Sigma, \mathrm{T}^*\Sigma)$ and that $\mathrm{d}_\Sigma\alpha, \mathrm{d}_\Sigma^*\alpha$ (defined as weak derivatives) are also of class L^p. We introduce the following functions

$$f := \mathrm{d}_\Sigma^*\alpha \;\in\; L^p(\Omega \times \Sigma), \qquad g := -*_\Sigma \mathrm{d}_\Sigma\alpha \;\in\; L^p(\Omega \times \Sigma),$$

and choose sequences $f^\nu, g^\nu \in \mathcal{C}^\infty(\Omega\times\Sigma)$, and $\alpha^\nu \in \mathcal{C}^\infty(\Omega\times\Sigma, \mathrm{T}^*\Sigma)$ that converge to f, g, and α respectively in the L^p-norm. Note that $\int_\Omega f = \int_\Omega g = 0$ in $L^p(\Sigma)$, so the f^ν and g^ν can be chosen such that their mean value over Ω

also vanishes for all $z \in \Sigma$. Then fix $z \in \Sigma$ and find $\xi^\nu, \zeta^\nu \in \mathcal{C}^\infty(\Omega \times \Sigma)$ such that

$$\begin{cases} \Delta_\Sigma \xi^\nu = f^\nu, \\ \xi^\nu(x,z) = 0 \quad \forall x \in \Omega, \end{cases} \qquad \begin{cases} \Delta_\Sigma \zeta^\nu = g^\nu, \\ \zeta^\nu(x,z) = 0 \quad \forall x \in \Omega. \end{cases}$$

These solutions are uniquely determined since $\Delta_\Sigma : W^{j+2,p}_z(\Sigma) \to W^{j,p}_m(\Sigma)$ is a bounded isomorphism for every $j \in \mathbb{N}_0$ depending smoothly on the metric, i.e. on $x \in \Omega$. Here $W^{j,p}_m(\Sigma)$ denotes the space of $W^{j,p}$-functions with mean value zero and $W^{j+2,p}_z(\Sigma)$ consists of those functions that vanish at $z \in \Sigma$.

Furthermore, let $\pi_x : \Omega^1(\Sigma) \to h^1(\Sigma, g_x)$ be the projection of the smooth 1-forms to the harmonic part $h^1(\Sigma) = \ker \Delta_\Sigma = \ker \mathrm{d}_\Sigma \cap \ker \mathrm{d}^*_\Sigma$ with respect to the metric g_x on Σ. Then π is a family of bounded operators from $L^p(\Sigma, \mathrm{T}^*\Sigma)$ to $W^{j,p}(\Sigma, \mathrm{T}^*\Sigma)$ for any $j \in \mathbb{N}_0$, and it depends smoothly on $x \in \Omega$. So the harmonic part of $\tilde{\alpha}^\nu$ is also smooth, $\pi \circ \tilde{\alpha}^\nu \in \mathcal{C}^\infty(\Omega \times \Sigma, \mathrm{T}^*\Sigma)$. Now consider

$$\alpha^\nu := \mathrm{d}_\Sigma \xi^\nu + *_\Sigma \mathrm{d}_\Sigma \zeta^\nu + \pi \circ \tilde{\alpha}^\nu \;\in\; \mathcal{C}^\infty(\Omega \times \Sigma, \mathrm{T}^*\Sigma).$$

We will show that the sequence α^ν of 1-forms converges to α in the L^p-norm and that moreover $\nabla_\Sigma \alpha^\nu$ is an L^p-Cauchy sequence. For that purpose we will use the following estimate. For all 1-forms $\beta \in W^{1,p}(\Sigma, \mathrm{T}^*\Sigma)$ abbreviating $\mathrm{d}_\Sigma = \mathrm{d}$

$$\begin{aligned} \|\beta\|_{W^{1,p}(\Sigma)} &\leq C\big(\|\mathrm{d}^*\beta\|_{L^p(\Sigma)} + \|\mathrm{d}\beta\|_{L^p(\Sigma)} + \|\pi(\beta)\|_{W^{1,p}(\Sigma)}\big) \\ &\leq C\big(\|\mathrm{d}^*\beta\|_{L^p(\Sigma)} + \|\mathrm{d}\beta\|_{L^p(\Sigma)} + \|\beta\|_{L^p(\Sigma)}\big). \end{aligned} \tag{6.7}$$

Here and in the following C denotes any finite constant that is uniform for all metrics g_x on Σ in a family of metrics that lies in a sufficiently small $\mathcal{C}^k$-neighbourhood of a fixed family of metrics. To prove (6.7) we use the Hodge decomposition $\beta = \mathrm{d}\xi + *\mathrm{d}\zeta + \pi(\beta)$. (See e.g. [Wa, Theorem 6.8] and recall that one can identify 2-forms on Σ with functions via the Hodge $*$ operator.) Here one chooses $\xi, \zeta \in W^{2,p}_z(\Sigma)$ such that they solve $\Delta\xi = \mathrm{d}^*\beta$ and $\Delta\zeta = *\mathrm{d}\beta$ respectively and concludes from proposition 3.5 for some uniform constant C

$$\begin{aligned} \|\mathrm{d}\xi\|_{W^{1,p}(\Sigma)} \leq \|\xi\|_{W^{2,p}(\Sigma)} \leq C\|\mathrm{d}^*\beta\|_{L^p(\Sigma)}, \\ \|{*\mathrm{d}\zeta}\|_{W^{1,p}(\Sigma)} \leq \|\zeta\|_{W^{2,p}(\Sigma)} \leq C\|\mathrm{d}\beta\|_{L^p(\Sigma)}. \end{aligned}$$

The second step in (6.7) moreover uses the fact that the projection to the harmonic part is bounded as map $\pi : L^p(\Sigma, \mathrm{T}^*\Sigma) \to W^{1,p}(\Sigma, \mathrm{T}^*\Sigma)$.

Now consider $\alpha - \alpha^\nu \in L^p(\Omega \times \Sigma, \mathrm{T}^*\Sigma)$. For almost all $x \in \Omega$ we have $\alpha(x, \cdot) - \alpha^\nu(x, \cdot) \in L^p(\Sigma, \mathrm{T}^*\Sigma)$ as well as $*\mathrm{d}_\Sigma(\alpha(x, \cdot) - \alpha^\nu(x, \cdot)) \in L^p(\Sigma)$ and $\mathrm{d}_\Sigma^*(\alpha(x, \cdot) - \alpha^\nu(x, \cdot)) \in L^p(\Sigma)$. Then for these $x \in \Omega$ one concludes from the Hodge decomposition that in fact $\alpha(x, \cdot) - \alpha^\nu(x, \cdot) \in W^{1,p}(\Sigma, \mathrm{T}^*\Sigma)$. So we can apply (6.7) and integrate over $x \in \Omega$ to obtain for all $\nu \in \mathbb{N}$

$$
\begin{aligned}
&\|\alpha - \alpha^\nu\|^p_{L^p(\Omega\times\Sigma)} \\
&\leq \int_\Omega \|\alpha(x, \cdot) - \alpha^\nu(x, \cdot)\|^p_{L^p(\Sigma, g_x)} \\
&\leq C \int_\Omega \big(\|\mathrm{d}_\Sigma^*(\alpha - \alpha^\nu)\|^p_{L^p(\Sigma)} + \|\mathrm{d}_\Sigma(\alpha - \alpha^\nu)\|^p_{L^p(\Sigma)} + \|\pi(\alpha - \tilde{\alpha}^\nu)\|^p_{W^{1,p}(\Sigma)} \big) \\
&\leq C \big(\|f - f^\nu\|^p_{L^p(\Omega\times\Sigma)} + \|g - g^\nu\|^p_{L^p(\Omega\times\Sigma)} + \|\alpha - \tilde{\alpha}^\nu\|^p_{L^p(\Omega\times\Sigma)} \big).
\end{aligned}
$$

In the last step we again used the continuity of π. This proves the convergence $\alpha^\nu \to \alpha$ in the L^p-norm, and hence $\nabla_\Sigma \alpha^\nu \to \nabla_\Sigma \alpha$ in the distributional sense. Next, we use (6.7) to estimate for all $\nu \in \mathbb{N}$

$$
\begin{aligned}
\|\nabla_\Sigma \alpha^\nu\|^p_{L^p(\Omega\times\Sigma)} &= \int_\Omega \|\nabla_\Sigma \alpha^\nu(x, \cdot)\|^p_{L^p(\Sigma, g_x)} \\
&\leq C \int_\Omega \big(\|\mathrm{d}_\Sigma^* \alpha^\nu\|_{L^p(\Sigma)} + \|\mathrm{d}_\Sigma \alpha^\nu\|_{L^p(\Sigma)} + \|\alpha^\nu\|_{L^p(\Sigma)} \big)^p \\
&\leq C \big(\|\mathrm{d}_\Sigma^* \alpha^\nu\|^p_{L^p(\Omega\times\Sigma)} + \|\mathrm{d}_\Sigma \alpha^\nu\|^p_{L^p(\Omega\times\Sigma)} + \|\alpha^\nu\|^p_{L^p(\Omega\times\Sigma)} \big).
\end{aligned}
$$

Here one deals with L^p-convergent sequences $\mathrm{d}_\Sigma^* \alpha^\nu = \Delta_\Sigma \xi^\nu = f^\nu \to f = \mathrm{d}_\Sigma^* \alpha$, $- * \mathrm{d}_\Sigma \alpha^\nu = \Delta_\Sigma \zeta^\nu = g^\nu \to g = - * \mathrm{d}_\Sigma \alpha$, and $\alpha^\nu \to \alpha$. So $(\nabla_\Sigma \alpha^\nu)_{\nu\in\mathbb{N}}$ is uniformly bounded in $L^p(\Omega \times \Sigma)$ and hence contains a weakly L^p-convergent subsequence. The limit is $\nabla_\Sigma \alpha$ since this already is the limit in the distributional sense. Thus we have proven the L^p-regularity of $\nabla_\Sigma \alpha$ on $\Omega \times \Sigma$, and moreover above estimate is preserved under the limit, which proves the lemma in the case $k = 0$,

$$
\begin{aligned}
\|\nabla_\Sigma \alpha\|_{L^p(\Omega\times\Sigma)} &\leq \liminf_{\nu\to\infty} \|\nabla_\Sigma \alpha^\nu\|_{L^p(\Omega\times\Sigma)} \\
&\leq \liminf_{\nu\to\infty} C \big(\|\mathrm{d}_\Sigma^* \alpha^\nu\|_{L^p(\Omega\times\Sigma)} + \|\mathrm{d}_\Sigma \alpha^\nu\|_{L^p(\Omega\times\Sigma)} + \|\alpha^\nu\|_{L^p(\Omega\times\Sigma)} \big) \\
&= C \big(\|\mathrm{d}_\Sigma^* \alpha\|_{L^p(\Omega\times\Sigma)} + \|\mathrm{d}_\Sigma \alpha\|_{L^p(\Omega\times\Sigma)} + \|\alpha\|_{L^p(\Omega\times\Sigma)} \big).
\end{aligned}
$$

In the case $k \geq 1$ one can now use the previous result to prove the lemma. Let $\alpha \in W^{k,p}(\Omega \times \Sigma, \mathrm{T}^*\Sigma)$ and suppose that $\mathrm{d}_\Sigma \alpha, \mathrm{d}_\Sigma^* \alpha$ are of class $W^{k,p}$. We denote by ∇ the covariant derivative on $\Omega\times\Sigma$. Then we have to show that $\nabla^k\nabla_\Sigma\alpha$ is of class L^p. So let $X_1,\dots,X_k$ be smooth vector fields on $\Omega\times\Sigma$ and introduce

$$\tilde\alpha := \nabla_{X_1}\dots\nabla_{X_k}\alpha \;\in\; L^p(\Omega\times\Sigma,\mathrm{T}^*\Sigma).$$

Both $\mathrm{d}_\Sigma\tilde\alpha$ and $\mathrm{d}_\Sigma^*\tilde\alpha$ are of class L^p since

$$\begin{aligned}\mathrm{d}_\Sigma\tilde\alpha &= [\mathrm{d}_\Sigma,\nabla_{X_1}\dots\nabla_{X_k}]\alpha+\nabla_{X_1}\dots\nabla_{X_k}\mathrm{d}_\Sigma\alpha,\\ \mathrm{d}_\Sigma^*\tilde\alpha &= [\mathrm{d}_\Sigma^*,\nabla_{X_1}\dots\nabla_{X_k}]\alpha+\nabla_{X_1}\dots\nabla_{X_k}\mathrm{d}_\Sigma^*\alpha.\end{aligned}$$

So the result for $k=0$ implies that $\nabla_\Sigma\tilde\alpha$ is of class L^p, hence $\nabla^k\nabla_\Sigma\alpha$ also is of class L^p since for all smooth vector fields

$$\nabla_{X_1}\dots\nabla_{X_k}\nabla_\Sigma\alpha = [\nabla_\Sigma,\nabla_{X_1}\dots\nabla_{X_k}]\alpha+\nabla_\Sigma\tilde\alpha.$$

With the same argument – using coordinate vector fields X_i and cutting them off – one obtains the estimate

$$\|\nabla^k\nabla_\Sigma\alpha\|_{L^p(\Omega\times\Sigma)} \le C\big(\|\nabla^k\mathrm{d}_\Sigma^*\alpha\|_{L^p(\Omega\times\Sigma)}+\|\nabla^k\mathrm{d}_\Sigma\alpha\|_{L^p(\Omega\times\Sigma)}+\|\alpha\|_{W^{k,p}(\Omega\times\Sigma)}\big).$$

Now this proves the lemma,

$$\begin{aligned}\|\nabla_\Sigma\alpha\|_{W^{k,p}(\Omega\times\Sigma)} &\le \|\nabla_\Sigma\alpha\|_{W^{k-1,p}(\Omega\times\Sigma)}+\|\nabla^k\nabla_\Sigma\alpha\|_{L^p(\Omega\times\Sigma)}\\ &\le C\big(\|\mathrm{d}_\Sigma^*\alpha\|_{W^{k,p}(\Omega\times\Sigma)}+\|\mathrm{d}_\Sigma\alpha\|_{W^{k,p}(\Omega\times\Sigma)}+\|\alpha\|_{W^{k,p}(\Omega\times\Sigma)}\big).\end{aligned}$$

□

Proof of theorem 6.8 :
Recall that a neighbourhood of the boundary ∂X is covered by embeddings $\bar\tau_{0,i}:\mathcal{U}_i\times\Sigma_i\hookrightarrow X$ such that $\bar\tau_{0,i}^*g_0=\mathrm{d}s^2+\mathrm{d}t^2+g_{0;s,t}$. (In the case (i) we put $g_0:=g$.) Since $K\subset X$ is compact one can cover it by a compact subset $K_{\rm int}\subset\operatorname{int}X$ and $K_{\rm bdy}:=\bigcup_{i=1}^n\bar\tau_{0,i}(I_{0,i}\times[0,\delta_0]\times\Sigma_i)$ for some $\delta_0>0$ and $I_{0,i}\subset\mathcal{S}_i$ that are either compact intervals in $\mathbb{R}$ or equal to S^1. Moreover, one can ensure that $K_{\rm bdy}\subset\operatorname{int}\mathcal{V}$ lies in the interior of the fixed neighbourhood of $K\cap\partial X$. Since X is exhausted by the compact submanifolds X_k one then finds $M:=X_k\subset X$ such that both $K_{\rm bdy}$ and $K_{\rm int}$ are contained in the interior of M (and thus also $K\subset M$). Now let $A\in\mathcal{A}^{1,p}(M)$ be a solution of the boundary value problem (6.2) with respect to a metric g that is compatible

with τ. Then we will prove its regularity and the corresponding estimates in the interior case on K_{int} and in the boundary case on K_{bdy} separately.

Interior case :

Firstly, since $K_{\text{int}} \subset \operatorname{int} M$ and $K_{\text{int}} \subset \operatorname{int} X = X \setminus \partial X$ we find a sequence of compact submanifolds $M_k \subset \operatorname{int} X$ such that $K_{\text{int}} \subset M_{k+1} \subset \operatorname{int} M_k \subset M$ for all $k \in \mathbb{N}$. We will prove inductively $A|_{M_k} \in \mathcal{A}^{k,p}(M_k)$ for all $k \in \mathbb{N}$ and thus $A|_{K_{\text{int}}} \in \mathcal{A}(K_{\text{int}})$ is smooth. Moreover, we inductively find constants $C_k, \delta_k > 0$ such that the additional assumptions of (ii) in the theorem imply

$$\|A - A_0\|_{W^{k,p}(M_k)} \le C_k. \tag{6.8}$$

Here we use the fixed smooth metric g_0 to define the Sobolev norms – for a sufficiently small $\mathcal{C}^k$-neighbourhood of metrics, the Sobolev norms are equivalent with a uniform constant independent of the metric. Moreover, recall that the reference connection A_0 is smooth.

To start the induction we observe that this regularity and estimate are satisfied for $k = 1$ by assumption. For the induction step assume this regularity and estimate to hold for some $k \in \mathbb{N}$. Then we will use proposition 6.9 on $A|_{M_k} \in \mathcal{A}^{k,p}(M_k)$ to deduce the regularity and estimate on M_{k+1}.

Every coordinate vector field on M_{k+1} can be extended to a vector field X on M_k that vanishes near the boundary ∂M_k. So it suffices to consider such vector fields, i.e. use $N = \emptyset$ in the proposition. Then $\alpha := A - A_0$ satisfies the assumption (6.5). For the weak equation (6.6) we calculate for all $\beta = \phi \cdot \iota_X g$ with $\phi \in \mathcal{T} = \mathcal{C}^\infty_\delta(M_k, \mathfrak{g})$

$$-\int_{M_k} \langle\, F_A \,,\, \mathrm{d}_A\beta \,\rangle \;=\; \int_{M_k} \langle\, \mathrm{d}_A(\phi \cdot \iota_X g) \wedge F_A \,\rangle \;=\; \int_{\partial M_k} \langle\, \phi \cdot \iota_X g \wedge F_A \,\rangle \;=\; 0.$$

We have used Stokes' theorem while approximating A by smooth connections $\tilde{A}$, for which the Bianchi identity $\mathrm{d}_{\tilde{A}} F_{\tilde{A}} = 0$ holds. Now proposition 6.9 and remark 6.10 imply that $A|_{M_{k+1}} \in \mathcal{A}^{k+1,p}(M_{k+1})$. In the case (ii) of the theorem the proposition moreover provides $\delta_{k+1} > 0$ and a uniform constant C for all metrics g with $\|g - g_0\|_{\mathcal{C}^{k+1}(M_k)} \le \delta_{k+1}$ such that the following holds: If (6.8) holds for some constant C_k, then

$$\begin{aligned} \|A - A_0\|_{W^{k+1,p}(M_{k+1})} &\le C\left(1 + \|A - A_0\|_{W^{k,p}(M_k)} + \|A - A_0\|^3_{W^{k,p}(M_k)}\right) \\ &\le C\left(1 + C_k + C_k^3\right) \;=:\; C_{k+1}. \end{aligned}$$

Here we have used the fact that the Sobolev norm of a 1-form is equivalent to an expression in terms of the Sobolev norms of its components in the coordinate charts. In case $k=1$ and $p\leq 4$, this uniform bound is not found directly but after finitely many iterations of proposition 6.9 that give estimates on manifolds $N_1 = M_1$ and $M_2 \subset N_{i+1} \subset \operatorname{int} N_i$. In each step one chooses a smaller $\delta_2 > 0$ and a bigger C_2. This iteration uses the same Sobolev embeddings as remark 6.10. This proves the induction step on the interior part $K_{\rm int}$.

Boundary case :

It remains to prove the regularity and estimates on $K_{\rm bdy}$ near the boundary. So consider a single boundary component $K' := \bar\tau_0(I_0 \times [0,\delta_0] \times \Sigma)$. We identify $I_0 = S^1 \cong \mathbb{R}/\mathbb{Z}$ or shift the compact interval such that $I_0 = [-r_0, r_0]$ and hence $K' = \bar\tau_0([-r_0,r_0] \times [0,\delta_0] \times \Sigma)$ for some $r_0 > 0$. Since $K_{\rm bdy}$ (and thus also K') lies in the interior of M as well as $\mathcal{V}$, one then finds $R_0 > r_0$ and $\Delta_0 > \delta_0$ such that $\bar\tau_0([-R_0,R_0] \times [0,\Delta_0] \times \Sigma) \subset M \cap \mathcal{V}$. Here $\bar\tau_0$ is the embedding that brings the metric g_0 into the standard form $\mathrm{d}s^2 + \mathrm{d}t^2 + g_{0;s,t}$. A different metric g compatible with τ defines a different embedding $\bar\tau$ such that $\bar\tau^* g = \mathrm{d}s^2 + \mathrm{d}t^2 + g_{s,t}$. However, if g is sufficiently $\mathcal{C}^1$-close to g_0, then the geodesics are $\mathcal{C}^0$-close and hence $\bar\tau$ is $\mathcal{C}^0$-close to $\bar\tau_0$. (These embeddings are fixed for $t=0$, and for $t>0$ given by the normal geodesics.) Thus for a sufficiently small choice of $\delta_2 > 0$ one finds $R > r > 0$ and $\Delta > \delta > 0$ such that for all τ-compatible metrics g in the δ_2-ball around g_0

$$K' \subset \bar\tau([-r,r] \times [0,\delta] \times \Sigma) \qquad \text{and} \qquad \bar\tau([-R,R] \times [0,\Delta] \times \Sigma) \subset M \cap \mathcal{V}.$$

(In the case (i) this holds with r_0, δ_0, R_0, and Δ_0 for the fixed metric $g = g_0$.) We will prove the regularity and estimates for $\bar\tau^* A$ on $[-r,r] \times [0,\delta] \times \Sigma$. This suffices because for $\mathcal{C}^{k+2}$-close metrics the embedding $\bar\tau$ will be $\mathcal{C}^{k+1}$-close to the fixed $\bar\tau_0$, so that one obtains uniform constants in the estimates between the $W^{k,p}$-norms of A and $\bar\tau^* A$. Furthermore, the families $g_{s,t}$ of metrics on Σ will be $\mathcal{C}^k$-close to $g_{0;s,t}$ for $(s,t) \in [-R,R] \times [0,\Delta]$ if δ_k is chosen sufficiently small. Now choose compact submanifolds $\Omega_k \subset \mathbb{H} := \{(s,t) \in \mathbb{R}^2 \,\big|\, t \geq 0\}$ such that for all $k \in \mathbb{N}$

$$[-r,r] \times [0,\delta] \;\subset\; \Omega_{k+1} \;\subset\; \operatorname{int} \Omega_k \;\subset\; [-R,R] \times [0,\Delta].$$

We will prove the theorem by establishing the regularity and estimates for $\bar\tau^* A$ on the $\Omega_k \times \Sigma$ in Sobolev spaces of increasing differentiability. We distinguish the cases $p > 4$ and $4 \geq p > 2$. In case $p > 4$ one uses the following induction.

I) *Let $p>2$ and suppose that $A\in\mathcal{A}^{1,2p}(M)$ solves (6.2). Then we will prove inductively that $\bar\tau^*A|_{\Omega_k\times\Sigma}\in\mathcal{A}^{k,q}(\Omega_k\times\Sigma)$ for all $k\in\mathbb{N}$ and with $q=p$ or $q=2p$ according to whether $k\geq 2$ or $k=1$. Moreover, we will find a constant $\delta>0$ and constants $C_k,\delta_k>0$ for all $k\geq 2$ such that the following holds:*

If in addition $\|g-g_0\|_{\mathcal{C}^{k+2}(M)}\leq\delta_k$ and

$$\|A-A_0\|_{W^{1,2p}(M)}\leq C_1,$$
$$\|\bar\tau_0^*(A-A_0)|_\Sigma\|_{L^\infty(\mathcal{U},\mathcal{A}^{0,p}(\Sigma))}\leq\delta,$$

then for all $k\in\mathbb{N}$

$$\|\bar\tau^*(A-A_0)\|_{W^{k,q}(\Omega_k\times\Sigma)}\leq C_k.$$

This is sufficient to conclude the theorem in case $p>4$ as follows. One uses I) with p replaced by $\frac{1}{2}p$ to obtain regularity and estimates of $A-A_0$ in $\mathcal{A}^{1,p}(\Omega_1\times\Sigma)$, $\mathcal{A}^{2,\frac{p}{2}}(\Omega_2\times\Sigma)$, and $\mathcal{A}^{k,\frac{p}{2}}(\Omega_k\times\Sigma)$ for all $k\geq 3$. Recall that the component K' of $K_{\rm bdy}$ is contained in each $\bar\tau(\Omega_k\times\Sigma)$. In addition, one has the Sobolev embeddings $W^{k+1,\frac{p}{2}}\hookrightarrow W^{k,p}\hookrightarrow\mathcal{C}^{k-1}$ on the compact 4-manifolds $\Omega_{k+1}\times\Sigma)$, c.f. [Ad, Theorem 5.4]. So this proves the regularity and estimates on $K_{\rm bdy}$.

In the case $4\geq p>2$ a preliminary iteration is required in order to achieve the regularity and estimates that are assumed in I). In contrast to I) the iteration is in p instead of k.

II) *Let $4\geq p>2$ and suppose that $A\in\mathcal{A}^{1,p}(M)$ solves (6.2). Then we will prove inductively that $\bar\tau^*A|_{\Omega_j\times\Sigma}\in\mathcal{A}^{1,p_j}(\Omega_j\times\Sigma)$ for a sequence (p_j) with $p_1=p$ and $p_{j+1}=\theta(p_j)\cdot p_j$, where $\theta:(2,4]\to(1,\frac{17}{16}]$ is monotonely increasing and thus the sequence terminates with $p_N>4$ for some $N\in\mathbb{N}$.*

Moreover, we will find constants $\delta>0$ and constants $C_{1,j},\delta_{1,j}>0$ for $j=2,\dots,N$ such that the following holds:

If for some $j=1,\dots,N$ in addition $\|g-g_0\|_{\mathcal{C}^3(M)}\leq\delta_{1,j}$ and

$$\|A-A_0\|_{W^{1,p}(M)}\leq C_1,$$
$$\|\bar\tau_0^*(A-A_0)|_\Sigma\|_{L^\infty(\mathcal{U},\mathcal{A}^{0,p}(\Sigma))}\leq\delta,$$

then

$$\|\bar\tau^*(A-A_0)\|_{W^{1,p_j}(\Omega_j\times\Sigma)}\leq C_{1,j}.$$

Assuming I) and II) we first prove the theorem for the case $4 \geq p > 2$. After finitely many steps the iteration of II) gives regularity and estimates in $\mathcal{A}^{1,p_N}(\Omega_N \times \Sigma)$ with $p_N > 4$ and under the assumption $\|g - g_0\|_{\mathcal{C}^3(M)} \leq \delta_{1,N}$ on the metric. Now if necessary decrease p_N slightly such that $2p \geq p_N > 4$, then one still has $\mathcal{A}^{1,p_N}$-regularity and estimates on all components of K_{bdy} as well as on K_{int} (from the previous argument on the interior). Thus the assumptions of I) are satisfied with p replaced by $\frac{1}{2}p_N$ and C_1 replaced by a combination of $C_{1,N}$ and a constant from the interior iteration (both of which only depend on C_1). One just has to choose $\delta_2 \leq \delta_{1,N}$ and choose the $\delta > 0$ in I) smaller than the $\delta > 0$ from II). Then the iteration in I) gives regularity and estimates of $A - A_0$ in $\mathcal{A}^{k,\frac{1}{2}p_N}(\Omega_k \times \Sigma)$ for all $k \geq 2$. This proves the theorem in case $2 < p \leq 4$ due to the Sobolev embeddings $W^{k+1,\frac{1}{2}p_N} \hookrightarrow W^{k,p} \hookrightarrow \mathcal{C}^{k-2}$. So it remains to establish I) and II).

Proof of I):
The start of the induction $k = 1$ is true by assumption (after replacing C_1 by a larger constant to make up for the effect of $\bar{\tau}^*$). For the induction step assume that the claimed regularity and estimates hold for some $k \in \mathbb{N}$ and consider the following decomposition of the connection A and its curvature:

$$\begin{aligned}\bar{\tau}^* A &= \Phi \, \mathrm{d}s + \Psi \, \mathrm{d}t + B, \\ \bar{\tau}^* F_A &= F_B + (\mathrm{d}_B \Phi - \partial_s B) \wedge \mathrm{d}s + (\mathrm{d}_B \Psi - \partial_t B) \wedge \mathrm{d}t \\ &\quad + (\partial_s \Psi - \partial_t \Phi + [\Phi, \Psi]) \, \mathrm{d}s \wedge \mathrm{d}t.\end{aligned} \tag{6.9}$$

Here $\Phi, \Psi \in W^{k,q}(\Omega_k \times \Sigma, \mathfrak{g})$, and $B \in W^{k,q}(\Omega_k \times \Sigma, \mathrm{T}^*\Sigma \otimes \mathfrak{g})$ is a 2-parameter family of 1-forms on Σ. Choose a further compact submanifold $\Omega \subset \operatorname{int} \Omega_k$ such that $\Omega_{k+1} \subset \operatorname{int} \Omega$. Now we shall use proposition 6.9 to deduce the higher regularity of Φ and Ψ on $\Omega \times \Sigma$. For this purpose one has to extend the vector fields ∂_s and ∂_t on $\Omega \times \Sigma$ to different vector fields on $\Omega_k \times \Sigma$, both denoted by X, and verify the assumptions (6.5) and (6.6) of proposition 6.9. These extensions will be chosen such that they vanish in a neighbourhood of $(\partial \Omega_k \setminus \partial \mathbb{H}) \times \Sigma$. Then $\alpha := \bar{\tau}^*(A - A_0)$ satisfies (6.5) on $M = \bar{\tau}(\Omega_k \times \Sigma)$ with $N = \bar{\tau}((\partial \Omega_k \cap \partial \mathbb{H}) \times \Sigma)$.

Choose a cutoff function $h \in \mathcal{C}^\infty(\Omega_k, [0,1])$ that equals 1 on Ω and vanishes in a neighbourhood of $\partial \Omega_k \setminus \partial \mathbb{H}$. Then firstly, $X := h \partial_t$ is a vector field as required that is perpendicular to the boundary $\partial \Omega_k \times \Sigma$. For this type of vector field we have to check the assumption (6.6) for all $\beta = \phi h \cdot \mathrm{d}t$ with $\phi \in \mathcal{C}^\infty_\delta(\Omega_k \times \Sigma, \mathfrak{g})$. Note that $\bar{\tau}_* \beta = (\phi \cdot h) \circ \bar{\tau}^{-1} \cdot \iota_{(\bar{\tau}_* \partial_t)} g$ can be trivially extended to M and then vanishes when restricted to ∂M. So we can use partial integration as in the interior case to obtain

$$\int_{\Omega_k\times\Sigma} \langle\, F_{\bar\tau^*A}\,,\, \mathrm{d}_{\bar\tau^*A}\beta\,\rangle \;=\; \int_M \langle\, F_A\,,\, \mathrm{d}_A\bar\tau_*\beta\,\rangle \;=\; -\int_{\partial M} \langle\, \bar\tau_*\beta\wedge F_A\,\rangle \;=\; 0.$$

Secondly, $X := h\partial_s$ also vanishes in a neighbourhood of $(\partial\Omega_k\setminus\partial\mathbb{H})\times\Sigma$ and is tangential to the boundary $\partial\Omega_k\times\Sigma$. So we have to verify (6.6) for all $\beta=\phi h\cdot\mathrm{d}s$ with $\phi\in\mathcal{T}=\mathcal{C}^\infty_\nu(\Omega_k\times\Sigma,\mathfrak{g})$. Again, $\bar\tau_*\beta$ extends trivially to M. Then the partial integration yields

$$\begin{aligned}\int_{\Omega_k\times\Sigma} \langle\, F_{\bar\tau^*A}\,,\, \mathrm{d}_{\bar\tau^*A}\beta\,\rangle &= -\int_{\bar\tau^{-1}(\partial M)} \langle\, \beta\wedge\bar\tau^*F_A\,\rangle \\ &= -\int_{(\Omega_k\cap\partial\mathbb{H})\times\Sigma} \langle\, \phi h\cdot\mathrm{d}s\wedge F_B\,\rangle \;=\; 0.\end{aligned}$$

The last step uses the fact that $B(s,0)=\tau^*A|_{\{s\}\times\Sigma}\in\mathcal{L}\subset\mathcal{A}^{0,p}_{\text{flat}}(\Sigma)$, and hence F_B vanishes on $\partial\mathbb{H}\times\Sigma$. However, we have to approximate A by smooth connections in order that Stokes' theorem holds and F_B is well defined. So this calculation crucially uses the fact that a $W^{1,p}$-connection with boundary values in the Lagrangian submanifold $\mathcal{L}$ can be $W^{1,p}$-approximated by smooth connections with boundary values in $\mathcal{L}\cap\mathcal{A}(\Sigma)$. This was proven in corollary 4.2. So we have verified the assumptions of proposition 6.9 for both $\Phi=\bar\tau^*A(\partial_s)$ and $\Psi=\bar\tau^*A(\partial_t)$ and thus can deduce $\Phi,\Psi\in W^{k+1,q}(\Omega\times\Sigma)$. Moreover, under the additional assumptions of (ii) in the theorem we have the estimates

$$\begin{aligned}\|\Phi-\Phi_0\|_{W^{k+1,q}(\Omega\times\Sigma)} &\le C_s\left(1+C_k+C_k^3\right) \;=:\; C^s_{k+1},\\ \|\Psi-\Psi_0\|_{W^{k+1,q}(\Omega\times\Sigma)} &\le C_t\left(1+C_k+C_k^3\right) \;=:\; C^t_{k+1}.\end{aligned}\tag{6.10}$$

The constants C_s and C_t are uniform for all metrics in some small $\mathcal{C}^{k+1}$-neighbourhood of $g_{0;s,t}$, so by a possibly smaller choice of $\delta_{k+1}>0$ they become independent of $g_{s,t}$. Note that in the above estimates we also have decomposed the reference connection in the tubular neighbourhood coordinates, $\bar\tau^*A_0=\Phi_0\,\mathrm{d}s+\Psi_0\,\mathrm{d}t+B_0$.

It remains to consider the Σ-component B in the tubular neighbourhood.

The boundary value problem (6.2) becomes in the coordinates (6.9)

$$\begin{cases} \mathrm{d}_{B_0}^*(B - B_0) = \nabla_s(\Phi - \Phi_0) + \nabla_t(\Psi - \Psi_0), \\ \qquad\quad *F_B = \partial_t\Phi - \partial_s\Psi + [\Psi, \Phi], \\ \partial_s B + *\partial_t B = \mathrm{d}_B\Phi + *\mathrm{d}_B\Psi, \\ B(s,0) \in \mathcal{L} \quad \forall (s,0) \in \partial\Omega_k. \end{cases} \tag{6.11}$$

Here d_B is the exterior derivative on Σ that is associated with the connection B, $\mathrm{d}_{B_0}^*$ is the coderivative associated with B_0, $*$ is the Hodge operator on Σ with respect to the metric $g_{s,t}$, and $\nabla_s\Phi := \partial_s\Phi + [\Phi_0, \Phi]$, $\nabla_t\Phi := \partial_t\Phi + [\Psi_0, \Phi]$. We rewrite the first two equations in (6.11) as a system of differential equations for $\alpha := B - B_0$ on Σ. For each $(s,t) \in \Omega_k$

$$\mathrm{d}_\Sigma^*\alpha(s,t) = \xi(s,t), \qquad\qquad \mathrm{d}_\Sigma\alpha(s,t) = *\zeta(s,t). \tag{6.12}$$

Here we have abbreviated

$$\begin{aligned} \xi &= *[B_0 \wedge *(B - B_0)] + \nabla_s(\Phi - \Phi_0) + \nabla_t(\Psi - \Psi_0), \\ \zeta &= - * \mathrm{d}_\Sigma B_0 - *\tfrac{1}{2}[B \wedge B] + \partial_t\Phi - \partial_s\Psi + [\Psi, \Phi]. \end{aligned}$$

These are both functions in $W^{k,q}(\Omega \times \Sigma, \mathfrak{g})$ due to the smoothness of A_0 and the previously established regularity of Φ and Ψ. (This uses the Sobolev embedding $W^{k,q} \cdot W^{k,q} \hookrightarrow W^{k,q}$ due to $W^{k,q} \hookrightarrow L^\infty$.) So lemma 6.11 asserts that $\nabla_\Sigma(B - B_0)$ is of class $W^{k,q}$ on $\Omega \times \Sigma$, and under the assumptions of (ii) in the theorem we obtain the estimate

$$\begin{aligned} &\|\nabla_\Sigma(B - B_0)\|_{W^{k,q}(\Omega\times\Sigma)} \\ &\le C\big(\|\xi\|_{W^{k,q}} + \|\zeta\|_{W^{k,q}} + \|B - B_0\|_{W^{k,q}}\big) \\ &\le C\big(1 + \|B - B_0\|_{W^{k,q}} + \|\Phi - \Phi_0\|_{W^{k+1,q}} + \|\Psi - \Psi_0\|_{W^{k+1,q}} \\ &\qquad + \|B - B_0\|_{W^{k,q}}^2 + \|\Phi - \Phi_0\|_{W^{k,q}}\|\Psi - \Psi_0\|_{W^{k,q}}\big) \\ &\le C\big(1 + C_k + C_{k+1}^s + C_{k+1}^t + C_k^2\big) \;=:\; C_{k+1}^\Sigma. \end{aligned} \tag{6.13}$$

Here C denotes any constant that is uniform for all metrics in a $\mathcal{C}^{k+1}$-neighbourhood of the fixed $g_{0;s,t}$, so this might again require a smaller choice of $\delta_{k+1} > 0$ in order that the constant C_{k+1}^Σ becomes independent of the metric $g_{s,t}$.

Now we have established the regularity and estimate for all derivatives of B of order $k+1$ containing at least one derivative in Σ-direction. (Note that in the case $k=1$ we even have L^q-regularity with $q=2p$ where only L^p-regularity was claimed. This additional regularity will be essential for the following argument.) It remains to consider the pure s- and t- derivatives of B and establish the L^p-regularity and -estimate for $\nabla_{\mathbb{H}}^{k+1}B$ on $\Omega_{k+1}\times\Sigma$, where $\nabla_{\mathbb{H}}$ is the standard covariant derivative on $\mathbb{H}$ with respect to the metric $\mathrm{d}s^2+\mathrm{d}t^2$. The reason for this regularity, as we shall show, is the fact that $B\in W^{k,q}(\Omega,\mathcal{A}^{0,p}(\Sigma))$ satisfies a Cauchy-Riemann equation with Lagrangian boundary conditions,

$$\begin{cases} \partial_s B + *\partial_t B = G, \\ B(s,0)\in\mathcal{L} \quad \forall (s,0)\in\partial\Omega. \end{cases} \tag{6.14}$$

The inhomogeneous term is

$$G := \mathrm{d}_B\Phi + *\mathrm{d}_B\Psi \ \in\ W^{k,q}(\Omega,\mathcal{A}^{0,p}(\Sigma)).$$

Here one uses the fact that $W^{k,q}(\Omega\times\Sigma,\mathrm{T}^*\Sigma\otimes\mathfrak{g})\subset W^{k,q}(\Omega,\mathcal{A}^{0,p}(\Sigma))$ since by lemma 5.8 the smooth 1-forms are dense in both spaces and the norm on the second space is weaker than the $W^{k,q}$-norm on $\Omega\times\Sigma$. Now one has to apply the regularity theorem 5.3 for the Cauchy-Riemann equation on the complex Banach space $(\mathcal{A}^{0,p}(\Sigma),J_0)$. As complex structure J_0 we use the Hodge operator $*$ on Σ with respect to the fixed family of metrics $g_{0;s,t}$ on Σ (that varies smoothly with $(s,t)\in\Omega$). The Lagrangian and hence totally real submanifold $\mathcal{L}\subset\mathcal{A}^{0,p}(\Sigma)$ is modelled on $Z=W^{1,p}_z(\Sigma,\mathfrak{g})\oplus\mathbb{R}^m$ (see lemma 4.1 (iii)), which is bounded isomorphic to a closed subspace of $L^p(\Sigma,\mathbb{R}^n)$ for some $n\in\mathbb{N}$. In the case (ii) of the theorem moreover a family of connections $B_0\in\mathcal{C}^\infty(\Omega,\mathcal{A}(\Sigma))$ is given such that $B_0(s,0)\in\mathcal{L}$ for all $(s,0)\in\partial\Omega$ and B satisfies

$$\begin{aligned}\|B-B_0\|_{L^\infty(\Omega,\mathcal{A}^{0,p}(\Sigma))} &= \|\bar\tau^*(A-A_0)|_\Sigma\|_{L^\infty(\Omega,\mathcal{A}^{0,p}(\Sigma))} \\ &\leq C\|\bar\tau_0^*(A-A_0)|_\Sigma\|_{L^\infty(\mathcal{U},\mathcal{A}^{0,p}(\Sigma))} \ \leq\ C\delta.\end{aligned}$$

Here one uses the fact that $\bar\tau(\Omega\times\Sigma)\subset\bar\tau_0(\mathcal{U}\times\Sigma)$ lies in a component of the fixed neighbourhood $\mathcal{V}$ of $K\cap\partial X$. The assumption of closeness to A_0 in $\mathcal{A}^{0,p}(\Sigma)$ was formulated for $\bar\tau_0^*(A-A_0)|_\Sigma$. However, for a metric g in a sufficiently small $\mathcal{C}^2$-neighbourhood of the fixed metric g_0 the extensions $\bar\tau$ and $\bar\tau_0$ are $\mathcal{C}^1$-close and one obtains the above estimate with a constant C independent of the metric. So $B\in W^{k,q}(\Omega,\mathcal{A}^{0,p}(\Sigma))$ satisfies the assumptions of theorem 5.3 if $\delta>0$ is chosen sufficiently small. (Note that this choice is independent of $k \in \mathbb{N}$.) In the case (i) of the theorem one can also choose such a smooth B_0 sufficiently close to B in the $L^\infty(\Omega, L^p(\Sigma))$-norm. In order to do this smooth approximation with boundary values in the Lagrangian, one uses the Banach submanifold coordinates in lemma 4.1 (iii) as in corollary 4.2.

Now theorem 5.3 asserts $B \in W^{k+1,p}(\Omega_{k+1}, \mathcal{A}^{0,p}(\Sigma))$. By lemma 5.8 this also proves $\nabla_{\mathbb{H}}^{k+1} B \in L^p(\Omega_{k+1}, \mathcal{A}^{0,p}(\Sigma)) = L^p(\Omega_{k+1} \times \Sigma, \mathrm{T}^*\Sigma \otimes \mathfrak{g})$, and this finishes the induction step $\bar{\tau}^* A|_{\Omega_{k+1}\times\Sigma} \in \mathcal{A}^{k+1,p}(\Omega_{k+1} \times \Sigma)$ for the regularity near the boundary. The induction step for the estimate in case (ii) of the theorem now follows from the estimate from theorem 5.3,

$$\begin{aligned}
&\|\nabla_{\mathbb{H}}^{k+1}(B - B_0)\|_{L^p(\Omega_{k+1}\times\Sigma)} \\
&\le \|B - B_0\|_{W^{k+1,p}(\Omega_{k+1},\mathcal{A}^{0,p}(\Sigma))} \\
&\le C\bigl(1 + \|G\|_{W^{k,q}(\Omega,\mathcal{A}^{0,p}(\Sigma))} + \|B - B_0\|_{W^{k,q}(\Omega,\mathcal{A}^{0,p}(\Sigma))}\bigr) \\
&\le C\bigl(1 + C_k + C_k^2 + C_{k+1}^s + C_{k+1}^t\bigr) \;=:\; C_{k+1}^{\mathbb{H}}.
\end{aligned} \tag{6.15}$$

Here the constant from theorem 5.3 is uniform for a sufficiently small $\mathcal{C}^{k+1}$-neighbourhood of complex structures. In this case, these are the families of Hodge operators on Σ that depend on the metric $g_{s,t}$. Thus for sufficiently small $\delta_{k+1} > 0$ that constant (and also the further Sobolev constants that come into the estimate) becomes independent of the metric. The final constant C_{k+1} then results from all the separate estimates, see the decomposition (6.9) and the estimates in (6.10), (6.13), and (6.15),

$$\|\bar{\tau}^*(A - A_0)\|_{W^{k+1,p}(\Omega_{k+1}\times\Sigma)} \;\le\; C_k + C_{k+1}^s + C_{k+1}^t + C_{k+1}^\Sigma + C_{k+1}^{\mathbb{H}}.$$

Proof of II):

Except for the higher differentiability of B in direction of $\mathbb{H}$ this iteration works by the same decomposition and equations as in I). The start of the induction $k = 1$ is given by assumption. For the induction step assume that the claimed W^{1,p_k}-regularity and -estimates hold for some $k \in \mathbb{N}$ with $p_k \le 4$. Then proposition 6.9 gives $\Phi, \Psi \in W^{2,q_k}(\Omega\times\Sigma)$ with corresponding estimates and

$$q_k = \begin{cases} \frac{4p_k}{8-p_k} & \text{if } p_k < 4, \\[2ex] 3 & \text{if } p_k = 4. \end{cases}$$

(In the case $p_k = 4$ one applies the proposition only assuming W^{1,p'_k}-regularity for $p'_k = \frac{24}{7} < 4$, then one obtains W^{2,q_k}-regularity with $q_k = 3$.) Now the right hand sides in (6.12) lie in $W^{1,q_k}(\Omega\times\Sigma)$, so lemma 6.11 gives W^{1,q_k}-regularity and -estimates for $\nabla_\Sigma B$ on $\Omega\times\Sigma$. Next, $B\in W^{1,p_k}(\Omega,\mathcal{A}^{0,p}(\Sigma))$ satisfies the Cauchy-Riemann equation (6.14) with the inhomogenous term $G\in W^{1,q_k}(\Omega\times\Sigma,\mathrm{T}^*(\Omega\times\Sigma)\otimes\mathfrak{g})$. Now we shall use the Sobolev embedding $W^{1,q_k}(\Omega\times\Sigma)\hookrightarrow L^{r_k}(\Omega\times\Sigma)$ with

$$r_k \;=\; \frac{4q_k}{4-q_k} \;=\; \begin{cases} \frac{2p_k}{4-p_k} & \text{if } p_k<4, \\ 12 & \text{if } p_k=4. \end{cases}$$

Note that $r_k>p_k\geq p$ due to $p_k>2$, so that we have $G\in L^{r_k}(\Omega,\mathcal{A}^{0,p}(\Sigma))$. We cannot apply theorem 5.3 directly because that would require the initial regularity $B\in W^{1,2p}(\Omega,\mathcal{A}^{0,p}(\Sigma))$ for some $p>2$. However, we still proceed as in its proof and introduce the coordinates from lemma 4.1 (iii) that straighten out the Lagrangian submanifold,

$$\Theta_{s,t}:\mathcal{W}_{s,t}\to\mathcal{A}^{0,p}(\Sigma).$$

Here $\mathcal{W}_{s,t}\subset Y\times Y$ is a neighbourhood of zero, Y is a closed subspace of $L^p(\Sigma,\mathbb{R}^m)$ for some $m\in\mathbb{N}$, Θ is in $\mathcal{C}^{k+1}$-dependence on (s,t) in a neighbourhood $U\subset\Omega$ of some $(s_0,0)\in\Omega\cap\partial\mathbb{H}$ and it maps diffeomorphically to a neighbourhood of the smooth connection $\Theta(0)=B_0(s_0,0)$. Thus one can write

$$B(s,t)=\Theta_{s,t}(v(s,t))\qquad\forall(s,t)\in U$$

with $v=(v_1,v_2)\in W^{1,p_k}(U,Y\times Y)$. Moreover, we have already seen that both B and $\nabla_\Sigma B$ are W^{1,q_k}-regular on $U\times\Sigma$, so we have the regularity $B\in W^{1,q_k}(U,\mathcal{A}^{1,q_k}(\Sigma))\subset W^{1,q_k}(U,\mathcal{A}^{0,s_k}(\Sigma))$ with corresponding estimates. Here we have used the Sobolev embedding $W^{1,q_k}(\Sigma)\hookrightarrow L^{s_k}(\Sigma)$, see [Ad, Theorem 5.4], for

$$s_k=\begin{cases} \frac{2q_k}{2-q_k} \;=\; \frac{4p_k}{8-3p_k} & \text{if } p_k<\frac{8}{3}, \\ \frac{44p_k-80}{8-p_k} & \text{if } p_k\geq\frac{8}{3}, \\ \frac{31}{2} & \text{if } p_k=4. \end{cases}$$

(Here we have chosen suitable values of s_k for later calculations in case $p_k\geq\frac{8}{3}$ and thus $q_k\geq 2$.) The special structure of the coordinates Θ in lemma 4.1 (iii) (it also is a local diffeomorphism between $\mathcal{A}^{0,s_k}(\Sigma)$ and a closed subset of $L^{s_k}(\Sigma,\mathbb{R}^{2m})$ since $s_k > p_k > 2$) implies that $v \in W^{1,q_k}(U, L^{s_k}(\Sigma,\mathbb{R}^{2m}))$, which will be important later on.

The Cauchy-Riemann equation (6.14) now becomes

$$\begin{cases} \partial_s v + I\partial_t v = f, \\ v_2(s,0) = 0 \quad \forall s \in \mathbb{R}. \end{cases}$$

Here $I = (\mathrm{d}_v\Theta)^{-1} * (\mathrm{d}_v\Theta) \in W^{1,p_k}(U, \mathrm{End}(Y\times Y))$ and

$$f \;=\; (\mathrm{d}_v\Theta)^{-1}(G - \partial_s\Theta(v) - *\partial_t\Theta(v)) \;\in\; L^{r_k}(U, Y\times Y).$$

We now approximate f in $L^{r_k}(U,Y\times Y)$ by smooth functions that vanish on ∂U, then partial integration in (5.5) yields for all $\phi \in \mathcal{C}^\infty(U, Y^*\times Y^*)$ and a smooth cutoff function as in the proof of theorem 5.3

$$\begin{aligned} \int_U \langle\, hv\,,\, \Delta\phi\,\rangle = \int_U \langle\, f\,,\, \partial_s(h\phi) - \partial_t(h\cdot I^*\phi)\,\rangle \;+ \int_U \langle\, \tilde{F}\,,\,\phi\,\rangle \\ + \int_{\partial U\cap\partial\mathbb{H}} \langle\, v_1\,,\, \partial_t(h\phi_1) + \partial_s(h\phi_2)\,\rangle. \end{aligned} \tag{6.16}$$

Here $\tilde{F} = (\Delta h)v + 2(\partial_s h)\partial_s v + 2(\partial_t h)\partial_t v + h(\partial_t I)\partial_s v - h(\partial_s I)\partial_t v$ contains the crucial terms $(\partial_t I)(\partial_s v)$ and $(\partial_s I)(\partial_t v)$ and thus lies in $L^{\frac{1}{2}p_k}(U, Y\times Y)$. This is a weak Laplace equation with Dirichlet boundary conditions for hv_2, Neumann boundary conditions for hv_1, and with the inhomogenous term in $W^{-1,r_k}(U,Y\times Y)$. The latter is the dual space of $W^{1,r'_k}(U,Y^*\times Y^*)$ with $\frac{1}{r_k} + \frac{1}{r'_k} = 1$. (The inclusion $L^{\frac{1}{2}p_k}(U) \hookrightarrow W^{-1,r_k}(U)$ is continuous as can be seen via the dual embedding that is due to $\frac{1}{2} - \frac{1}{r'_k} \geq -1 + \frac{1}{p_k/2}$.) Recall that $Y \subset L^p(\Sigma,\mathbb{R}^m)$ is a closed subspace. Since $r_k > p$ the special regularity theorem 5.7 for the Laplace equation with values in a Banach space cannot be applied to deduce $hv \in W^{1,r_k}(U, Y\times Y)$. However, the general regularity theory for the Laplace equation extends to functions with values in a Hilbert space (c.f. [We]). So we use the embedding $L^p(\Sigma) \hookrightarrow L^2(\Sigma)$. Then (6.16) is a weak Laplace equation with the inhomogenous term in $W^{-1,r_k}(U, L^2(\Sigma,\mathbb{R}^{2m}))$ and enables us to deduce $hv \in W^{1,r_k}(U, L^2(\Sigma,\mathbb{R}^{2m}))$ and thus $v \in W^{1,r_k}(\tilde{U}, L^2(\Sigma,\mathbb{R}^{2m}))$ with the corresponding estimates for some smaller domain $\tilde{U}$ (a finite union of these still covers a neighbourhood of $\Omega\cap\partial\mathbb{H}$). Furthermore, recall that $v \in W^{1,q_k}(U, L^{s_k}(\Sigma,\mathbb{R}^{2m}))$. Now we claim that the following inclusion with the corresponding estimates holds for some suitable p_{k+1}

$$W^{1,r_k}(\tilde{U}, L^2(\Sigma)) \;\cap\; W^{1,q_k}(\tilde{U}, L^{s_k}(\Sigma)) \;\subset\; W^{1,p_{k+1}}(\tilde{U}, L^{p_{k+1}}(\Sigma)). \tag{6.17}$$

To show (6.17) it suffices to estimate the $L^{p_{k+1}}(\tilde{U} \times \Sigma)$-norm of a smooth function by its $L^{r_k}(\tilde{U}, L^2(\Sigma))$- and $L^{q_k}(\tilde{U}, L^{s_k}(\Sigma))$-norms. Let $\alpha > 2$ and $t \in [1, 2)$, then the Hölder inequality gives for all $f \in \mathcal{C}^\infty(\tilde{U} \times \Sigma, \mathbb{R}^{2m})$

$$\begin{aligned}
\|f\|^\alpha_{L^\alpha(\tilde{U}\times\Sigma)} &= \int_{\tilde{U}} \int_\Sigma |f|^t |f|^{\alpha-t} \\
&\le \int_{\tilde{U}} \|f\|^t_{L^2(\Sigma)} \|f\|^{\alpha-t}_{L^{2\frac{\alpha-t}{2-t}}(\Sigma)} \\
&\le \|f\|^t_{L^r(\tilde{U},L^2(\Sigma))} \|f\|^{\alpha-t}_{L^{r\frac{\alpha-t}{r-t}}(\tilde{U},L^{2\frac{\alpha-t}{2-t}}(\Sigma))} \\
&\le \|f\|^\alpha_{L^r(\tilde{U},L^2(\Sigma))} + \|f\|^\alpha_{L^{r\frac{\alpha-t}{r-t}}(\tilde{U},L^{2\frac{\alpha-t}{2-t}}(\Sigma))}.
\end{aligned}$$

Here we abbreviated $r := r_k > p_k > 2$. Now we want

$$q_k = \frac{r_k(\alpha - t)}{r_k - t} \qquad \text{and} \qquad s_k = \frac{2(\alpha - t)}{2 - t}. \tag{6.18}$$

Indeed, in the case $p_k = 4$ our choices $q_k = 3$, $r_k = 12$, and $s_k = \frac{31}{2}$ together with $t := \frac{5}{3}$ and $\alpha := \frac{17}{4}$ solve these equations. So we obtain $p_{k+1} = \alpha = \frac{17}{16} p_k$. In case $p_k < 4$ the first equation gives

$$\alpha = \frac{4 + t}{8 - p_k} p_k. \tag{6.19}$$

If moreover $p_k \ge \frac{8}{3}$, then we choose $t := \frac{5}{3}$ to obtain $\alpha = \frac{17}{24-3p_k} p_k \ge \frac{17}{16} p_k$. This also solves (6.18) with our choice $s_k = \frac{44p_k - 80}{8 - p_k}$, so we obtain $p_{k+1} = \frac{17}{16} p_k$. Finally, in case $\frac{8}{3} > p_k > 2$ one obtains from (6.18)

$$t \;=\; \frac{p_k^2}{-p_k^2 + 7p_k - 8} \;\in\; [1, 2).$$

Inserting this in (6.19) yields $\alpha = \theta(p_k) \cdot p_k$ with

$$\theta(p_k) = \frac{3p_k - 4}{-p_k^2 + 7p_k - 8}.$$

One then checks that $\theta(2) = 1$ and $\theta'(p) > 0$ for $p > 2$, thus $\theta(p) > 1$ for $p > 2$. Moreover, $\theta(\frac{8}{3}) = \frac{9}{8}$, so $\theta(p') = \frac{17}{16}$ for some $p' \in (2, \frac{8}{3})$. Now for $p \geq p'$ we extend the function constantly to obtain a monotonely increasing function $\theta : (2, 4] \to (1, \frac{17}{16}]$. With this modified function we finally choose $p_{k+1} = \theta(p_k) \cdot p_k$ for all $2 < p_k \leq 4$. This finishes the proof of (6.17) and thus shows that $v \in W^{1,p_{k+1}}(\tilde{U}, L^{p_{k+1}}(\Sigma))$.

In addition, note that our choice of $p_{k+1} \leq \alpha$ will always satisfy $p_{k+1} \leq r_k$. In case $p_k = 4$ see the actual numbers, in case $p_k < 4$ this is due to (6.19), $t \leq 2$, and $p_k > 2$,

$$\alpha \;\leq\; \frac{6}{8 - p_k} p_k \;\leq\; \frac{2}{4 - p_k} p_k \;=\; r_k.$$

Now we again use the special structure of the coordinates Θ in lemma 4.1 (iii) to deduce that $B = \Theta \circ v \in W^{1,p_{k+1}}(\tilde{U}, \mathcal{A}^{0,p_{k+1}}(\Sigma))$ with the corresponding estimates. Above, we already established the W^{1,r_k}- and thus $W^{1,p_{k+1}}$-regularity and -estimates for Φ and Ψ as well as $B \in L^{p_{k+1}}(\tilde{U}, \mathcal{A}^{1,p_{k+1}}(\Sigma))$. (Recall the Sobolev embedding $W^{1,q_k} \hookrightarrow L^{r_k}$, and that $p_k \geq q_k$ and $r_k \geq p_{k+1}$, so we have $L^{r_k}(\tilde{U}, L^{r_k}(\Sigma))$-regularity of B as well as $\nabla_\Sigma B$.) Putting all this together we have established the $W^{1,p_{k+1}}$-regularity and -estimates for $\bar{\tau}^* A$ over $\tilde{U}_i \times \Sigma$, where the $\tilde{U}_i$ cover a neighbourhood of $\Omega_{k+1} \cap \partial\mathbb{H}$. The interior regularity again follows directly from proposition 6.9.

This iteration gives a sequence (p_k) with $p_{k+1} = \theta(p_k) \cdot p_k \geq \theta(p) \cdot p_k$. So this sequence grows at a rate greater or equal to $\theta(p) > \theta(2) = 1$ and hence reaches $p_N > 4$ after finitely many steps. This finishes the proof of II) and the theorem. □

Proof of theorem 6.1 :
Fix a solution $A \in \mathcal{A}^{1,p}_{\text{loc}}(X)$ of (6.1) with $p > 2$. We have to find a gauge transformation $u \in \mathcal{G}^{2,p}_{\text{loc}}(X)$ such that $u^* A \in \mathcal{A}(X)$ is smooth. Recall that the manifold $X = \bigcup_{k \in \mathbb{N}} X_k$ is exhausted by compact submanifolds X_k meeting the assumptions of proposition 6.3. So it suffices to prove for every $k \in \mathbb{N}$ that there exists a gauge transformation $u \in \mathcal{G}^{2,p}(X_k)$ such that $u^* A|_{X_k}$ is smooth.

For that purpose fix $k \in \mathbb{N}$ and choose a compact submanifold $M \subset X$ that is large enough such that theorem 6.8 applies to the compact subset $K := X_k \subset M$. Next, choose $A_0 \in \mathcal{A}(M)$ such that $\|A - A_0\|_{W^{1,p}(M)}$ and $\|A - A_0\|_{L^q(M)}$ are sufficiently small for the local slice theorem, proposition 6.4, to apply to A_0 with the reference connection $\hat{A} = A$. Here due to $p > 2$ one can choose $q > 4$ in the local slice theorem such that the Sobolev embedding $W^{1,p}(M) \hookrightarrow L^q(M)$ holds. Then by proposition 6.4 and remark 6.5 (i) one obtains a gauge transformation $u \in \mathcal{G}^{2,p}(M)$ such that u^*A is in relative Coulomb gauge with respect to A_0. Moreover, u^*A also solves (6.1) since both the anti-self-duality equation and the Lagrangian submanifolds $\mathcal{L}_i$ are gauge invariant. The latter is due to the fact that u restricts to a gauge transformation in $\mathcal{G}^{1,p}(\Sigma_i)$ on each boundary slice $\tau_i(\{s\} \times \Sigma_i)$ due to the Sobolev embedding $\mathcal{G}^{2,p}(\mathcal{U}_i \times \Sigma) \subset W^{1,p}(\mathcal{U}_i, \mathcal{G}^{1,p}(\Sigma_i)) \hookrightarrow \mathcal{C}^0(\mathcal{U}_i, \mathcal{G}^{1,p}(\Sigma_i))$. So $u^*A \in \mathcal{A}^{1,p}(M)$ is a solution of (6.2) and theorem 6.8 (i) asserts that $u^*A|_{X_k} \in \mathcal{A}(X_k)$ is indeed smooth.

Such a gauge transformation $u \in \mathcal{G}^{2,p}(X_k)$ can be found for every $k \in \mathbb{N}$, hence proposition 6.3 (i) asserts that there exists a gauge transformation $u \in \mathcal{G}^{2,p}_{\mathrm{loc}}(X)$ on the full noncompact manifold such that $u^*A \in \mathcal{A}(X)$ is smooth as claimed. □

Proof of theorem 6.2 :
Fix a smoothly convergent sequence of metrics $g^\nu \to g$ that are compatible to τ and let $A^\nu \in \mathcal{A}^{1,p}_{\mathrm{loc}}(X)$ be a sequence of solutions of (6.1) with respect to the metrics g^ν. Recall that the manifold $X = \bigcup_{k\in\mathbb{N}} X_k$ is exhausted by compact submanifolds X_k meeting the assumptions of proposition 6.3. We will find a subsequence (again denoted A^ν) and a sequence of gauge transformations $u^\nu \in \mathcal{G}^{2,p}_{\mathrm{loc}}(X)$ such that the sequence ${u^\nu}^* A^\nu$ is bounded in the $W^{\ell,p}$-norm on X_k for all $\ell \in \mathbb{N}$ and $k \in \mathbb{N}$. Then due to the compact Sobolev embeddings $W^{\ell,p}(X_k) \hookrightarrow \mathcal{C}^{\ell-2}(X_k)$ one finds a further (diagonal) subsequence that converges uniformly with all derivatives on every compact subset of X.

By proposition 6.3 (ii) it suffices to construct the gauge transformations and establish the claimed uniform bounds over X_k for all $k \in \mathbb{N}$ and for any subsequence of the connections (again denoted A^ν). So fix $k \in \mathbb{N}$ and choose a compact submanifold $M \subset X$ such that theorem 6.8 holds with $K = X_k \subset M$. Choose a further compact submanifold $M' \subset X$ such that theorem 6.8 holds with $K = M \subset M'$. Then by assumption of the theorem $\|F_{A^\nu}\|_{L^p(M')}$ is uniformly bounded. So the weak Uhlenbeck compactness theorem, proposition 6.6, provides a subsequence (still denoted A^ν), a limit connection $A_0 \in \mathcal{A}^{1,p}(M')$, and gauge transformations $u^\nu \in \mathcal{G}^{2,p}(M')$ such that ${u^\nu}^* A^\nu \to A_0$ in the weak $W^{1,p}$-topology. The limit A_0 then satisfies the boundary value problem (6.1) with respect to the limit metric g. So as in the proof of theorem 6.1 one finds a gauge transformation $u_0 \in \mathcal{G}^{2,p}(M)$ such that $u_0^*A_0 \in \mathcal{A}(M)$ is smooth. Now replace A_0 by $u_0^*A_0$ and u^ν by $u^\nu u_0 \in \mathcal{G}^{2,p}(M)$, then one still has a $W^{1,p}$-bound, $\|u^{\nu\,*}A^\nu - A_0\|_{W^{1,p}(M)} \leq c_0$ for some constant c_0, see lemma A.5.

Due to $p > 2$ one can now choose $q > 4$ in the local slice theorem such that the Sobolev embedding $W^{1,p}(M) \hookrightarrow L^q(M)$ is compact. Hence for a further subsequence of the connections $u^{\nu\,*}A^\nu \to A_0$ in the L^q-norm. Let $\varepsilon > 0$ be the constant from proposition 6.4 for the reference connection $\hat{A} = A_0$, then one finds a further subsequence such that $\|u^{\nu\,*}A^\nu - A_0\|_{L^q(M)} \leq \varepsilon$ for all $\nu \in \mathbb{N}$. So the local slice theorem provides further gauge transformations $\tilde{u}^\nu \in \mathcal{G}^{2,p}(M)$ such that the $\tilde{u}^{\nu\,*}A^\nu$ are in relative Coulomb gauge with respect to A_0. The gauge transformed connections still solve (6.1), hence the $\tilde{u}^{\nu\,*}A^\nu$ are solutions of (6.2). Moreover, we have $\|\tilde{u}^{\nu\,*}A^\nu - A_0\|_q \leq C_{CG}\|u^{\nu\,*}A^\nu - A_0\|_q$, hence $\tilde{u}^{\nu\,*}A^\nu \to A_0$ in the L^q-norm, and

$$\left\|\tilde{u}^{\nu\,*}A^\nu - A_0\right\|_{W^{1,p}(M)} \leq C_{CG}c_0.$$

The higher $W^{k,p}$-bounds will now follow from theorem 6.8, so we first have to verify its assumptions. Fix the metric $g_0 := g$ and a compact neighbourhood $\mathcal{V} = \bigcup_{i=1}^n \bar{\tau}_{0,i}(\mathcal{U}_i \times \Sigma_i)$ of $K \cap \partial X$. Then the $\bar{\tau}_{0,i}^*(\tilde{u}^{\nu\,*}A^\nu - A_0)|_{\Sigma_i}$ are uniformly $W^{1,p}$-bounded and converge to zero in the L^q-norm on $\mathcal{U}_i \times \mathcal{S}_i$ as seen above. Now the embedding

$$W^{1,p}(\mathcal{U}_i \times \Sigma_i, \mathrm{T}^*\Sigma_i \otimes \mathfrak{g}) \hookrightarrow L^\infty(\mathcal{U}_i, \mathcal{A}^{0,p}(\Sigma_i))$$

is compact by lemma 6.7. Thus one finds a subsequence such that the $\bar{\tau}_{0,i}^*(\tilde{u}^{\nu\,*}A^\nu)|_{\Sigma_i}$ converge in $L^\infty(\mathcal{U}_i, \mathcal{A}^{0,p}(\Sigma_i))$. The limit can only be $\bar{\tau}_{0,i}^*A_0|_{\Sigma_i}$ since this already is the L^q-limit. Now in theorem 6.8 (ii) choose the constant $C_1 = C_{CG}c_0$ and let $\delta > 0$ be the constant determined from C_1. Then one can take a subsequence such that

$$\|\bar{\tau}_{0,i}^*(\tilde{u}^{\nu\,*}A^\nu - A_0)_{\Sigma_i}\|_{L^\infty(\mathcal{U}_i, \mathcal{A}^{0,p}(\Sigma_i))} \leq \delta \qquad \forall i = 1, \ldots, n, \forall \nu.$$

Now theorem 6.8 (ii) provides the claimed uniform bounds as follows. Fix $\ell \in \mathbb{N}$, then $\|g^\nu - g\|_{\mathcal{C}^{\ell+2}(M)} \leq \delta_\ell$ for all $\nu \geq \nu_\ell$ with some $\nu_\ell \in \mathbb{N}$, and thus

$$\left\|\tilde{u}^{\nu\,*}A^\nu - A_0\right\|_{W^{\ell,p}(X_k)} \leq C_\ell \qquad \forall \nu \geq \nu_\ell.$$

This finally implies the uniform bound for this subsequence,

$$\sup_{\nu \in \mathbb{N}}\left\|\tilde{u}^{\nu\,*}A^\nu\right\|_{W^{\ell,p}(X_k)} < \infty.$$

Here the gauge transformations $\tilde{u}^\nu \in \mathcal{G}^{2,p}(X_k)$ still depend on $k \in \mathbb{N}$ and are only defined on X_k. But now proposition 6.3 (ii) provides a subsequence of (A^ν) and gauge transformations $u^\nu \in \mathcal{G}^{2,p}_{\mathrm{loc}}(X)$ defined on the full noncompact manifold such that $u^{\nu\,*}A^\nu$ is uniformly bounded in every $W^{\ell,p}$-norm on every compact submanifold X_k. Now one can iteratively use the compact Sobolev embeddings $W^{\ell+2,p}(X_\ell) \hookrightarrow \mathcal{C}^\ell(X_\ell)$ for each $\ell \in \mathbb{N}$ to find a further subsequence of the connections that converges in $\mathcal{C}^\ell(X_\ell)$. If in each step one fixes one further element of the sequence, then this iteration finally yields a sequence of connections that converges uniformly with all derivatives on every compact subset of X to a smooth connection $A \in \mathcal{A}(X)$. $\Box$

Chapter 7

Fredholm theory

This chapter concerns the linearization of the boundary value problem (1.2) in the special case of a compact 4-manifold of the form $X = S^1 \times Y$, where Y is a compact orientable 3-manifold whose boundary $\partial Y = \Sigma$ is a disjoint union of connected Riemann surfaces. The aim of this chapter is to prove theorem C. The parts (i), (ii), and (iii) of theorem C are restated and proven below as theorem 7.1, lemma 7.2, and lemma 7.3.

So we equip $S^1 \times Y$ with a product metric $\tilde{g} = \mathrm{d}s^2 + g_s$ (where g_s is an S^1-family of metrics on Y) and assume that this is compatible with the natural space-time splitting of the boundary $\partial X = S^1 \times \Sigma$. This means that for some $\Delta > 0$ there exists an embedding

$$\tau : S^1 \times [0, \Delta) \times \Sigma \hookrightarrow S^1 \times Y$$

preserving the boundary, $\tau(s, 0, z) = (s, z)$ for all $s \in S^1$ and $z \in \Sigma$, such that

$$\tau^* \tilde{g} = \mathrm{d}s^2 + \mathrm{d}t^2 + g_{s,t}.$$

Here $g_{s,t}$ is a smooth family of metrics on Σ. This assumption on the metric implies that the normal geodesics are independent of $s \in S^1$ in a neighbourhood of the boundary. So in fact, the embedding is given by $\tau(s, t, z) = (s, \gamma_z(t))$, where γ is the normal geodesic starting at $z \in \Sigma$. This seems like a very restrictive assumption, but it suffices for our application to Riemannian 4-manifolds with a boundary space-time splitting. Indeed, the neighbourhoods of the compact boundary components are isometric to $S^1 \times Y$ with $Y = [0, \Delta] \times \Sigma$ and a metric $\mathrm{d}s^2 + \mathrm{d}t^2 + g_{s,t}$.

Now fix $p > 2$ and let $\mathcal{L} \subset \mathcal{A}^{0,p}_{\mathrm{flat}}(\Sigma)$ be a gauge invariant Lagrangian submanifold of $\mathcal{A}^{0,p}(\Sigma)$ as in chapter 4. Then for $\tilde{A} \in \mathcal{A}^{1,p}(S^1 \times Y)$ we consider the boundary value problem

$$\begin{cases} *F_{\tilde{A}} + F_{\tilde{A}} = 0, \\ \tilde{A}|_{\{s\}\times\partial Y} \in \mathcal{L} \quad \forall s \in S^1. \end{cases} \tag{7.1}$$

Fix a smooth connection $\tilde{A} \in \mathcal{A}(S^1 \times Y)$ with Lagrangian boundary values (but not necessarily a solution of this boundary value problem). It can be decomposed as $\tilde{A} = A + \Phi \mathrm{d}s$ with $\Phi \in \mathcal{C}^\infty(S^1 \times Y, \mathfrak{g})$ and with $A \in \mathcal{C}^\infty(S^1 \times Y, \mathrm{T}^*Y \otimes \mathfrak{g})$ satisfying $A_s := A(s)|_{\partial Y} \in \mathcal{L}$ for all $s \in S^1$. Similarly, a tangent vector $\tilde{\alpha}$ to $\mathcal{A}^{1,p}(S^1 \times Y)$ decomposes as $\tilde{\alpha} = \alpha + \varphi \mathrm{d}s$ with $\varphi \in W^{1,p}(S^1 \times Y, \mathfrak{g})$ and $\alpha \in W^{1,p}(S^1 \times Y, \mathrm{T}^*Y \otimes \mathfrak{g})$. With respect to this splitting the linearization of the Lagrangian boundary condition in (7.1) at $\tilde{A}$ is $\alpha(s)|_{\partial Y} \in \mathrm{T}_{A_s}\mathcal{L}$ for all $s \in S^1$. Moreover, the linearization of the anti-self-duality equation $F_{\tilde{A}}^+ = \frac{1}{2}(F_{\tilde{A}} + *F_{\tilde{A}}) = 0$ at $\tilde{A}$ can be expressed as

$$\mathrm{d}_{\tilde{A}}^+\tilde{\alpha} \;=\; \tfrac{1}{2} * \bigl(\nabla_s\alpha - \mathrm{d}_A\varphi + *\mathrm{d}_A\alpha\bigr) - \tfrac{1}{2}\bigl(\nabla_s\alpha - \mathrm{d}_A\varphi + *\mathrm{d}_A\alpha\bigr) \wedge \mathrm{d}s \;=\; 0.$$

Here d_A denotes the exterior derivative corresponding to the connection $A(s)$ on Y for all $s \in S^1$, $*$ denotes the Hodge operator on Y with respect to the s-dependent metric g_s on Y, and we use the notation $\nabla_s\alpha := \partial_s\alpha + [\Phi, \alpha]$. Now the linearized operator for the boundary value problem (7.1) has to be augmented with the local slice condition at $\tilde{A}$, i.e. the condition that the tangent vectors $\tilde{\alpha}$ to the space of connections at $\tilde{A}$ lie in a complement of the gauge orbit through $\tilde{A}$. This complement is fixed by the Coulomb gauge conditions

$$\mathrm{d}_{\tilde{A}}^*\tilde{\alpha} \;=\; -\nabla_s\varphi + \mathrm{d}_A^*\alpha \;=\; 0 \qquad \text{and} \qquad * \,\tilde{\alpha}|_{S^1\times\partial Y} \;=\; - * \alpha|_{\partial Y} \wedge \mathrm{d}s \;=\; 0.$$

Now let $E_A^{1,p} \subset W^{1,p}(S^1 \times Y, \mathrm{T}^*Y \otimes \mathfrak{g})$ be the subspace of S^1-families of 1-forms α that satisfy the above boundary conditions from the linearization of (7.1) and the Coulomb gauge,

$$*\alpha(s)|_{\partial Y} = 0 \qquad \text{and} \qquad \alpha(s)|_{\partial Y} \in \mathrm{T}_{A_s}\mathcal{L} \qquad \text{for all } s \in S^1.$$

Then the linearized operator for the study of the moduli space of gauge equivalence classes of solutions of (7.1) is

$$D_{(A,\Phi)} : E_A^{1,p} \times W^{1,p}(S^1 \times Y, \mathfrak{g}) \longrightarrow L^p(S^1 \times Y, \mathrm{T}^*Y \otimes \mathfrak{g}) \times L^p(S^1 \times Y, \mathfrak{g})$$

given by

$$D_{(A,\Phi)}(\alpha, \varphi) \;=\; \bigl(\nabla_s\alpha - \mathrm{d}_A\varphi + *\mathrm{d}_A\alpha \,,\, \nabla_s\varphi - \mathrm{d}_A^*\alpha\bigr).$$

Our main result, theorem C (i), is the following.

Theorem 7.1 *Let $p > 2$ and assume that $A + \Phi \mathrm{d}s \in \mathcal{A}(S^1 \times Y)$ satisfies the boundary condition $A(s)|_{\partial Y} \in \mathcal{L}$ for all $s \in S^1$. Then $D_{(A,\Phi)}$ is a Fredholm operator.*

We now give an outline of the proof of this theorem. The first crucial point of this Fredholm theory is the following estimate, theorem C (ii), which ensures that $D_{(A,\Phi)}$ has a closed image and a finite dimensional kernel.

Lemma 7.2 *There is a constant C such that for all $\tilde{\alpha} \in W^{1,p}(X, \mathrm{T}^*X \otimes \mathfrak{g})$ satisfying*

$$*\tilde{\alpha}|_{\partial X} = 0 \qquad \text{and} \qquad \tilde{\alpha}|_{\{s\}\times\partial Y} \in \mathrm{T}_{A_s}\mathcal{L} \quad \forall s \in S^1$$

one has the estimate

$$\|\tilde{\alpha}\|_{W^{1,p}} \leq C\big(\|\mathrm{d}^+_{\tilde{A}}\tilde{\alpha}\|_p + \|\mathrm{d}^*_{\tilde{A}}\tilde{\alpha}\|_p + \|\tilde{\alpha}\|_p\big).$$

Postponing the proof we first note that by the above calculations the estimate in this lemma is equivalent to the following estimate for all $\alpha \in E^{1,p}_A$ and $\varphi \in W^{1,p}(S^1 \times Y, \mathfrak{g})$

$$\begin{aligned} \|(\alpha,\varphi)\|_{W^{1,p}} &= \|\alpha\|_{W^{1,p}} + \|\varphi\|_{W^{1,p}} \\ &\leq C\big(\|\nabla_s\alpha - \mathrm{d}_A\varphi + *\mathrm{d}_A\alpha\|_p + \|\nabla_s\varphi - \mathrm{d}^*_A\alpha\|_p + \|\alpha\|_p + \|\varphi\|_p\big) \\ &= C\big(\|D_{(A,\Phi)}(\alpha,\varphi)\|_p + \|(\alpha,\varphi)\|_p\big). \end{aligned}$$

The second part of the proof of theorem 7.1 is the identification of the cokernel of the operator with the kernel of a slightly modified linearized operator, which will be used to prove that the cokernel is finite dimensional. To be more precise let $\sigma : S^1 \times Y \to S^1 \times Y$ denote the reflection given by $\sigma(s,y) := (-s,y)$, where $S^1 \cong \mathbb{R}/\mathbb{Z}$. Then we will establish the following duality:

$$(\beta,\zeta) \in (\operatorname{im} D_{(A,\Phi)})^\perp \iff (\beta\circ\sigma, \zeta\circ\sigma) \in \ker D_{\sigma^*(A,\Phi)},$$

where $D_{\sigma^*(A,\Phi)}$ is the linearized operator at the connection $\sigma^*\tilde{A} = A\circ\sigma - \Phi\circ\sigma$ with respect to the metric $\sigma^*\tilde{g}$ on $S^1 \times Y$. Once we know that $\operatorname{im} D_{(A,\Phi)}$ is closed, this gives an isomorphism between $(\operatorname{coker} D_{(A,\Phi)})^* \cong (\operatorname{im} D_{(A,\Phi)})^\perp$ and

$\ker D_{\sigma^*(A,\Phi)}$. Here Z^* denotes the dual space of a Banach space Z, and for a subspace $Y \subset Z$ we denote by $Y^\perp \subset Z^*$ the space of linear functionals that vanish on Y. Now the estimate in lemma 7.2 will also apply to $D_{\sigma^*(A,\Phi)}$, and this implies that its kernel – and hence the cokernel of $D_{(A,\Phi)}$ – is of finite dimension. The main difficulty in establishing the above duality is the following regularity result, theorem C (iii). As before we shall use the notation $\frac{1}{p} + \frac{1}{p^*} = 1$.

Lemma 7.3 *Let $q \geq p^*$ such that $q \neq 2$. Let $\beta \in L^q(S^1 \times Y, \mathrm{T}^*Y \otimes \mathfrak{g})$, $\zeta \in L^q(S^1 \times Y, \mathfrak{g})$, and suppose that there exists a constant C such that for all $\alpha \in E^{1,p}_A$ and $\varphi \in W^{1,p}(S^1 \times Y, \mathfrak{g})$*

$$\left| \int_{S^1 \times Y} \langle\, D_{(A,\Phi)}(\alpha, \varphi)\, ,\, (\beta, \zeta)\, \rangle \right| \;\leq\; C\, \|(\alpha, \varphi)\|_{q^*}.$$

Then $\beta \in W^{1,q}(S^1 \times Y, \mathrm{T}^*Y \otimes \mathfrak{g})$ and $\zeta \in W^{1,q}(S^1 \times Y, \mathfrak{g})$.

This regularity as well as the estimate in lemma 7.2 will be proven analogously to the nonlinear regularity and estimates in chapter 6. Again, the interior regularity and estimate is standard elliptic theory, and one has to use a splitting near the boundary. We shall show that the S^1- and the normal component both satisfy a Laplace equation with Neumann and Dirichlet boundary conditions respectively. The Σ-component will again gives rise to a (weak) Cauchy-Riemann equation in a Banach space, only this time the boundary values will lie in the tangent space of the Lagrangian. In contrast to the required L^p-estimates we shall first show that the L^2-estimate for L^p-regular 1-forms can be obtained by more elementary methods. These were already outlined in [Sa1] as a first indication for the Fredholm property of the boundary value problem (7.1).

Let $\tilde\alpha \in W^{1,p}(X, \mathrm{T}^*X \otimes \mathfrak{g})$ be as supposed for some $p > 2$. From the first boundary condition $*\tilde\alpha|_{\partial X} = 0$ one obtains

$$\|\nabla\tilde\alpha\|_2^2 = \|\mathrm{d}\tilde\alpha\|_2^2 + \|\mathrm{d}^*\tilde\alpha\|_2^2 - \int_{\partial X} \tilde g(Y_{\tilde\alpha}, \nabla_{Y_{\tilde\alpha}}\nu).$$

Here one has $\int_{\partial X} \tilde g(Y_{\tilde\alpha}, \nabla_{Y_{\tilde\alpha}}\nu) \geq -C\|\tilde\alpha\|^2_{L^2(\partial X)}$ since the vector field $Y_{\tilde\alpha}$ is given by $\iota_{Y_{\tilde\alpha}}\tilde g = \tilde\alpha$. In this last term one uses the following estimate for general $1 < p < \infty$.

Let $\tau : [0,\Delta)\times\partial X \to X$ be a diffeomorphism to a tubular neighbourhood of ∂X in X. Then for all $\delta > 0$ one finds a constant C_δ such that for all $f \in W^{1,p}(X)$

$$\begin{aligned}
&\|f\|^p_{L^p(\partial X)} \\
&= \int_{\partial X}\int_0^1 \frac{\mathrm{d}}{\mathrm{d}s}\Bigl((s-1)|f(\tau(s,z))|^p\Bigr)\,\mathrm{d}s\,\mathrm{d}^3z \\
&\leq \int_{\partial X}\int_0^1 |f(\tau(s,z))|^p\,\mathrm{d}s\,\mathrm{d}^3z + \int_{\partial X}\int_0^1 p|f(\tau(s,z))|^{p-1}|\partial_s f(\tau(s,z))|\,\mathrm{d}s\,\mathrm{d}^3z \\
&\leq C\bigl(\|f\|^p_{L^p(X)} + \|f\|^{p-1}_{L^p(X)}\|\nabla f\|_{L^p(X)}\bigr) \\
&\leq \bigl(\delta\|f\|_{W^{1,p}(X)} + C_\delta\|f\|_{L^p(X)}\bigr)^p. \qquad (7.2)
\end{aligned}$$

This uses the fact that for all $x, y \geq 0$ and $\delta > 0$

$$x^{p-1}y \;\leq\; \left\{\begin{array}{ll} \delta^p y^p & ; x \leq \delta^{\frac{p}{p-1}}y \\ \delta^{-\frac{p}{p-1}}x^p & ; x \geq \delta^{\frac{p}{p-1}}y \end{array}\right\} \;\leq\; \bigl(\delta y + \delta^{-\frac{1}{p-1}}x\bigr)^p.$$

So we obtain

$$\|\tilde\alpha\|_{W^{1,2}} \leq C\bigl(\|\mathrm{d}_{\tilde A}\tilde\alpha\|_2 + \|\mathrm{d}^*_{\tilde A}\tilde\alpha\|_2 + \|\tilde\alpha\|_2\bigr). \qquad (7.3)$$

In fact, the analogous $W^{1,p}$-estimates hold true for general p, as is proven e.g. in [We, Theorem 6.1]. However, in the case $p = 2$ one can calculate further for all $\delta > 0$

$$\begin{aligned}
\|\mathrm{d}_{\tilde A}\tilde\alpha\|_2^2 &= \int_X \langle\, \mathrm{d}_{\tilde A}\tilde\alpha\,,\, 2\mathrm{d}^+_{\tilde A}\tilde\alpha\,\rangle - \int_X \langle\, \mathrm{d}_{\tilde A}\tilde\alpha \wedge \mathrm{d}_{\tilde A}\tilde\alpha\,\rangle \\
&= 2\|\mathrm{d}^+_{\tilde A}\tilde\alpha\|_2^2 - \int_X \langle\, \tilde\alpha \wedge [F_{\tilde A}\wedge\tilde\alpha]\,\rangle - \int_{\partial X}\langle\, \tilde\alpha\wedge \mathrm{d}_{\tilde A}\tilde\alpha\,\rangle \\
&\leq 2\|\mathrm{d}^+_{\tilde A}\tilde\alpha\|_2^2 + C_\delta\|\tilde\alpha\|_2^2 + \delta\|\tilde\alpha\|^2_{W^{1,2}}. \qquad (7.4)
\end{aligned}$$

Here the boundary term above is estimated as follows. We use the universal covering of $S^1 = \mathbb{R}/\mathbb{Z}$ to integrate over $[0,1]\times\partial Y$ instead of $\partial X = S^1\times\partial Y$. Introduce $A := (A_s)_{s\in S^1}$, which is a smooth path in $\mathcal{L}$. Then using the splitting $\tilde\alpha|_{\partial X} = \alpha + \varphi\mathrm{d}s$ with $\alpha : S^1\times\Sigma \to \mathrm{T}^*\Sigma\otimes\mathfrak{g}$ and $\varphi : S^1\times\Sigma\to\mathfrak{g}$

one obtains

$$\begin{aligned}
&-\int_{\partial X}\langle\,\tilde\alpha\wedge \mathrm{d}_{\tilde A}\tilde\alpha\,\rangle \\
&= -\int_0^1\int_\Sigma \langle\,\varphi\,,\,\mathrm{d}_A\alpha\,\rangle\,\mathrm{dvol}_\Sigma\wedge\mathrm{d}s\; -\int_0^1\int_\Sigma\langle\,\alpha\wedge(\mathrm{d}_A\varphi-\nabla_s\alpha)\,\rangle\wedge\mathrm{d}s \\
&= \int_0^1\int_\Sigma\langle\,\alpha\wedge\nabla_s\alpha\,\rangle\wedge\mathrm{d}s \\
&\le \delta\|\tilde\alpha\|^2_{W^{1,2}(X)}+C'_\delta\|\tilde\alpha\|^2_{L^2(X)}.
\end{aligned}$$

Firstly, we have used the fact that $\mathrm{d}_A\alpha|_\Sigma=0$ since $\alpha(s)\in\mathrm{T}_{A_s}\mathcal{L}\subset\ker\mathrm{d}_{A_s}$ for all $s\in S^1$. Secondly, we have also used that both α and $\mathrm{d}_A\varphi$ lie in $\mathrm{T}_A\mathcal{L}$, hence the symplectic form $\int_\Sigma\langle\,\alpha\wedge\mathrm{d}_A\varphi\,\rangle$ vanishes for all $s\in S^1$. This is not strictly true since $\tilde\alpha$ only restricts to an L^p-regular 1-form on ∂X. However, as 1-form on $[0,1]\times Y$ it can be approximated as follows by smooth 1-forms that meet the Lagrangian boundary condition on $[0,1]\times\Sigma$.

We use the linearization of the coordinates in lemma 4.1 (iii) at A_s for all $s\in[0,1]$. Since the path $s\mapsto A_s\in\mathcal{L}\cap\mathcal{A}(\Sigma)$ is smooth, this gives a smooth path of diffeomorphisms Θ_s for any $q>2$,

$$\Theta_s:\;\begin{array}{ccc} Z\times Z & \longrightarrow & L^q(\Sigma,\mathrm{T}^*\Sigma\otimes\mathfrak{g}) \\ (\xi,v,\zeta,w) & \longmapsto & \mathrm{d}_{A_s}\xi+\sum_{i=1}^m v^i\gamma_i(s)+*\mathrm{d}_{A_s}\zeta+\sum_{i=1}^m w^i*\gamma_i(s), \end{array}$$

where $Z:=W^{1,q}_z(\Sigma,\mathfrak{g})\times\mathbb{R}^m$ and the $\gamma_i\in\mathcal{C}^\infty([0,1]\times\Sigma,\mathrm{T}^*\Sigma\otimes\mathfrak{g})$ satisfy $\gamma_i(s)\in\mathrm{T}_{A_s}\mathcal{L}$ for all $s\in[0,1]$. We perform above estimate on $[0,1]\times Y$ since we can not necessarily achieve $\gamma_i(0)=\gamma_i(1)$. In these coordinates, we mollify to obtain the required smooth approximations of $\tilde\alpha$ near the boundary. Furthermore, we use these coordinates for $q=3$ to write the smooth approximations on the boundary as $\alpha(s)=\mathrm{d}_{A_s}\xi(s)+\sum_{i=1}^m v^i(s)\gamma_i(s)$ with $\|\xi(s)\|_{W^{1,3}(\Sigma)}+|v(s)|\le C\|\alpha(s)\|_{L^3(\Sigma)}$. Then for all $s\in[0,1]$

$$\begin{aligned}
\int_\Sigma\langle\,\alpha(s)\wedge\nabla_s\alpha(s)\,\rangle &= \int_\Sigma\langle\,\alpha\wedge\big(\mathrm{d}_{A_s}\partial_s\xi+\textstyle\sum_{i=1}^m\partial_s v^i\cdot\gamma_i\big)\,\rangle \\
&\quad+\int_\Sigma\langle\,\alpha\wedge\big([\Phi,\alpha]+[\partial_sA,\xi]+\textstyle\sum_{i=1}^m v^i\cdot\partial_s\gamma_i\big)\,\rangle \\
&\le C\|\alpha(s)\|_{L^2(\Sigma)}\|\alpha(s)\|_{L^3(\Sigma)}.
\end{aligned}$$

Here the crucial point is that $\mathrm{d}_A\partial_s\xi$ and $\partial_s v^i\cdot\gamma_i$ are tangent to the Lagrangian, hence the first term vanishes. Now one uses (7.2) for $p=2$ and the Sobolev trace theorem (the restriction $W^{1,2}(X) \to L^3(\partial X)$ is continuous by e.g. [Ad, Theorem 6.2]) to obtain the estimate,

$$\begin{aligned}\int_0^1 \int_\Sigma \langle\, \alpha \wedge \nabla_s \alpha \,\rangle \wedge \mathrm{d}s &\leq C\|\tilde{\alpha}\|_{L^2(\partial X)}\|\tilde{\alpha}\|_{L^3(\partial X)} \\ &\leq \tfrac{\delta}{2}\|\tilde{\alpha}\|^2_{W^{1,2}(X)} + C_\delta\|\tilde{\alpha}\|_{L^2(X)}\|\tilde{\alpha}\|_{W^{1,2}(X)} \\ &\leq \delta\|\tilde{\alpha}\|^2_{W^{1,2}(X)} + C'_\delta\|\tilde{\alpha}\|^2_{L^2(X)}.\end{aligned}$$

This proves (7.4). Now $\delta > 0$ can be chosen arbitrarily small, so the term $\|\tilde{\alpha}\|_{W^{1,2}}$ can be absorbed into the left hand side of (7.3), and thus one obtains the claimed estimate

$$\|\tilde{\alpha}\|_{W^{1,2}} \;\leq\; C\big(\|\mathrm{d}^+_{\tilde{A}}\tilde{\alpha}\|_2 + \|\mathrm{d}^*_{\tilde{A}}\tilde{\alpha}\|_2 + \|\tilde{\alpha}\|_2\big).$$

Proof of lemma 7.2 :
We will use lemma 3.3 for the manifold $M := S^1 \times Y$ in several different cases to obtain the estimate for different components of $\tilde{\alpha}$. The first weak equation in lemma 3.3 is the same in all cases. For all $\eta \in \mathcal{C}^\infty(M;\mathfrak{g})$

$$\begin{aligned}\int_M \langle\, \tilde{\alpha} \,,\, \mathrm{d}\eta \,\rangle &= \int_M \langle\, \mathrm{d}^*\tilde{\alpha} \,,\, \eta \,\rangle + \int_{\partial M} \langle\, \eta \,,\, *\tilde{\alpha} \,\rangle \\ &= \int_M \langle\, \mathrm{d}^*_{\tilde{A}}\tilde{\alpha} + *[\tilde{A} \wedge *\tilde{\alpha}] \,,\, \eta \,\rangle \quad = \int_M \langle\, f \,,\, \eta \,\rangle.\end{aligned}$$

Here one uses the fact that $*\tilde{\alpha}|_{\partial M} = 0$. Then $f \in L^p(M,\mathfrak{g})$ and

$$\|f\|_p \leq \|\mathrm{d}^*_{\tilde{A}}\tilde{\alpha}\|_p + 2\|\tilde{A}\|_\infty\|\tilde{\alpha}\|_p. \tag{7.5}$$

For the second weak equation lemma 3.3 we obtain for all $\lambda \in \Omega^1(M;\mathfrak{g})$

$$\begin{aligned}\int_M \langle\, \tilde{\alpha} \,,\, \mathrm{d}^*\mathrm{d}\lambda \,\rangle &= \int_M \langle\, \tilde{\alpha} \,,\, \mathrm{d}^*\mathrm{d}\lambda + \mathrm{d}^* * \mathrm{d}\lambda \,\rangle \qquad (7.6) \\ &= \int_M \langle\, \gamma \,,\, \mathrm{d}\lambda \,\rangle - \int_{S^1\times\partial Y} \langle\, \tilde{\alpha} \wedge *\mathrm{d}\lambda \,\rangle - \int_{S^1\times\partial Y} \langle\, \tilde{\alpha} \wedge \mathrm{d}\lambda \,\rangle,\end{aligned}$$

where $\gamma \;=\; \mathrm{d}\tilde{\alpha} + *\mathrm{d}\tilde{\alpha} \;=\; 2\mathrm{d}^+_{\tilde{A}}\tilde{\alpha} - 2[\tilde{A} \wedge \tilde{\alpha}]^+ \;\in L^p(M, \Lambda^2\mathrm{T}^*M \otimes \mathfrak{g})$ and

$$\|\gamma\|_p \leq 2\|\mathrm{d}^+_{\tilde{A}}\tilde{\alpha}\|_p + 4\|\tilde{A}\|_\infty\|\tilde{\alpha}\|_p. \tag{7.7}$$

Now recall that there is an embedding $\tau : S^1 \times [0,\Delta) \times \Sigma \hookrightarrow S^1 \times Y$ to a tubular neighbourhood of $S^1 \times \partial Y$ such that $\tau^*\tilde{g} = \mathrm{d}s^2 + \mathrm{d}t^2 + g_{s,t}$ for a family

$g_{s,t}$ of metrics on Σ. One can then cover $M = S^1 \times Y$ with $\tau(S^1 \times [0, \frac{\Delta}{2}] \times \Sigma)$ and a compact subset $V \subset M \setminus \partial M$.

For the claimed estimate of $\tilde{\alpha}$ over V it suffices to use lemma 3.3 for vector fields $X \in \Gamma(\mathrm{T}M)$ that equal to coordinate vector fields on V and vanish on ∂M. So one has to consider (7.6) for $\lambda = \phi \cdot \iota_X \tilde{g}$ with $\phi \in \mathcal{C}^\infty_\delta(M, \mathfrak{g})$. Then both boundary terms vanish and hence lemma 3.3 directly asserts, with some constants C and C_V, that

$$\begin{aligned} \|\tilde{\alpha}\|_{W^{1,p}(V)} &\leq C\big(\|f\|_{L^p(M)} + \|\gamma\|_{L^p(M)} + \|\tilde{\alpha}\|_{L^p(M)}\big) \\ &\leq C_V\big(\|\mathrm{d}^+_{\tilde{A}}\tilde{\alpha}\|_{L^p(M)} + \|\mathrm{d}^*_{\tilde{A}}\tilde{\alpha}\|_{L^p(M)} + \|\tilde{\alpha}\|_{L^p(M)}\big). \end{aligned}$$

So it remains to prove the estimate for $\tilde{\alpha}$ near the boundary $\partial M = S^1 \times \Sigma$. For that purpose we can use the decomposition $\tau^*\tilde{\alpha} = \varphi \mathrm{d}s + \psi \mathrm{d}t + \alpha$, where $\varphi, \psi \in W^{1,p}(S^1 \times [0, \Delta) \times \Sigma, \mathfrak{g})$ and $\alpha \in W^{1,p}(S^1 \times [0, \Delta) \times \Sigma, \mathrm{T}^*\Sigma \otimes \mathfrak{g})$. Let $\Omega := S^1 \times [0, \frac{3}{4}\Delta]$ and let $K := S^1 \times [0, \frac{\Delta}{2}]$. Then we will prove the estimate for φ and ψ on $\Omega \times \Sigma$ and for α on $K \times \Sigma$.

Firstly, note that $\psi = \tilde{\alpha}(\tau_*\partial_t) \circ \tau$, where $-\tau_*\partial_t|_{\partial M} = \nu$ is the outer unit normal to ∂M. So one can cut off $\tau_*\partial_t$ outside of $\tau(\Omega \times \Sigma)$ to obtain a vector field $X \in \Gamma(\mathrm{T}M)$ that satisfies the assumption of lemma 3.3, that is $X|_{\partial M} = -\nu$ is perpendicular to the boundary. Then one has to test (7.6) with $\lambda = \phi \cdot \iota_X \tilde{g}$ for all $\phi \in \mathcal{C}^\infty_\delta(M, \mathfrak{g})$. Again both boundary terms vanish. Indeed, on $S^1 \times \partial Y$ we have $\phi \equiv 0$ and $\iota_X \tilde{g} = \tau_*\mathrm{d}t$, hence $\mathrm{d}\lambda|_{\mathbb{R}\times\partial Y} = 0$ and $*\mathrm{d}\lambda|_{\mathbb{R}\times\partial Y} = -\frac{\partial\phi}{\partial\nu} * \tau_*(\mathrm{d}t \wedge \mathrm{d}t) = 0$. Thus lemma 3.3 yields the following estimate.

$$\begin{aligned} \|\psi\|_{W^{1,p}(\Omega\times\Sigma)} &\leq C\|\tilde{\alpha}(X)\|_{W^{1,p}(M)} \\ &\leq C\big(\|f\|_{L^p(M)} + \|\gamma\|_{L^p(M)} + \|\tilde{\alpha}\|_{L^p(M)}\big) \\ &\leq C_t\big(\|\mathrm{d}^+_{\tilde{A}}\tilde{\alpha}\|_{L^p(M)} + \|\mathrm{d}^*_{\tilde{A}}\tilde{\alpha}\|_{L^p(M)} + \|\tilde{\alpha}\|_{L^p(M)}\big). \end{aligned}$$

Here C denotes any finite constant and the bounds on the derivatives of τ enter into the constant C_t.

Next, for the regularity of $\varphi = \tilde{\alpha}(\partial_s) \circ \tau$ one can apply lemma 3.3 with the tangential vector field $X = \partial_s$. Recall that τ preserves the S^1-coordinate. One has to verify the second weak equation for all $\phi \in \mathcal{C}^\infty_\nu(M, \mathfrak{g})$, i.e. consider (7.6) for $\lambda = \phi \cdot \iota_X \tilde{g} = \phi \cdot \mathrm{d}s$. The first boundary term vanishes since one has $*\mathrm{d}\lambda|_{S^1\times\partial Y} = -\frac{\partial\phi}{\partial\nu}\,\mathrm{dvol}_{\partial Y} = 0$. For the second term one can choose any $\delta > 0$

and then finds a constant C_δ such that for all $\phi \in \mathcal{C}^\infty_\nu(M,\mathfrak{g})$

$$\begin{aligned}\left|\int_{S^1\times\partial Y}\langle\,\tilde\alpha\wedge \mathrm{d}\lambda\,\rangle\right| &= \left|\int_{S^1}\int_\Sigma\langle\,\alpha(s,0)\wedge \mathrm{d}_\Sigma(\phi\circ\tau)(s,0)\,\rangle\wedge \mathrm{d}s\right|\\ &= \left|\int_{S^1\times\partial Y}\langle\,\tilde\alpha\wedge[\tilde A,\phi]\,\rangle\wedge \mathrm{d}s\right|\\ &\le \|\tilde\alpha\|_{L^p(\partial M)}\|\tilde A\|_\infty\|\phi\|_{L^{p^*}(\partial M)}\\ &\le \big(\delta\|\tilde\alpha\|_{W^{1,p}(M)}+C_\delta\|\tilde\alpha\|_{L^p(M)}\big)\|\phi\|_{W^{1,p^*}(M)}.\end{aligned}$$

This uses the fact that $\alpha(s,0)$ and $\mathrm{d}_{A_s}(\phi\circ\tau)|_{(s,0)\times\Sigma}$ both lie in the tangent space $\mathrm{T}_{A_s}\mathcal{L}$ to the Lagrangian, on which the symplectic form vanishes, that is $\int_\Sigma\langle\,\alpha\wedge \mathrm{d}_A(\phi\circ\tau)\,\rangle = 0$. Moreover, we have used the trace theorem for Sobolev spaces, in particular the estimate (7.2). Now lemma 3.3 and remark 3.4 yield with $c_1 = \|f\|_p$, $c_2 = \|\gamma\|_{L^p(M)} + \delta\|\tilde\alpha\|_{W^{1,p}(M)} + C_\delta\|\tilde\alpha\|_{L^p(M)}$, and using (7.5), (7.7)

$$\begin{aligned}&\|\varphi\|_{W^{1,p}(\Omega\times\Sigma)}\\ &\le C\big(\|f\|_{L^p(M)}+c_2+\|\tilde\alpha\|_{L^p(M)}\big)\\ &\le \delta\|\tilde\alpha\|_{W^{1,p}(M)}+C_s(\delta)\big(\|\mathrm{d}^+_{\tilde A}\tilde\alpha\|_{L^p(M)}+\|\mathrm{d}^*_{\tilde A}\tilde\alpha\|_{L^p(M)}+\|\tilde\alpha\|_{L^p(M)}\big).\end{aligned}$$

Here again $\delta > 0$ can be chosen arbitrarily small and the constant $C_s(\delta)$ depends on this choice.

It remains to establish the estimate for the Σ-component α near the boundary. In the coordinates τ on $\Omega\times\Sigma$, the forms $\mathrm{d}^*_{\tilde A}\tilde\alpha$ and $\mathrm{d}^+_{\tilde A}\tilde\alpha$ become

$$\begin{aligned}\tau^*\mathrm{d}^*_{\tilde A}\tilde\alpha &= -\partial_s\varphi-\partial_t\psi+\mathrm{d}^*_\Sigma\alpha-\tau^*(*[\tilde A\wedge *\tilde\alpha]),\\ \tau^*\mathrm{d}^+_{\tilde A}\tilde\alpha &= \tfrac12\big(-(\partial_s\alpha+*_\Sigma\partial_t\alpha)\wedge \mathrm{d}s+*_\Sigma(\partial_s\alpha+*_\Sigma\partial_t\alpha)\wedge \mathrm{d}t\big)\\ &\qquad+\tfrac12\big(\mathrm{d}_\Sigma\alpha+(*_\Sigma \mathrm{d}_\Sigma\alpha)\mathrm{d}s\wedge \mathrm{d}t\big)+\tau^*([\tilde A\wedge\tilde\alpha]^+).\end{aligned}$$

So one obtains the following bounds: The components in the mixed direction of Ω and Σ of the second equation yields for some constant C_1

$$\begin{aligned}\|\partial_s\alpha+*_\Sigma\partial_t\alpha\|_{L^p(\Omega\times\Sigma)} &\le \big\|\tau^*\mathrm{d}^+_{\tilde A}\tilde\alpha\big\|_{L^p(\Omega\times\Sigma)}+\big\|\tau^*([\tilde A\wedge\tilde\alpha]^+)\big\|_{L^p(\Omega\times\Sigma)}\\ &\le C_1\big(\|\mathrm{d}^+_{\tilde A}\tilde\alpha\|_{L^p(M)}+\|\tilde\alpha\|_{L^p(M)}\big).\end{aligned}$$

Similarly, a combination of the first equation and the Σ-component of the second equation can be used for every $\delta > 0$ to find a constant $C_2(\delta)$ such that

$$\begin{aligned}
&\|\mathrm{d}_\Sigma\alpha\|_{L^p(\Omega\times\Sigma)} + \|\mathrm{d}_\Sigma^*\alpha\|_{L^p(\Omega\times\Sigma)} \\
&\leq C\big(\|\mathrm{d}_{\tilde{A}}^+\tilde{\alpha}\|_{L^p(M)} + \|\mathrm{d}_{\tilde{A}}^*\tilde{\alpha}\|_{L^p(M)} + \|\tilde{\alpha}\|_{L^p(M)} \\
&\qquad\qquad\qquad + \|\varphi\|_{W^{1,p}(\Omega\times\Sigma)} + \|\psi\|_{W^{1,p}(\Omega\times\Sigma)}\big) \\
&\leq \delta\|\tilde{\alpha}\|_{W^{1,p}(M)} + C_2(\delta)\big(\|\mathrm{d}_{\tilde{A}}^+\tilde{\alpha}\|_{L^p(M)} + \|\mathrm{d}_{\tilde{A}}^*\tilde{\alpha}\|_{L^p(M)} + \|\tilde{\alpha}\|_{M)}\big).
\end{aligned}$$

Now firstly, lemma 6.11 provides an L^p-estimate for the derivatives of α in Σ-direction,

$$\begin{aligned}
&\|\nabla_\Sigma\alpha\|_{L^p(\Omega\times\Sigma)} \\
&\leq C\big(\|\mathrm{d}_\Sigma\alpha\|_{L^p(\Omega\times\Sigma)} + \|\mathrm{d}_\Sigma^*\alpha\|_{L^p(\Omega\times\Sigma)} + \|\alpha\|_{L^p(\Omega\times\Sigma)}\big) \\
&\leq \delta\|\tilde{\alpha}\|_{W^{1,p}(M)} + C_\Sigma(\delta)\big(\|\mathrm{d}_{\tilde{A}}^+\tilde{\alpha}\|_{L^p(M)} + \|\mathrm{d}_{\tilde{A}}^*\tilde{\alpha}\|_{L^p(M)} + \|\tilde{\alpha}\|_{L^p(M)}\big),
\end{aligned}$$

where again $C_\Sigma(\delta)$ depends on the choice of $\delta > 0$. For the derivatives in s- and t-direction, we will now apply theorem 5.4 on the Banach space $X = L^p(\Sigma, \mathrm{T}^*\Sigma\otimes\mathfrak{g})$ with the complex structure $*_\Sigma$ determined by the metric $g_{s,t}$ on Σ and hence depending smoothly on $(s,t) \in \Omega$. The Lagrangian and hence totally real submanifold $\mathcal{L} \subset X$ is modelled on a Banach space $Z = W_z^{1,p}(\Sigma, \mathfrak{g})\oplus\mathbb{R}^m$ as seen in lemma 4.1, and this is bounded isomorphic to a closed subspace of $L^p(\Sigma, \mathbb{R}^n)$. Now $\alpha \in W^{1,p}(\Omega, X)$ satisfies the Lagrangian boundary condition $\alpha(s,0) \in \mathrm{T}_{A_s}\mathcal{L}$ for all $s \in S^1$, where $s \mapsto A_s$ is a smooth loop in $\mathcal{L}$. Thus corollary 5.5 yields a constant C such that the following estimate holds,

$$\begin{aligned}
\|\nabla_\Omega\alpha\|_{L^p(K\times\Sigma)} &\leq \|\alpha\|_{W^{1,p}(K,X)} \\
&\leq C\big(\|\partial_s\alpha + *_\Sigma\partial_t\alpha\|_{L^p(\Omega,X)} + \|\alpha\|_{L^p(\Omega,X)}\big) \\
&\leq C_K\big(\|\mathrm{d}_{\tilde{A}}^+\tilde{\alpha}\|_{L^p(M)} + \|\tilde{\alpha}\|_{L^p(M)}\big).
\end{aligned}$$

Here C_K also includes the above constant C_1. Now adding up all the estimates for the different components of $\tilde{\alpha}$ gives for all $\delta > 0$

$$\begin{aligned}
\|\tilde{\alpha}\|_{W^{1,p}} \leq{}& (C_V + C_t + C_s(\delta) + C_\Sigma(\delta) + C_K)\big(\|\mathrm{d}_{\tilde{A}}^+\tilde{\alpha}\|_p + \|\mathrm{d}_{\tilde{A}}^*\tilde{\alpha}\|_p + \|\tilde{\alpha}\|_p\big) \\
&+ 2\delta\|\tilde{\alpha}\|_{W^{1,p}}.
\end{aligned}$$

Finally, choose $\delta = \frac{1}{4}$, then the term $\|\tilde{\alpha}\|_{W^{1,p}}$ can be absorbed into the left hand side and this finishes the proof of the lemma. □

Proof of lemma 7.3 :
Let $\beta \in L^q(S^1 \times Y, \mathrm{T}^*Y \otimes \mathfrak{g})$ and $\zeta \in L^q(S^1 \times Y, \mathfrak{g})$ be as supposed in the lemma. Then there exists a constant C such that for all $\alpha \in E_A^{1,p}$ and $\varphi \in W^{1,p}(S^1 \times Y, \mathfrak{g})$

$$\begin{aligned}
&\left| \int_{S^1} \int_Y \langle\, \nabla_s \alpha - \mathrm{d}_A \varphi + *\mathrm{d}_A \alpha \,,\, \beta \,\rangle \;+ \int_{S^1} \int_Y \langle\, \nabla_s \varphi - \mathrm{d}_A^* \alpha \,,\, \zeta \,\rangle \right| \\
&= \left| \int_{S^1 \times Y} \langle\, D_{(A,\Phi)}(\alpha, \varphi) \,,\, (\beta, \zeta) \,\rangle \right| \\
&\leq C \| (\alpha, \varphi) \|_{q^*}.
\end{aligned} \tag{7.8}$$

The higher regularity of ζ is most easily seen if we go back to the notation $\tilde{\alpha} = \alpha + \varphi \mathrm{d}s$ with $D_{(A,\Phi)}(\alpha, \varphi) = (2\gamma \,,\, -\mathrm{d}_{\tilde{A}}^* \tilde{\alpha})$ for $\mathrm{d}_{\tilde{A}}^+ \tilde{\alpha} = *\gamma - \gamma \wedge \mathrm{d}s$. Abbreviate $M := S^1 \times Y$, then we have for all $\tilde{\alpha} \in \mathcal{C}^\infty(M, \mathrm{T}^*M \otimes \mathfrak{g})$ with $*\tilde{\alpha}|_{\partial M} = 0$ and $\tilde{\alpha}|_{\{s\} \times \partial Y} \in \mathrm{T}_{A_s} \mathcal{L}$ for all $s \in S^1$

$$\left| \int_M \langle\, 2\mathrm{d}_{\tilde{A}}^+ \tilde{\alpha} \,,\, \beta \wedge \mathrm{d}s \,\rangle + \int_M \langle\, \mathrm{d}_{\tilde{A}}^* \tilde{\alpha} \,,\, \zeta \,\rangle \right| \leq C \| \tilde{\alpha} \|_{q^*}.$$

Now use the embedding $\tau : S^1 \times [0, \Delta) \times \Sigma \hookrightarrow M$ to construct a connection $\hat{A} \in \mathcal{A}(M)$ such that $\tau^* \hat{A}(s,t,z) = A_s(z)$ near the boundary (this can be cut off and then extends trivially to all of M). Then $\tilde{\alpha} := \mathrm{d}_{\hat{A}} \phi$ satisfies the above boundary conditions for all $\phi \in \mathcal{C}_\nu^\infty(M, \mathfrak{g})$ since $\mathrm{d}_{\hat{A}} \phi(\nu) = \frac{\partial \phi}{\partial \nu} + [\hat{A}(\nu), \phi] = 0$ and $\mathrm{d}_{\hat{A}} \phi|_{\{s\} \times \partial Y} = \mathrm{d}_{A_s} \phi \in \mathrm{T}_{A_s} \mathcal{L}$ for all $s \in S^1$. Thus we obtain for all $\phi \in \mathcal{C}_\nu^\infty(M, \mathfrak{g})$ in view of $\Delta \phi = \mathrm{d}^*(\tilde{\alpha} - [\hat{A}, \phi])$, denoting all constants by C,

$$\begin{aligned}
&\left| \int_M \langle\, \Delta \phi \,,\, \zeta \,\rangle \right| \\
&= \left| \int_M \langle\, \mathrm{d}_{\tilde{A}}^* \tilde{\alpha} + *[\tilde{A} \wedge *\tilde{\alpha}] - \mathrm{d}^*[\hat{A}, \phi] \,,\, \zeta \,\rangle \right| \\
&\leq C \| \tilde{\alpha} \|_{q^*} + \left| \int_M \langle\, -2\mathrm{d}_{\tilde{A}}^+ \mathrm{d}_{\hat{A}} \phi \,,\, \beta \wedge \mathrm{d}s \,\rangle \right| + \left| \int_M \langle\, *[\tilde{A} \wedge *\mathrm{d}_{\hat{A}} \phi] - \mathrm{d}^*[\hat{A}, \phi] \,,\, \zeta \,\rangle \right| \\
&\leq C \| \phi \|_{W^{1,q^*}}.
\end{aligned}$$

The regularity theory for the Neumann problem, e.g. proposition 3.5, then asserts that $\zeta \in W^{1,q}(M)$.

To deduce the higher regularity of β we will mainly use lemma 3.3. The first weak equation in the lemma is given by choosing $\alpha = 0$ in (7.8). For all $\eta \in \mathcal{C}^\infty(M, \mathfrak{g})$

$$\begin{aligned}\left|\int_M \langle \beta, \mathrm{d}\eta \rangle\right| &= \left|\int_{S^1}\int_Y \langle \beta, \mathrm{d}_A\eta - [A,\eta] \rangle\right| \\ &\le C\|\eta\|_{q^*} + \left|\int_{S^1}\int_Y \langle \nabla_s\zeta, \eta \rangle\right| + \left|\int_{S^1}\int_Y \langle [\beta\wedge *A], \eta \rangle\right| \\ &\le C\|\eta\|_{q^*}.\end{aligned}$$

For the second weak equation let $\varphi = 0$ and $\alpha = *\mathrm{d}\lambda - \partial_s\lambda$ for $\lambda = \phi\cdot\iota_X\tilde{g}$ with $\phi \in \mathcal{T}$ in the function space $\mathcal{C}^\infty_\delta(M,\mathfrak{g})$ or $\mathcal{C}^\infty_\nu(M,\mathfrak{g})$ corresponding to the vector field $X \in \mathcal{C}^\infty(M, \mathrm{T}Y)$. If the boundary conditions for $\alpha \in E^{1,p}_A$ are satisfied, then we obtain with $\mathrm{d} = \mathrm{d}_Y$

$$\begin{aligned}&\left|\int_M \langle \beta, \mathrm{d}_M^*\mathrm{d}_M\lambda \rangle\right| \\ &= \left|\int_{S^1}\int_Y \langle \beta, *\mathrm{d}*\mathrm{d}\lambda - \partial_s^2\lambda - *(\partial_s*)\partial_s\lambda \rangle\right| \\ &= \left|\int_{S^1}\int_Y \langle \beta, *\mathrm{d}_A\alpha - *[A\wedge *\mathrm{d}\lambda] + *\mathrm{d}_A\partial_s\lambda\right. \\ &\qquad\qquad \left. + \nabla_s\alpha - [\Phi,\partial_s\lambda] - \nabla_s*\mathrm{d}\lambda - *(\partial_s*)\partial_s\lambda \rangle\right| \\ &\le C\|\lambda\|_{W^{1,q^*}} + \left|\int_{S^1}\int_Y \langle \zeta, \mathrm{d}_A^*\alpha \rangle\right| + \left|\int_{S^1}\int_Y \langle \beta, *\mathrm{d}_A\partial_s\lambda - \nabla_s*\mathrm{d}\lambda \rangle\right| \\ &\le C\|\phi\|_{W^{1,q^*}}.\end{aligned}$$

Here we have used the identity

$$*\mathrm{d}_A\partial_s\lambda - \nabla_s*\mathrm{d}\lambda = *[A\wedge\partial_s\lambda] - [\Phi, *\mathrm{d}\lambda] - (\partial_s*)\mathrm{d}\lambda.$$

Moreover, we have used partial integration with vanishing boundary term $*\alpha|_{\partial Y} = 0$ to obtain

$$\int_{S^1}\int_Y \langle \zeta, \mathrm{d}_A^*\alpha \rangle = \int_{S^1}\int_Y \langle \mathrm{d}_A\zeta, *\mathrm{d}\lambda - \partial_s\lambda \rangle.$$

Now let $X \in \mathcal{C}^\infty(M, \mathrm{T}Y)$ be perpendicular to the boundary $\partial M = S^1\times\partial Y$, then for all $\phi \in \mathcal{C}^\infty_\delta(M)$ the boundary conditions for $\alpha = *\mathrm{d}\lambda - \partial_s\lambda \in E^{1,p}_A$

are satisfied. Indeed, on the boundary $\partial M = S^1 \times \partial Y$ the 1-form $\lambda = \phi \cdot \iota_X \tilde{g}$ vanishes, we have $\iota_X \tilde{g} = h \cdot \tau_* \mathrm{d}t$ for some smooth function h, and moreover $\mathrm{d}\phi = -\frac{\partial \phi}{\partial \nu} \cdot \tau_* \mathrm{d}t$. Hence

$$\begin{aligned} *\alpha|_{\partial Y} &= \mathrm{d}\lambda|_{\partial Y} - *\partial_s \lambda|_{\partial Y} = 0, \\ \alpha|_{\partial Y} &= *\mathrm{d}\lambda|_{\partial Y} - \partial_s \lambda|_{\partial Y} = -\tfrac{\partial \phi}{\partial \nu} h * (\tau_* \mathrm{d}t \wedge \tau_* \mathrm{d}t) = 0. \end{aligned}$$

Thus for all vector fields $X \in \mathcal{C}^\infty(M, \mathrm{T}Y)$ that are perpendicular to the boundary, lemma 3.3 asserts that $\beta(X) \in W^{1,q}(M, \mathfrak{g})$. In particular, this implies $W^{1,q}$-regularity of β on all compact subsets $K \subset \operatorname{int} M$. So it remains to prove the regularity on the neighbourhood $\tau(S^1 \times [0, \frac{\Delta}{2}) \times \Sigma)$ of the boundary ∂M. In these coordinates we decompose

$$\tau^* \beta = \xi \mathrm{d}t + \hat{\beta}.$$

Now firstly, lemma 3.3 applies as described above to assert the regularity $\xi = \beta(\tau_* \partial_t) \circ \tau \in W^{1,q}(\Omega \times \Sigma, \mathfrak{g})$ on $\Omega := S^1 \times [0, \frac{3}{4}\Delta)$. Here a vector field X that is perpendicular to the boundary is constructed by cutting off $\tau_* \partial_t$ outside of $\tau(\Omega \times \Sigma)$.

So it remains to consider $\hat{\beta} \in L^q(\Omega \times \Sigma, \mathrm{T}^*\Sigma \otimes \mathfrak{g})$ and establish its $W^{1,q}$-regularity. In order to derive a weak inequality for $\hat{\beta}$ on $\Omega \times \Sigma$ from (7.8) we use $\tilde{\alpha} = \tau_*(\varphi \mathrm{d}s + \psi \mathrm{d}t + \hat{\alpha})$ with $\varphi \in \mathcal{C}^\infty_\delta(\Omega \times \Sigma, \mathfrak{g})$, $\psi \in \mathcal{C}^\infty_\delta(\Omega \times \Sigma, \mathfrak{g})$, and $\hat{\alpha} \in W^{1,p}(\Omega \times \Sigma, \mathrm{T}^*\Sigma \otimes \mathfrak{g})$ such that $\hat{\alpha}(s, \frac{3}{4}\Delta, \cdot) = 0$ and $\hat{\alpha}(s, 0, \cdot) \in \mathrm{T}_{A_s}\mathcal{L}$ for all $s \in S^1$. This $\tilde{\alpha}$ satisfies the boundary conditions for (7.8) and it can be extended trivially to a $W^{1,p}$-regular 1-form on all of M. Thus we obtain

$$\begin{aligned} &\left| \int_{\Omega \times \Sigma} \langle\, \nabla_s \hat{\alpha} + *\nabla_t \hat{\alpha} - \mathrm{d}_A \varphi - *\mathrm{d}_A \psi \,,\, \hat{\beta} \,\rangle \right| \\ &\le \left| \int_{\Omega \times \Sigma} \langle\, -\nabla_s \psi + \nabla_t \varphi - *\mathrm{d}_A \hat{\alpha} \,,\, \xi \,\rangle \, + \int_{\Omega \times \Sigma} \langle\, \nabla_s \varphi + \nabla_t \psi - \mathrm{d}_A^* \hat{\alpha} \,,\, \zeta \,\rangle \right| \\ &\quad + C \| \varphi \, \mathrm{d}s + \psi \, \mathrm{d}t + \hat{\alpha} \|_{q^*}. \end{aligned}$$

Here we have introduced the decomposition $\tau^* \tilde{A} = \Phi \mathrm{d}s + \Psi \mathrm{d}t + A$, where $A \in \mathcal{C}^\infty(\Omega \times \Sigma, \mathrm{T}^*\Sigma \otimes \mathfrak{g})$ with $A(s, 0) = A_s \in \mathcal{L}$ for all $s \in S^1$. We have also used the notation $\nabla_t \varphi = \partial_t \varphi + [\Psi, \varphi]$, and moreover d_A and $*$ denote the differential and Hodge operator on Σ. Now firstly put $\hat{\alpha} = 0$, then we obtain for all $\varphi,\psi\in\mathcal{C}^\infty_\delta(\Omega\times\Sigma,\mathfrak{g})$ by partial integration

$$\left|\int_{\Omega\times\Sigma}\langle\,\mathrm{d}_A\varphi\,,\,\hat\beta\,\rangle\right|\le\bigl(C+\|\nabla_t\xi-\nabla_s\zeta\|_{L^q(\Omega\times\Sigma)}\bigr)\|\varphi\|_{L^{q^*}(\Omega\times\Sigma)},$$
$$\left|\int_{\Omega\times\Sigma}\langle\,*\mathrm{d}_A\psi\,,\,\hat\beta\,\rangle\right|\le\bigl(C+\|\nabla_s\xi+\nabla_t\zeta\|_{L^q(\Omega\times\Sigma)}\bigr)\|\psi\|_{L^{q^*}(\Omega\times\Sigma)}.$$

This shows that the weak derivatives $\mathrm{d}_\Sigma^*\hat\beta$ and $\mathrm{d}_\Sigma\hat\beta$ are of class L^q on $\Omega\times\Sigma$. So we have verified the assumptions of lemma 6.11 for $\hat\beta$ and conclude that $\nabla_\Sigma\hat\beta$ also is of class L^q. So it remains to deduce the L^q-regularity of $\partial_s\hat\beta$ and $\partial_t\hat\beta$ on $S^1\times[0,\frac{\Delta}{2}]\times\Sigma$ from the above inequality for $\varphi=\psi=0$, namely from

$$\left|\int_{\Omega\times\Sigma}\langle\,\nabla_s\hat\alpha+*\nabla_t\hat\alpha\,,\,\hat\beta\,\rangle\right|\le\bigl(C+\|\mathrm{d}_A\zeta+*\mathrm{d}_A\xi\|_{L^q(\Omega\times\Sigma)}\bigr)\|\hat\alpha\|_{L^{q^*}(\Omega\times\Sigma)}.\quad(7.9)$$

This holds for all $\hat\alpha\in W^{1,p}(\Omega\times\Sigma,\mathrm{T}^*\Sigma\otimes\mathfrak{g})$ such that $\hat\alpha(s,\frac{3}{4}\Delta,\cdot)=0$ and $\hat\alpha(s,0,\cdot)\in\mathrm{T}_{A_s}\mathcal{L}$ for all $s\in S^1$. We now have to employ different arguments according to whether $q>2$ or $q<2$.

Case q $>$ 2 :

In this case the regularity of $\partial_s\hat\beta$ and $\partial_t\hat\beta$ will follow from theorem 5.4 on the Banach space $X=L^q(\Sigma,\mathrm{T}^*\Sigma\otimes\mathfrak{g})$ with the complex structure given by the Hodge operator on Σ with respect to the metric $g_{s,t}$. From (7.9) one obtains the following estimate for some constant C and all $\hat\alpha$ as above:

$$\left|\int_\Omega\int_\Sigma\langle\,\hat\beta\,,\,\partial_s\hat\alpha+\partial_t(*\hat\alpha)\,\rangle\right|\le C\|\hat\alpha\|_{L^{q^*}(\Omega,X^*)}.\qquad(7.10)$$

Note that this extends to the $W^{1,q^*}(\Omega,L^{q^*}(\Sigma))$-closure of the admissible $\hat\alpha$ from above. In particular the estimate above holds for all $\hat\alpha\in W^{1,q}(\Omega,X)$ that are supported in Ω and satisfy $\hat\alpha(s,0,\cdot)\in\mathrm{T}_{A_s}\mathcal{L}$ for all $s\in S^1$. To see that these can be approximated by smooth $\hat\alpha$ with Lagrangian boundary conditions one uses the Banach submanifold coordinates for $\mathcal{L}$ given by lemma 4.1 as before. Here the Lagrangian $\mathcal{L}\subset X$ is the L^q-restriction or -completion of the original Lagrangian in $\mathcal{A}^{0,p}(\Sigma)$. It is modelled on $W^{1,q}_z(\Sigma,\mathfrak{g})\times\mathbb{R}^m$ as seen in chapter 4. However, in order to be able to apply theorem 5.4 (i), we need to extend this estimate to all $\hat\alpha\in\mathcal{C}^\infty(\Omega,X^*)$ with $\operatorname{supp}\alpha\subset\Omega$ and $\alpha(s,0)\in(*\mathrm{T}_{A_s}\mathcal{L})^\perp$ for all $s\in S^1$. This is possible since any such $\hat\alpha$ can be approximated in $W^{1,q^*}(\Omega,X^*)$ by $\hat\alpha_i\in\mathcal{C}^\infty(\Omega,X)$ that are compactly supported in Ω and satisfy the above stronger boundary condition $\hat\alpha_i(s,0)\in \mathrm{T}_{A_s}\mathcal{L}$ for all $s\in S^1$.

Indeed, lemma 5.8 provides such an approximating sequence α_i without the Lagrangian boundary conditions. From the proof via mollifiers one sees that the approximating sequence can be chosen with compact support in Ω. Now for all $s\in S^1$ one has the topological splitting $X=\mathrm{T}_{A_s}\mathcal{L}\oplus *\mathrm{T}_{A_s}\mathcal{L}$ and thus $X^*=(\mathrm{T}_{A_s}\mathcal{L})^\perp\oplus(*\mathrm{T}_{A_s}\mathcal{L})^\perp$. Since $q>2$ the embedding $X\hookrightarrow X^*$ iThens continuous. This identification uses the L^2-inner product on X which equals the metric $\omega(\cdot,*\cdot)$ given by the symplectic form ω and the complex structure $*$. So due to the Lagrangian condition this embedding maps $\mathrm{T}_{A_s}\mathcal{L}\hookrightarrow(*\mathrm{T}_{A_s}\mathcal{L})^\perp$ and $*\mathrm{T}_{A_s}\mathcal{L}\hookrightarrow(\mathrm{T}_{A_s}\mathcal{L})^\perp$. We write $\hat\alpha=\gamma+\delta$ and $\alpha_i=\gamma_i+\delta_i$ according to these splittings to obtain $\gamma,\delta\in\mathcal{C}^\infty(\Omega,X^*)$ and $\gamma_i,\delta_i\in\mathcal{C}^\infty(\Omega,X)$ such that $*\mathrm{T}_A\mathcal{L}\ni\gamma_i\to\gamma\in(\mathrm{T}_A\mathcal{L})^\perp$ and $\mathrm{T}_A\mathcal{L}\ni\delta_i\to\delta\in(*\mathrm{T}_A\mathcal{L})^\perp$ with convergence in $W^{1,q^*}(\Omega,X^*)$. The boundary condition on $\hat\alpha$ gives $\gamma|_{t=0}\equiv 0$. Moreover, $\partial_t\gamma$ is uniformly bounded in X^*, so one can find a constant C such that $\|\gamma(s,t)\|_{X^*}\le Ct$ for all $t\in[0,\frac34\Delta]$ and hence for sufficiently small $\varepsilon>0$

$$\|\gamma\|_{L^{q^*}(S^1\times[0,\varepsilon],X^*)}\le \tfrac{C}{1+q^*}\varepsilon^{1+\frac{1}{q^*}}.$$

Now let $\delta>0$ be given and choose $1>\varepsilon>0$ such that $\|\gamma\|_{L^{q^*}(S^1\times[0,\varepsilon],X^*)}\le\varepsilon\delta$ and $\|\gamma\|_{W^{1,q^*}(S^1\times[0,\varepsilon],X^*)}\le\delta$. Next, choose $i\in\mathbb{N}$ sufficiently large such that $\|\gamma_i-\gamma\|_{W^{1,q^*}(\Omega,X^*)}\le\varepsilon\delta$, and let $h\in\mathcal{C}^\infty([0,\frac34\Delta],[0,1])$ be a cutoff function with $h(0)=0$, $h|_{t\ge\varepsilon}\equiv 0$, and $|h'|\le\frac{2}{\varepsilon}$. Then $\hat\alpha_i:=h\gamma_i+\delta_i\in\mathcal{C}^\infty(\Omega,X)$ satisfies the Lagrangian boundary condition $\hat\alpha_i(s,0)\in\mathrm{T}_{A_s}\mathcal{L}$ and approximates $\hat\alpha$ in view of the following estimate,

$$\begin{aligned}\|\hat\alpha_i-\hat\alpha\|_{W^{1,q^*}(\Omega,X^*)}&\le\|h(\gamma_i-\gamma)\|_{W^{1,q^*}(\Omega,X^*)}+\|(1-h)\gamma\|_{W^{1,q^*}(\Omega,X^*)}\\&\le\|\gamma_i-\gamma\|_{W^{1,q^*}(\Omega,X^*)}+\tfrac{2}{\varepsilon}\|\gamma_i-\gamma\|_{L^{q^*}(\Omega,X^*)}\\&\qquad+\|\gamma\|_{W^{1,q^*}(S^1\times[0,\varepsilon],X^*)}+\tfrac{2}{\varepsilon}\|\gamma\|_{L^{q^*}(S^1\times[0,\varepsilon],X^*)}\\&\le 6\delta.\end{aligned}$$

This approximation shows that (7.10) holds indeed true for all $\hat\alpha\in\mathcal{C}^\infty(\Omega,X^*)$ with $\operatorname{supp}\alpha\subset\Omega$ and $\alpha(s,0)\in(*\mathrm{T}_{A_s}\mathcal{L})^\perp$ for all $s\in S^1$. Thus theorem 5.4 asserts that $\hat\beta\in W^{1,q}(K,X)$ for $K:=S^1\times[0,\frac{\Delta}{2}]$, and hence $\partial_s\hat\beta$ and $\partial_t\hat\beta$ are of class L^q on $S^1\times[0,\frac{\Delta}{2}]\times\Sigma$ as claimed.

Case q < 2 :
In this case we cover S^1 by two intervals, $S^1 = I_1 \cup I_2$ such that there are isometric embeddings $(0,1) \hookrightarrow S^1$ identifying $[\frac{1}{4}, \frac{3}{4}]$ with I_1 and I_2 respectively. Abbreviate $K := [\frac{1}{4}, \frac{3}{4}] \times [0, \frac{\Delta}{2}]$ and let $\Omega' \subset (0,1) \times [0, \frac{3}{4}\Delta]$ be a compact submanifold of the half space $\mathbb{H}$ such that $K \subset \operatorname{int}\Omega'$. Then for each of the above identifications $S^1 \setminus \{pt\} \cong (0,1)$ one has L^q-regularity of $\hat\beta$ on $\Omega' \times \Sigma$ by assumption and of $*\mathrm{d}_A\xi + \mathrm{d}_A\zeta$ from above. Now the task is to establish in both cases the L^q-regularity of $\partial_s\hat\beta$ and $\partial_t\hat\beta$ on $K \times \Sigma$ using (7.9). For that purpose choose a cutoff function $h \in \mathcal{C}^\infty(\mathbb{H}, [0,1])$ supported in Ω' such that $h|_K \equiv 1$. Then it suffices to find a constant C such that for all $\gamma \in \mathcal{C}_0^\infty(\Omega' \times \Sigma, \mathrm{T}^*\Sigma \otimes \mathfrak{g})$ (these are compactly supported in $\operatorname{int}(\Omega') \times \Sigma$)

$$\left| \int_{\Omega' \times \Sigma} \langle\, \partial_s\gamma \,,\, h\hat\beta \,\rangle \right| \leq C\|\gamma\|_{q^*}.$$

This gives L^q-regularity of the weak derivative $\partial_s(h\hat\beta)$ and hence of $\partial_s\hat\beta$ on $K \times \Sigma$. For the regularity of $\partial_t\hat\beta$ one has to replace $\partial_s\gamma$ by $\partial_t\gamma$, then the argument is the same as the following argument for $\partial_s\hat\beta$.

We linearize the submanifold chart maps along $(A_s)_{s\in(0,1)} \in \mathcal{L} \cap \mathcal{A}(\Sigma)$ given by lemma 4.1 (iii) for the Lagrangian $\mathcal{L} \subset \mathcal{A}^{0,q^*}(\Sigma)$. Note that this uses the L^{q^*}-completion of the actual Lagrangian in $\mathcal{A}^{0,p}(\Sigma)$. Abbreviate $Z := W_z^{1,q^*}(\Sigma, \mathfrak{g}) \times \mathbb{R}^m$ and let $*_{s,t}$ denote the Hodge operator on Σ with respect to the metric $g_{s,t}$. Then one obtains a smooth family of bounded isomorphisms

$$\Theta_{s,t} :\; Z \times Z \xrightarrow{\sim} L^{q^*}(\Sigma, \mathrm{T}^*\Sigma \otimes \mathfrak{g}) \;=:\; X$$

defined for all $(s,t) \in \Omega'$ by

$$\Theta_{s,t}(\xi, v, \zeta, w) \;=\; \mathrm{d}_{A_s}\xi + \textstyle\sum_{i=1}^m v^i\gamma_i(s) + *_{s,t}\mathrm{d}_{A_s}\zeta + \sum_{i=1}^m w^i *_{s,t} \gamma_i(s).$$

Here $\gamma_i \in \mathcal{C}^\infty((0,1) \times \Sigma, \mathrm{T}^*\Sigma \otimes \mathfrak{g})$ with $\gamma_i(s) \in \mathrm{T}_{A_s}\mathcal{L}$ for all $s \in (0,1)$. Abbreviate $Z^\infty := \mathcal{C}_z^\infty(\Sigma, \mathfrak{g}) \times \mathbb{R}^m \subset Z$, then $\Theta_{s,t}$ maps $Z^\infty \times Z^\infty$ into the set of smooth 1-forms $\Omega^1(\Sigma, \mathfrak{g})$. So given any $\gamma \in \mathcal{C}_0^\infty(\Omega' \times \Sigma, \mathrm{T}^*\Sigma \otimes \mathfrak{g})$ we have $f := \Theta^{-1} \circ \gamma \in \mathcal{C}_0^\infty(\Omega', Z^\infty \times Z^\infty)$ and for some constant C

$$\|f\|_{L^{q^*}(\Omega', Z\times Z)} \;\leq\; C\|\gamma\|_{L^{q^*}(\Omega', X)} \;=\; C\|\gamma\|_{L^{q^*}(\Omega'\times\Sigma)}.$$

Write $f = (f_1, f_2)$ with $f_i \in \mathcal{C}_0^\infty(\Omega', Z^\infty)$ and note that $\int_{\Omega'} \partial_s f_1 = 0$ due to the compact support. So one can solve $\Delta_{\Omega'}\phi_1 = \partial_s f_1$ by $\phi_1 \in \mathcal{C}_\nu^\infty(\Omega', Z^\infty)$

with $\int_{\Omega'} \phi_1 = 0$ and $\Delta_{\Omega'}\phi_2 = \partial_s f_2$ by $\phi_2 \in \mathcal{C}^\infty_\delta(\Omega', Z^\infty)$. (For the $\mathcal{C}^\infty_z(\Sigma,\mathfrak{g})$-component of Z^∞ one has solutions of the Laplace equation on every $\Omega'\times\{x\}$ that depend smoothly on $x \in \Sigma$.) Now let $\Phi := (\phi_1, \phi_2) \in \mathcal{C}^\infty(\Omega', Z \times Z)$ and consider the 1-form

$$\hat\alpha_\gamma \;:=\; h \cdot \Theta(-\partial_s\Phi + J_0\partial_t\Phi) \;\in\; \mathcal{C}^\infty(\Omega', X).$$

This extends to a 1-form on $\Omega \times \Sigma$ that is admissible in (7.9). Indeed, $\hat\alpha_\gamma$ vanishes for s close to 0 or 1 and thus trivially extends to $s \in S^1$. The Lagrangian boundary condition is met since for all $s \in S^1$

$$\hat\alpha_\gamma(s,0) = h(s,0)\cdot\Theta_{s,0}(-\partial_s\phi_1 - \partial_t\phi_2, -\partial_s\phi_2 + \partial_t\phi_1) \in \Theta_{s,0}(Z,0) = \mathrm{T}_{A_s}\mathcal{L}.$$

So (7.9) provides a constant C such that for all $\hat\alpha_\gamma$ of the above form

$$\left| \int_{\Omega'\times\Sigma} \langle\, \hat\beta\,,\, \partial_s\hat\alpha_\gamma + \partial_t(*\hat\alpha_\gamma)\,\rangle \right| \;\le\; C\|\hat\alpha_\gamma\|_{L^{q^*}(\Omega,X)}$$

Moreover, one has for all $\gamma \in \mathcal{C}^\infty_0(\Omega'\times\Sigma, \mathrm{T}^*\Sigma\otimes\mathfrak{g})$ and the associated f, Φ and $\hat\alpha_\gamma$ and denoting all constants by C

$$\|\hat\alpha_\gamma\|_{L^{q^*}(\Omega,X)} \;\le\; C\|\Phi\|_{W^{1,q^*}(\Omega',Z\times Z)} \;\le\; C\|f\|_{L^{q^*}(\Omega',Z\times Z)} \;\le\; C\|\gamma\|_{L^{q^*}(\Omega'\times\Sigma)}.$$

Here the second inequality follows from $\Delta_{\Omega'}\Phi = \partial_s f$ and lemma 5.7 (iii) and (iv) as follows. In the $\mathbb{R}^m$-component of Z, this is the usual elliptic estimate for the Dirichlet or Neumann problem; for the components in the infinite dimensional part $Y := W^{1,q^*}_z(\Sigma,\mathfrak{g})$ of Z (still denoted by ϕ_i and f_i) this uses the following estimate. For all $\psi \in \mathcal{C}^\infty_\nu(\Omega', Y^*)$ in the case $i=1$ and for all $\psi \in \mathcal{C}^\infty_\delta(\Omega', Y^*)$ in the case $i=2$

$$\begin{aligned}
\left|\int_{\Omega'\times\Sigma} \langle\, \phi_i\,,\, \Delta_{\Omega'}\psi\,\rangle\right| \;&=\; \left|\int_{\Omega'\times\Sigma} \langle\, \Delta_{\Omega'}\phi_i\,,\, \psi\,\rangle\right| \;=\; \left|\int_{\Omega'\times\Sigma} \langle\, \partial_s f_i\,,\, \psi\,\rangle\right| \\
&=\; \left|\int_{\Omega'\times\Sigma} \langle\, f_i\,,\, \partial_s\psi\,\rangle\right| \;\le\; \|f_i\|_{L^{q^*}(\Omega',Y)}\|\psi\|_{W^{1,q}(\Omega',Y^*)}.
\end{aligned}$$

Now a calculation shows that

$$\partial_s\hat\alpha_\gamma + \partial_t(*\hat\alpha_\gamma) = h\cdot\Theta(\Delta\Phi) + \partial_s(h\cdot\Theta)(-\partial_s\Phi + J_0\partial_t\Phi) + \partial_t(h\cdot\Theta)(\partial_t\Phi - J_0\partial_s\Phi).$$

We then use $\Delta\Phi = \partial_s f$ to obtain, denoting all constants by C,

$$\begin{aligned}
&\left| \int_{\Omega'\times\Sigma} \langle\, h\cdot\hat{\beta}\,,\, \partial_s\gamma\,\rangle \right| \\
&= \left| \int_{\Omega'\times\Sigma} \langle\, \hat{\beta}\,,\, h\cdot\Theta(\Delta\Phi) + h\cdot\partial_s\Theta(f)\,\rangle \right| \\
&\le \left| \int_{\Omega'\times\Sigma} \langle\, \hat{\beta}\,,\, \partial_s\hat{\alpha}_\gamma + \partial_t(*\hat{\alpha}_\gamma)\,\rangle \right| \\
&\quad + C\|\hat{\beta}\|_{L^q(\Omega',X^*)}\big(\|-\partial_s\Phi + J_0\partial_t\Phi)\|_{L^{q^*}(\Omega',Z\times Z)} + \|f\|_{L^{q^*}(\Omega',Z\times Z)}\big) \\
&\le C\|\gamma\|_{L^{q^*}(\Omega'\times\Sigma)}.
\end{aligned}$$

This proves the L^q-regularity of $\partial_s\hat{\beta}$ (and analogously of $\partial_t\hat{\beta}$) on $S^1\times[0,\frac{\Delta}{2}]\times\Sigma$ in the case $q<2$ and thus finishes the proof of the lemma. □

Proof of theorem 7.1 :
Lemma 7.2 and the subsequent remark imply that for some constant C and for all (α,φ) in the domain of $D_{(A,\Phi)}$

$$\|(\alpha,\varphi)\|_{W^{1,p}} \le C\big(\|D_{(A,\Phi)}(\alpha,\varphi)\|_p + \|(\alpha,\varphi)\|_p\big).$$

Note that the embedding $W^{1,p}(X)\hookrightarrow L^p(X)$ is compact, so this estimate asserts that $\ker D_{(A,\Phi)}$ is finite dimensional and $\operatorname{im} D_{(A,\Phi)}$ is closed (see e.g. [Z, 3.12]). So it remains to consider the cokernel of $D_{(A,\Phi)}$. We abbreviate $Z := L^p(S^1\times Y, \mathrm{T}^*Y\otimes\mathfrak{g})\times L^p(S^1\times Y,\mathfrak{g})$, then $\operatorname{coker} D_{(A,\Phi)} = Z/\operatorname{im} D_{(A,\Phi)}$ is a Banach space since $\operatorname{im} D_{(A,\Phi)}$ is closed. So it has the same dimension as its dual space $(Z/\operatorname{im} D_{(A,\Phi)})^* \cong (\operatorname{im} D_{(A,\Phi)})^\perp$. Now let $\sigma : S^1\times Y\to S^1\times Y$ denote the reflection $\sigma(s,y) := (-s,y)$ on $S^1\cong\mathbb{R}/\mathbb{Z}$, then we claim that there is an isomorphism

$$\begin{array}{ccc} (\operatorname{im} D_{(A,\Phi)})^\perp & \xrightarrow{\sim} & \ker D_{\sigma^*(A,\Phi)} \\ (\beta,\zeta) & \longmapsto & (\beta\circ\sigma,\zeta\circ\sigma). \end{array} \tag{7.11}$$

Here $D_{\sigma^*(A,\Phi)} = D_{(A',\Phi')}$ is the linearized operator at the reflected connection $\sigma^*\tilde{A} = A' + \Phi'\mathrm{d}s$ with respect to the metric $\sigma^*\tilde{g}$ on X. Note that $\ker D_{\sigma^*(A,\Phi)}$ is finite dimensional since the estimate in lemma 7.2 also holds for the operator $D_{\sigma^*(A,\Phi)}$. So this would indeed prove that $\operatorname{coker} D_{\tilde{A}}$ is of finite dimension and hence $D_{\tilde{A}}$ is a Fredholm operator.

To establish the above isomorphism consider any $(\beta,\zeta) \in (\operatorname{im} D_{(A,\Phi)})^{\perp}$, that is $\beta \in L^{p^*}(S^1 \times Y, \mathrm{T}^*Y \otimes \mathfrak{g})$ and $\zeta \in L^{p^*}(S^1 \times Y, \mathfrak{g})$ such that for all $\alpha \in E_A^{1,p}$ and $\varphi \in W^{1,p}(S^1 \times Y, \mathfrak{g})$

$$\int_{S^1 \times Y} \langle\, D_{(A,\Phi)}(\alpha,\varphi)\,,\,(\beta,\zeta)\,\rangle = 0.$$

Iteration of lemma 7.3 implies that β and ζ are in fact $W^{1,p}$-regular: We start with $q = p^* < 2$, then the lemma asserts W^{1,p^*}-regularity. Next, the Sobolev embedding theorem gives L^{q_1}-regularity for some $q_1 \in (\frac{4}{3}, 2)$ with $q_1 > p^*$. Indeed, the Sobolev embedding holds for any $q_1 \le \frac{4p^*}{4-p^*}$, and $\frac{4}{3} < \frac{4p^*}{4-p^*}$ as well as $p^* < \frac{4p^*}{4-p^*}$ holds due to $p^* > 1$. So the lemma together with the Sobolev embeddings can be iterated to give $L^{q_{i+1}}$-regularity for $q_{i+1} = \frac{4q_i}{4-q_i}$ as long as $4 > q_i > 2$ or $2 > q_i \ge p^*$. This iteration yields $q_2 \in (2,4)$ and $q_3 > 4$. Thus another iteration of the lemma gives W^{1,q_3}- and thus also L^p-regularity of β and ζ. Finally, since $p > 2$ the lemma applies again and asserts the claimed $W^{1,p}$-regularity of β and ζ. Now by partial integration

$$\begin{aligned}
0 &= \int_{S^1 \times Y} \langle\, D_{(A,\Phi)}(\alpha,\varphi)\,,\,(\beta,\zeta)\,\rangle \\
&= \int_{S^1}\int_Y \langle\, \nabla_s\alpha - \mathrm{d}_A\varphi + *\mathrm{d}_A\alpha\,,\,\beta\,\rangle \;+ \int_{S^1}\int_Y \langle\, \nabla_s\varphi - \mathrm{d}_A^*\alpha\,,\,\zeta\,\rangle \\
&= \int_{S^1}\int_Y \langle\, \alpha\,,\, -\nabla_s\beta - \mathrm{d}_A\zeta + *\mathrm{d}_A\beta\,\rangle \;+ \int_{S^1}\int_Y \langle\, \varphi\,,\, -\nabla_s\zeta - \mathrm{d}_A^*\beta\,\rangle \\
&\quad - \int_{S^1}\int_\Sigma \langle\, \alpha \wedge \beta\,\rangle \;- \int_{S^1}\int_\Sigma \langle\, \varphi\,,\, *\beta\,\rangle.
\end{aligned} \tag{7.12}$$

Testing this with all $\alpha \in \mathcal{C}_0^\infty(S^1 \times Y, \mathrm{T}^*Y \otimes \mathfrak{g}) \subset E_A^{1,p}$ and $\varphi \in \mathcal{C}_0^\infty(S^1 \times Y, \mathfrak{g})$ implies $-\nabla_s\beta - \mathrm{d}_A\zeta + *\mathrm{d}_A\beta = 0$ and $-\nabla_s\zeta - \mathrm{d}_A^*\beta = 0$. Then furthermore we deduce $*\beta(s)|_{\partial Y} = 0$ for all $s \in S^1$ from testing with φ that run through all of $\mathcal{C}^\infty(S^1 \times \Sigma, \mathfrak{g})$ on the boundary. Finally, $\int_{S^1}\int_\Sigma \langle\, \alpha \wedge \beta\,\rangle = 0$ remains from (7.12). Since both α and β restricted to $S^1 \times \Sigma$ are continuous paths in $\mathcal{A}^{0,p}(\Sigma)$, this implies that for all $s \in S^1$ and every $\alpha \in \mathrm{T}_{A_s}\mathcal{L}$

$$0 \;=\; \int_\Sigma \langle\, \alpha \wedge \beta(s)\,\rangle \;=\; \omega(\alpha, \beta(s)),$$

where ω is the symplectic structure on $\mathcal{A}^{0,p}(\Sigma)$ from (4.1). Since $\mathrm{T}_{A_s}\mathcal{L}$ is a Lagrangian subspace, this proves $\beta(s)|_{\partial Y} \in \mathrm{T}_{A_s}\mathcal{L}$ for all $s \in S^1$ and thus

$\beta \in E_A^{1,p}$, or equivalently $\beta \circ \sigma \in E_{A\circ\sigma}$. So $(\beta \circ \sigma, \zeta \circ \sigma)$ lies in the domain of $D_{\sigma^*(A,\Phi)}$. Now note that $\sigma^*\tilde{A} = A \circ \sigma - (\Phi \circ \sigma)\mathrm{d}s$, thus one obtains $(\beta \circ \sigma, \zeta \circ \sigma) \in \ker D_{\sigma^*(A,\Phi)}$ since

$$D_{\sigma^*(A,\Phi)}(\beta\circ\sigma,\zeta\circ\sigma) \;=\; \bigl((-\nabla_s\beta - \mathrm{d}_A\zeta + *\mathrm{d}_A\beta)\circ\sigma\,,\,(-\nabla_s\zeta - \mathrm{d}_A^*\beta)\circ\sigma\bigr) \;=\; 0.$$

This proves that the map in (7.11) indeed maps into $\ker D_{\sigma^*(A,\Phi)}$. To see the surjectivity of this map consider any $(\beta,\zeta) \in \ker D_{\sigma^*(A,\Phi)}$. Then the same partial integration as in (7.12) shows that $(\beta \circ \sigma, \zeta \circ \sigma) \in (\mathrm{im}\, D_{(A,\Phi)})^{\perp}$, and thus (β, ζ) is the image of this element under the map (7.11). So this establishes the isomorphism (7.11) and thus shows that $D_{(A,\Phi)}$ is Fredholm. $\square$

Chapter 8

Bubbling

In this chapter we discuss the possible bubbling phenomena for sequences of anti-self-dual instantons with Lagrangian boundary conditions and bounded Yang-Mills energy. So as in chapter 6 let (X, τ, g) be a Riemannian 4-manifold with a boundary space-time splitting and consider a G-bundle over X for a compact Lie group G. Let $(A^\nu)_{\nu\in\mathbb{N}} \subset \mathcal{A}(X)$ be a sequence of solutions of

$$\begin{cases} *F_A + F_A = 0, \\ \tau_i^* A|_{\{s\}\times\Sigma_i} \in \mathcal{L}_i \quad \forall s \in \mathcal{S}_i,\ i = 1, \dots, n. \end{cases} \tag{8.1}$$

Suppose that this sequence has bounded Yang-Mills energy,

$$\mathcal{E}(A^\nu) := \int_X |F_{A^\nu}|^2 \;\le\; E \;<\; \infty \qquad \forall \nu \in \mathbb{N}.$$

This means that the curvature of the connections is globally L^2-bounded. If the curvature was in fact locally L^p-bounded for some $p > 2$, then the compactness theorem B would imply that modulo gauge there exists a $\mathcal{C}^\infty_{\mathrm{loc}}$-convergent subsequence of the connections. So in order to compactify the moduli space of solutions of (8.1) with bounded energy, it remains to analyse the possible local blow-up of the curvature. We first discuss the standard approach of local rescaling at a point $x \in X$ where the curvature blows up. This is the situation of the following lemma, in which we use a local trivialization and geodesic normal coordinates near x – either in $\mathbb{R}^4$ if $x \in \operatorname{int} X$ or in $\mathbb{H}^4 = \{y \in \mathbb{R}^4 \mid y_0 \ge 0\}$ if $x \in \partial X$. In the latter case due to the compatibility of the metric with the space-time splitting the coordinates can be chosen such that the slices $\{y_0 = 0, y_1 = c\}$ correspond to the time-slices of the boundary ∂X.

Lemma 8.1 *Let $x \in X$ such that $\sup_{\nu\in\mathbb{N}}\|F_{A^\nu}\|_{L^\infty(B_\varepsilon(x))} = \infty$ for all $\varepsilon > 0$. Then there exists a subsequence (again denoted (A^ν)), a sequence $x^\nu \to x$, and sequences $0 < \varepsilon^\nu \to 0$, $R^\nu \to \infty$ such that $\varepsilon^\nu R^\nu \to \infty$ and*

$$\big|F_{A^\nu}(x^\nu)\big| = (R^\nu)^2, \qquad \sup_{y\in B_{\varepsilon^\nu}(x^\nu)} \big|F_{A^\nu}(y)\big| \le 4(R^\nu)^2.$$

(i) If $x \in \operatorname{int} X$ or $x \in \partial X$ and $\limsup_{\nu\to\infty} R^\nu x_0^\nu = \infty$, then there exists a subsequence (again denoted (A^ν)) and possibly smaller $0 < \varepsilon^\nu \le x_0^\nu$ such that still $r^\nu := \varepsilon^\nu R^\nu \to \infty$. Define a sequence of embeddings $\phi_\nu : B_{r^\nu} \to X$ in the coordinates near x by $\phi_\nu(y) := x^\nu + \frac{1}{R^\nu} y$. Then the rescaled connections $\tilde{A}^\nu := \phi_\nu^* A^\nu \in \mathcal{A}(B_{r^\nu})$ are anti-self-dual with respect to the metrics $\tilde{g}^\nu := (R^\nu)^{-2}\phi_\nu^* g$. Now for a further subsequence there exist gauge transformations $u^\nu \in \mathcal{G}(B_{r^\nu})$ such that

$$u^{\nu\,*}\tilde{A}^\nu \underset{\nu\to\infty}{\longrightarrow} A^\infty \;\in\; \mathcal{A}(\mathbb{R}^4)$$

converges uniformly with all derivatives on every compact set. The limit $A^\infty \in \mathcal{A}(\mathbb{R}^4)$ is a nontrivial anti-self-dual instanton on $\mathbb{R}^4$ of finite energy $\mathcal{E}(A^\infty) \le E$, hence

$$\lim_{\nu\to\infty} \mathcal{E}(A^\nu|_{B_{\varepsilon^\nu}(x^\nu)}) \;=\; \lim_{\nu\to\infty} \mathcal{E}(\tilde{A}^\nu) \;=\; \mathcal{E}(A^\infty) \;\ge\; 8\pi^2.$$

(ii) If $x \in \partial X$ and $\limsup_{\nu\to\infty} R^\nu x_0^\nu = \Delta < \infty$, then there exists a subsequence (again denoted (A^ν)) such that the ε^ν can be replaced by $\sqrt{(\varepsilon^\nu)^2 - (x_0^\nu)^2} > 0$ and still $r^\nu := \varepsilon^\nu R^\nu \to \infty$. Define the embeddings $\phi_\nu : D^\nu := B_{r^\nu} \cap \mathbb{H}^4 \to X$, metrics, and rescaled connections $\tilde{A}^\nu \in \mathcal{A}(D^\nu)$ as in (i) with the exception that $[\phi_\nu(y)]_0 := \frac{1}{R^\nu} y_0$. Fix $p > 4$, then for a further subsequence there exists a sequence of gauge transformations $u^\nu \in \mathcal{G}^{1,p}(D^\nu)$ such that

$$u^{\nu\,*}\tilde{A}^\nu \underset{\nu\to\infty}{\rightharpoonup} A^\infty \;\in\; \mathcal{A}^{1,p}_{\mathrm{loc}}(\mathbb{H}^4)$$

converges $W^{1,p}$-weakly on every compact subset. Moreover, the convergence is uniformly with all derivatives on every interior compact subset of $\mathbb{H}^4$. The limit $A^\infty \in \mathcal{A}^{1,p}_{\mathrm{loc}}(\mathbb{H}^4) \cap \mathcal{A}(\operatorname{int}\mathbb{H}^4)$ is an anti-self-dual instanton of finite energy $\mathcal{E}(A^\infty) \le E$ and with weakly flat boundary conditions on each time-slice $\{y_0 = 0, y_1 = c\}$. In case $\Delta > 0$ it is necessarily nontrivial. Moreover,

$$\limsup_{\nu\to\infty} \mathcal{E}(A^\nu|_{B_{\varepsilon^\nu}(x^\nu)}) \;=\; \limsup_{\nu\to\infty} \mathcal{E}(\tilde{A}^\nu) \;\ge\; \mathcal{E}(A^\infty).$$

Before we give the proof let us point out the significance of this lemma. In case (i) the rescaled and gauge transformed connections converge to a connection that extends to an anti-self-dual instanton on $S^4 = \mathbb{R}^4 \cap \{\infty\}$. One then says that 'an instanton on S^4 bubbles off at x'. Note that this can happen in the interior as well as on the boundary of X. If case (ii) was ruled out by some reason, then this would show that a minimum energy of $8\pi^2$ concentrates at every blow-up point of the curvature. Hence for a suitable subsequence of the connections this bubbling off only happens at finitely many points. On the complement of these points, the curvature is locally L^∞-bounded. Now theorem B applies on this complement (in the case of bubbling at the boundary, one has to take away the full time-slice in which the bubbling point lies). It asserts that a further subsequence of the connections converges modulo gauge with all derivatives on every compact subset of this complement of the bubbling points. At the interior bubbling points, Uhlenbeck's removable singularity theorem, proposition 8.3, below asserts that the limit connection extends over the bubbling point – to a connection on a different bundle. (The gauge transformation in the proposition can be seen as transition function between the trivialization of the original bundle on a punctured neighbourhood of the bubbling point and the trivialization of the new bundle on the full neighbourhood.) So if there was only interior bubbling, then this would provide the desired compactification of the moduli spaces of solutions of (8.1) of bounded energy. However, the bubbling at the boundary poses two difficulties.

Firstly, in the case of instantons on S^4 bubbling off at the boundary one needs a generalization of Uhlenbeck's removable singularity theorem to points at the boundary.

Secondly, if case (ii) in lemma 8.1 occurs, then one does not even have a minimum energy concentrating at the blow-up point of the curvature. This is due to substantial flaws of the local rescaling: By cutting out only small balls in each time-slice of the boundary one loses the global Lagrangian information in the boundary condition. The slicewise flatness at the boundary does not suffice to make the boundary value problem elliptic and hence obtain higher regularity at the boundary. In the case $\Delta > 0$ one still obtains a nontrivial instanton in the limit due to the $\mathcal{C}^\infty$-convergence in the interior of the half space. For such instantons Salamon conjectured in [Sa1, Conjecture 3.2] a quantization of the energy that would imply that this instanton on the half space bubbles off with a minimum energy of $8\pi^2$. In case $\Delta = 0$, however, one does not have $\mathcal{C}^0$-convergence of the curvature at the rescaled blow-up points, so the limit connection might be trivial and thus have no energy although some energy bubbles off 'very close to the boundary'.

So the results of the standard bubbling analysis in lemma 8.1 are far from a complete understanding of the possible bubbling phenomena for the boundary value problem (8.1). We still indicate its proof in the following and give a further discussion of the bubbling at the boundary in a subsection below. Lemma 8.1 uses Hofer's trick, [HZ, Ch.6,Lemma 5], as well as Uhlenbeck's removable singularity theorem, [U1, Theorem 4.1] (which holds for Yang-Mills connections as well as for anti-self-dual connections).

Lemma 8.2 (Hofer's trick)
Let $f : X \to [0,\infty)$ be a continuous function on a complete metric space X. Assume that $x_0 \in X$ and $\varepsilon_0 > 0$ are given. Then there exists $x \in B_{2\varepsilon_0}(x_0)$ and $\varepsilon \in (0,\varepsilon_0]$ such that

$$\varepsilon f(x) \geq \varepsilon_0 f(x_0) \qquad \textit{and} \qquad \sup_{y\in B_\varepsilon(x)} f(y) \leq 2f(x).$$

Proposition 8.3 (Uhlenbeck's removable singularity theorem)
Let $A \in \mathcal{A}(B^4\setminus\{0\})$ be an anti-self-dual connection over the punctured 4-ball. If $\mathcal{E}(A) < \infty$ then there exists a gauge transformation $u \in \mathcal{G}(B^4 \setminus \{0\})$ such that u^*A extends to a smooth anti-self-dual connection over B^4.

Proof of lemma 8.1 :
Fix sequences $0 < \varepsilon_0^\nu \to 0$ and $R_0^\nu \to \infty$ such that $\varepsilon_0^\nu R_0^\nu \to \infty$. By assumption one finds a subsequence (again denoted (A^ν)) and $x_0^\nu \in B_{\varepsilon_0^\nu}(x)$ such that $|F_{A^\nu}(x_0^\nu)| \geq (R_0^\nu)^2$. Now apply Hofer's trick to the function $\sqrt{|F_{A^\nu}|}$ on X for each $\nu \in \mathbb{N}$ to obtain the required $x^\nu \in B_{2\varepsilon_0^\nu}(x_0^\nu)$, $\varepsilon^\nu \in (0,\varepsilon_0^\nu]$, and $R^\nu := \sqrt{|F_{A^\nu}(x^\nu)|}$.

Next, note that the metrics $\tilde{g}^\nu$ smoothly converge to the Euclidean metric on $\mathbb{R}^4$ since we used normal geodesic coordinates centered at $x = \lim x^\nu$. For the rescaled connections one obtains $F_{\tilde{A}^\nu}(y) = (R^\nu)^{-2} F_{A^\nu}(x^\nu + \frac{1}{R^\nu}y)$. Hence for the norms with respect to the metrics $\tilde{g}^\nu$ one has

$$\left|F_{\tilde{A}^\nu}(0)\right| = 1, \qquad \left\|F_{\tilde{A}^\nu}\right\|_{L^\infty(D^\nu)} \leq 4.$$

(In the rescaling of case (ii) only the first identity has to be replaced by $\left|F_{\tilde{A}^\nu}(t^\nu R^\nu,0,0,0)\right| = 1$.) The energy, however, is preserved by this rescaling,

$$\mathcal{E}(\tilde{A}^\nu) = \mathcal{E}(A^\nu|_{B_{\varepsilon^\nu}(x^\nu)}).$$

Now in case (i) the domains of $\tilde{A}^\nu$ can be chosen as balls that exhaust $\mathbb{R}^4$. So the strong Uhlenbeck compactness theorem for noncompact manifolds without boundary (see e.g. [We, Theorem 11.3] or its generalization, theorem B) applies and proves the convergence claimed in (i). The nontriviality of the limit connection is due to

$$|F_{A^\infty}(0)| \;=\; \lim_{\nu\to\infty} |(u^\nu)^{-1}F_{A^\nu}u^\nu(0)| \;=\; 1.$$

Next, $\mathbb{R}^4$ is conformally equivalent to $S^4 \setminus \{pt\}$, so A^∞ can be seen as anti-self-dual connection on the punctured S^4 with finite energy. Uhlenbeck's removable singularity theorem, proposition 8.3, then asserts that A^∞ extends to an anti-self-dual connection $\bar{A}^\infty$ on a bundle $P \to S^4$. Hence its energy is a multiple of the second Chern number of the bundle P over S^4,

$$\mathcal{E}(A^\infty) \;=\; \int_{S^4} \langle\, F_{\bar{A}^\infty} \wedge *F_{\bar{A}^\infty}\,\rangle \;=\; -\int_{S^4} \langle\, F_{\bar{A}^\infty} \wedge F_{\bar{A}^\infty}\,\rangle \;=\; 8\pi^2 c_2(P).$$

Finally, this energy is at least $8\pi^2$ since the Chern number is integral and the limit connection A^∞ is nontrivial, so $\mathcal{E}(A^\infty) > 0$.

In case (ii) the weak $W^{1,p}$-convergence is due to the weak Uhlenbeck compactness theorem for noncompact manifolds, see e.g. [We, Theorem 8.5]. This weak convergence preserves the anti-self-duality equation as well as the slicewise weak flatness at the boundary, and it gives the bound on the energy of the limit connection. The interior $\mathcal{C}^\infty$-convergence again follows from the strong Uhlenbeck compactness theorem. In case $\Delta > 0$ this interior convergence implies the nontriviality of the instanton since

$$|F_{A^\infty}(\Delta,0,0,0)| \;=\; \lim_{\nu\to\infty} |F_{A^\nu}(t^\nu R^\nu,0,0,0)| \;=\; 1.$$

□

Bubbling at the boundary

For the discussion of the possible bubbling phenomena at the boundary we restrict our attention to the possible blow-up of the curvature at one space-time slice $\tau(\{pt\}\times\Sigma)$ of the boundary. We use the decomposition induced by the extended space-time splitting $\bar{\tau} : \mathcal{U}\times\Sigma \to X$. Here one can choose $\mathcal{U} \subset \mathbb{H} = \{(s,t)\in\mathbb{R}^2 \,\big|\, t \geq 0\}$ as a neighbourhood of 0 and then examine the possible blow-up of the curvature at $\{0\}\times\Sigma$. So instead of the A^ν above we consider a sequence of connections

$$\Phi^\nu \mathrm{d}s + \Psi^\nu \mathrm{d}t + A^\nu \;\in\; \mathcal{A}(\mathcal{U}\times\Sigma)$$

with $\Phi^\nu, \Psi^\nu \in \mathcal{C}^\infty(\mathcal{U}\times\Sigma, \mathfrak{g})$, and $A^\nu \in \mathcal{C}^\infty(\mathcal{U}\times\Sigma, \mathrm{T}^*\Sigma\otimes\mathfrak{g})$ for all $\nu\in\mathbb{N}$. The boundary value problem (8.1) then becomes

$$\begin{cases} B_s^\nu + *B_t^\nu = 0, \\ \partial_t\Phi^\nu - \partial_s\Psi^\nu + [\Psi^\nu, \Phi^\nu] = *F_{A^\nu}, \\ A^\nu|_{(s,0)\times\Sigma} \in \mathcal{L} \qquad \forall (s,0)\in\mathcal{U}. \end{cases}$$

Here we have introduced the following notation for the components of the curvature (dropping the superscript ν):

$$B_s := \partial_s A - \mathrm{d}_A\Phi, \qquad\qquad B_t := \partial_t A - \mathrm{d}_A\Psi.$$

The Bianchi identity then takes the form

$$\begin{aligned} &\nabla_s F_A = \mathrm{d}_A B_s, \qquad \nabla_t F_A = \mathrm{d}_A B_t, \\ &\nabla_s B_t - \nabla_t B_s = \mathrm{d}_A(\partial_t\Phi - \partial_s\Psi + [\Psi,\Phi]). \end{aligned}$$

Moreover, the bound on the energy becomes

$$\sup_{\nu\in\mathbb{N}} \int |B_s^\nu|^2 + |F_{A^\nu}|^2 \;\le\; \tfrac{1}{2}E \;<\; \infty.$$

Note that $|B_s| = |B_t|$ and $|F_A| = |\partial_t\Phi - \partial_s\Psi + [\Psi,\Phi]|$. Now a first observation is that $|F_A|$ cannot blow up faster that $|B_s|$, in particular if B_s^ν is L^p-bounded near $\{0\}\times\Sigma$ for $p>4$, then there is no bubbling at all. This is made precise by the following lemma.

Lemma 8.4 *Let $x^\nu \to (0,z)\in\mathcal{U}\times\Sigma$ be such that*

$$|F_{A^\nu}(x^\nu)| = (R^\nu)^2 \to \infty.$$

Then for all $p>4$ and $0<\varepsilon^\nu\to 0$ with $\varepsilon^\nu R^\nu\to\infty$ there exists a constant $c>0$ such that

$$\left\|B_s^\nu\right\|_{L^p(B_{\varepsilon^\nu}(x^\nu))} \ge c(R^\nu)^{\frac{2p-4}{p}}.$$

Proof: Let x_0^ν, R_0^ν, and ε_0^ν be as supposed. Assume in contradiction that for some subsequence and $\delta^\nu \to 0$

$$\|B_s^\nu\|_{L^p(B_{\varepsilon_0^\nu}(x_0^\nu))} \leq \delta^\nu (R_0^\nu)^{\frac{2p-4}{p}}.$$

Use Hofer's lemma 8.2 for the functions $\sqrt{|F_{A^\nu}|}$ to obtain $x^\nu \in B_{\frac{2}{3}\varepsilon_0^\nu}(x_0^\nu)$, $\varepsilon \in (0, \frac{1}{3}\varepsilon_0^\nu]$, and $R^\nu := \sqrt{|F_{A^\nu}(x^\nu)|}$ such that $r^\nu := \varepsilon^\nu R^\nu \to \infty$,

$$|F_{A^\nu}(x^\nu)| = (R^\nu)^2, \qquad \|F_{A^\nu}\|_{L^\infty(B_{\varepsilon^\nu}(x^\nu))} \leq 4(R^\nu)^2,$$

and one still has

$$\|B_s^\nu\|_{L^p(B_{\varepsilon^\nu}(x^\nu))} \leq \delta^\nu (R_0^\nu)^{\frac{2p-4}{p}} \leq \delta^\nu (R^\nu)^{\frac{2p-4}{p}}.$$

Then one can pullback the connections and rescale the metrics as in lemma 8.1 to obtain $\tilde{\Phi}^\nu \mathrm{d}s + \tilde{\Psi}^\nu \mathrm{d}t + \tilde{A}^\nu \in \mathcal{A}(D^\nu)$ with $D^\nu = B_{r^\nu}$ or $D^\nu = B_{r^\nu} \cap \mathbb{H}^4$ if $\limsup t^\nu R^\nu < \infty$. Then $\|F_{\tilde{A}^\nu}\|_\infty$ is bounded and $\tilde{B}_s^\nu$ converges to zero in the L^p-norm on every compact set. (Here one writes $x^\nu = (s^\nu, t^\nu, z^\nu)$ and uses local coordinates for $z^\nu \in \Sigma$, so $\mathbb{H}^4 = \{(s,t,z) \in \mathbb{R}^4 \mid t \geq 0\}$.) Now Uhlenbeck's weak compactness theorem on noncompact manifolds (e.g. [We, Theorem 8.5]) gives a subsequence of gauge equivalent connections that converge $W^{1,p}$-weakly on all compact subsets. The limit $\Phi \mathrm{d}s + \Psi \mathrm{d}t + A$ is a connection on $\mathbb{R}^4$ (if $\limsup t^\nu R^\nu = \infty$) or on $\mathbb{H}^4$. The convergence is actually in $\mathcal{C}^\infty$ on every compact subset of the interior by Uhlenbeck's strong compactness theorem for anti-self-dual instantons (e.g. [We, Theorem 11.3]). Furthermore, the proof of theorem B also provides a $W^{1,p}$-bound on the F_{A^ν}. (The proof only breaks down at the higher estimates in theorem 6.8 for the s- and t-derivatives of the A^ν, which require the Lagrangian boundary condition.) So for a further subsequence, the curvature converges locally in $\mathcal{C}^0$, and hence the limit connection is nontrivial, $|F_A(0)| = 0$ or $|F_A(0,\Delta,0,0)| = 1$ if $\lim t^\nu R^\nu = \Delta < \infty$. The limit connection moreover satisfies

$$\begin{cases} \partial_s A - \mathrm{d}_A \Phi = 0, \\ \partial_t A - \mathrm{d}_A \Psi = 0, \\ \partial_t \Phi - \partial_s \Psi + [\Psi, \Phi] = *F_A. \end{cases}$$

Here one can choose a gauge in which $\Psi \equiv 0$, then A is t-independent and hence $|F_A(0,t,0,0)| = 1$ for all $t \geq 0$ in contradiction to the finite energy.

Indeed, an a priori estimate for Yang-Mills connections with small energy asserts that the curvature must decay as $t \to \infty$, see [U1, Theorem 3.5] . □

Now let us assume that $\|B_s^\nu(s,t)\|_{L^p(\Sigma)}$ blows up at $(s,t) = 0$ for some $p > 4$. By Hofer's lemma 8.2 one finds $(s^\nu, t^\nu) \to 0$ and $0 < \varepsilon^\nu \to 0$ such that $r^\nu := \varepsilon^\nu R^\nu \to \infty$ and

$$\|B_s^\nu(s^\nu,t^\nu)\|_{L^p(\Sigma)} = R^\nu \to \infty, \quad \sup_{(s,t)\in B_{2\varepsilon^\nu}(s^\nu,t^\nu)} \|B_s^\nu(s,t)\|_{L^p(\Sigma)} \le 4R^\nu. \tag{8.2}$$

Here we apply the Hofer trick to the functions $\|B_s^\nu\|_{L^p(\Sigma)}^{\frac{1}{2}}$, so we in fact even have $\varepsilon^\nu\sqrt{R^\nu} \to \infty$. Then as a direct consequence of the above lemma one has a bound on the rate at which F_{A^ν} can blow up.

Corollary 8.5 *Suppose that $(s^\nu, t^\nu) \to 0$, $R^\nu \to \infty$, and $0 < \varepsilon^\nu \to 0$ such that $r^\nu := \varepsilon^\nu R^\nu \to \infty$ and (8.2). Then*

$$(R^\nu)^{-\frac{2p}{2p-4}} \|F_{A^\nu}\|_{L^\infty(B_{\varepsilon^\nu}(s^\nu,t^\nu)\times\Sigma)} \underset{\nu\to\infty}{\longrightarrow} 0.$$

Proof: Assume in contradiction that (for a subsequence) there exist $\delta > 0$ and $x^\nu \in B_{\varepsilon^\nu}(s^\nu,t^\nu)\times\Sigma$ such that $|F_{A^\nu}(x^\nu)| \ge (\delta R^\nu)^{\frac{2p}{2p-4}}$. Since Σ is compact one finds a further subsequence such that $x^\nu \to (0,z)$ for some $z \in \Sigma$. Then lemma 8.4 applies with R^ν replaced by $\tilde{R}^\nu := (\delta R^\nu)^{\frac{p}{2p-4}}$. (Note that $\varepsilon^\nu \tilde{R}^\nu \ge \varepsilon^\nu\sqrt{\delta R^\nu}$ for sufficiently large ν, and thus converges to ∞.) Thus one obtains for some constant $c > 0$ the following contradiction:

$$c\delta R^\nu \;=\; c(\tilde{R}^\nu)^{\frac{2p-4}{p}} \;\le\; \|B_s^\nu\|_{L^p(B_{\varepsilon^\nu}(x^\nu))} \;\le\; 4R^\nu \mathrm{Vol}(B_{2\varepsilon^\nu}(s^\nu,t^\nu))^{\frac{1}{p}}.$$

□

Let us try a 2-dimensional rescaling only in the (s,t)-variables. Define embeddings $\phi_\nu : D^\nu \times \Sigma \to \mathcal{U} \times \Sigma$ for $D^\nu := [-r^\nu, r^\nu] \times [-\min(r^\nu, t^\nu R^\nu), r^\nu]$ by $\phi_\nu(\sigma,\tau,z) := (s^\nu + \frac{1}{R^\nu}\sigma, t^\nu + \frac{1}{R^\nu}\tau, z)$. (In the case $\limsup t^\nu R^\nu \to \infty$ this will have to be adapted by replacing $t^\nu + \frac{1}{R^\nu}\tau$ with $\frac{1}{R^\nu}\tau$.) Then define the rescaled metrics $\tilde{g}^\nu := g(s^\nu + \frac{1}{R^\nu}\sigma, t^\nu + \frac{1}{R^\nu}\tau, z)$ on $D^\nu \times \Sigma$ and the following connections in $\mathcal{A}(D^\nu \times \Sigma)$:

$$\begin{aligned}\tilde{\Phi}^\nu \mathrm{d}s + \tilde{\Psi}^\nu \mathrm{d}t + \tilde{A}^\nu &:= \phi_\nu^*\big(\Phi^\nu \mathrm{d}s + \Psi^\nu \mathrm{d}t + A^\nu\big) \\ &= \big(\tfrac{1}{R^\nu}\Phi^\nu \mathrm{d}s + \tfrac{1}{R^\nu}\Psi^\nu \mathrm{d}t + A^\nu\big)\circ\phi_\nu. \end{aligned} \tag{8.3}$$

These now satisfy the boundary value problem [1]

$$\begin{cases} \partial_s \tilde{A}^\nu - \mathrm{d}_{\tilde{A}^\nu} \tilde{\Phi}^\nu = - * \big(\partial_t \tilde{A}^\nu - \mathrm{d}_{\tilde{A}^\nu} \tilde{\Psi}^\nu \big), \\ \partial_t \tilde{\Phi}^\nu - \partial_s \tilde{\Psi}^\nu + [\tilde{\Psi}^\nu, \tilde{\Phi}^\nu] = (R^\nu)^{-2} * F_{\tilde{A}^\nu}, \\ \tilde{A}^\nu|_{(s,-t^\nu R^\nu)\times\Sigma} \in \mathcal{L} \qquad \forall s \in [-r^\nu, r^\nu]. \end{cases}$$

Moreover, one has $\|(\partial_s \tilde{A}^\nu - \mathrm{d}_{\tilde{A}^\nu} \tilde{\Phi}^\nu)(0,0)\|_{L^p(\Sigma)} = 1$ for all $\nu \in \mathbb{N}$ and the following estimates for some $\delta^\nu \to 0$

$$\begin{aligned} \|F_{\tilde{A}^\nu}\|_{L^\infty(D^\nu, L^p(\Sigma))} &\leq \delta^\nu (R^\nu)^{\frac{2p}{2p-4}}, \\ \|\partial_s \tilde{A}^\nu - \mathrm{d}_{\tilde{A}^\nu} \tilde{\Phi}^\nu\|_{L^\infty(D^\nu, L^p(\Sigma))} &\leq 4, \\ \|\partial_t \tilde{\Phi}^\nu - \partial_s \tilde{\Psi}^\nu + [\tilde{\Psi}^\nu, \tilde{\Phi}^\nu]\|_{L^\infty(D^\nu, L^p(\Sigma))} &\leq \delta^\nu (R^\nu)^{\frac{2p}{2p-4}-2} \;=\; \delta^\nu (R^\nu)^{-\frac{p-4}{p-2}}. \end{aligned} \tag{8.4}$$

Note that due to $p > 4$ the last norm converges to 0 as $\nu \to \infty$. This hints at a holomorphic curve in the space of connections as limit object. If the connections would converge in some gauge, then in the limit one could find a gauge such that $\tilde{\Phi}$ and $\tilde{\Psi}$ vanish simultaneously, and thus be left with $\partial_s \tilde{A} + *\partial_t \tilde{A} = 0$. This should yield a nontrivial finite energy holomorphic curve in the complex Hilbert space $(\mathcal{A}^{0,2}(\Sigma), *)$. If $\limsup_{\nu\to\infty} t^\nu R^\nu = \infty$, this plane should extend to a holomorphic sphere by a removable singularity theorem. Otherwise one would obtain holomorphic halfplanes with Lagrangian boundary conditions that should extend to a holomorphic disc with Lagrangian boundary conditions. There are two indications that this might be the right approach to the bubbling phenomena at the boundary.

Firstly, one expects for general reasons that there should not be any holomorphic spheres bubbling off. All spheres in the space of connections are contractible, so any holomorphic sphere $u : S^2 \to \mathcal{A}^{0,2}(\Sigma)$ (extending to $\bar{u} : B^2 \to \mathcal{A}^{0,2}(\Sigma)$) has zero energy $\int_{S^2} |\partial_s u|^2 = \int_{S^2} u^*\omega = \int_{B^2} \bar{u}^* \mathrm{d}\omega = 0$. Moreover, if the bubbling off of holomorphic spheres was possible, then this should also happen at interior Σ-slices, where so far all blowing up of the curvature has been explained as bubbling off of instantons on S^4. In fact, the 2-dimensional rescaling does not lead to holomorphic spheres ($t^\nu R^\nu$ is always bounded) if one excludes the cases where instantons on S^4 bubble off at the same boundary slice (or similarly at the same interior slice), as will

[1] This can be seen as anti-self-duality equation, where the metric on Σ is conformally rescaled by the factor $(R^\nu)^2$. An analogous equation without boundary conditions was studied in [DS2, (3.5)], where the roles of A and $\Phi \mathrm{d}s + \Psi \mathrm{d}t$ are interchanged.

be shown in the subsequent lemma 8.6. Secondly, the convergence of the rescaled connections to a holomorphic curve in the space of connections can be made rigorous if one assumes an L^∞-bound on F_{A^ν} near the boundary, see lemma 8.8 below.

So firstly, assume that there is no instanton on S^4 bubbling off at $\{0\}\times\Sigma$ (as described in lemma 8.1 (i)). This is equivalent to the assumption that for all subsequences of the connections (still denoted (A^ν)) and all $(s^\nu,t^\nu)\to 0$ there is a bound

$$\sup_{\nu\in\mathbb{N}} (t^\nu)^2 \bigl\| \bigl(F_{A^\nu} + B_s^\nu \wedge \mathrm{d}s\bigr)(s^\nu,t^\nu)\bigr\|_{L^\infty(B_{\frac{1}{2}t^\nu}(s^\nu,t^\nu)\times\Sigma)} < \infty .$$

Under this assumption the following lemma proves that in the above 2-dimensional rescaling one always finds $\limsup t^\nu R^\nu < \infty$, which would give rise to a holomorphic disc with Lagrangian boundary conditions – not a holomorphic sphere. (In fact, this lemma also holds if (s^ν,t^ν) converges to some interior point (s,t) with $t>0$, and that shows that there are no holomorphic spheres bubbling off in the interior.)

Lemma 8.6 *Let* $(s^\nu,t^\nu)\to 0$ *and suppose that*

$$\sup_{\nu\in\mathbb{N}} (t^\nu)^2 \|F_{A^\nu}\|_{L^\infty(B_{\frac{1}{2}t^\nu}(s^\nu,t^\nu)\times\Sigma)} < \infty .$$

Then there exists a constant C *such that for all* $\nu\in\mathbb{N}$

$$(t^\nu)^2 \|B_s^\nu(s^\nu,t^\nu)\|_{L^\infty(\Sigma)} \le C .$$

Proof: In a sufficiently small neighbourhood of $(0,0)$ one can assume that the metric on Σ is independent of (s,t). (One only gets small additional terms below that can be absorbed into others.) Then use the Bianchi identities and the anti-self-duality equation to calculate (dropping the superscript ν) $\nabla_s B_s + \nabla_t B_t = -*\mathrm{d}_A * F_A$ and thus

$$\begin{aligned}
(\nabla_s^2+\nabla_t^2)B_s &= \nabla_s\bigl(-\nabla_t B_t - *\mathrm{d}_A * F_A\bigr) + \nabla_t\bigl(\nabla_s B_t - \mathrm{d}_A * F_A\bigr)\\
&= \bigl(\nabla_t\nabla_s - \nabla_s\nabla_t\bigr)B_t - *\bigl(\nabla_s \mathrm{d}_A - \mathrm{d}_A\nabla_s\bigr) * F_A - *\mathrm{d}_A * \nabla_s F_A\\
&\quad - \bigl(\nabla_t \mathrm{d}_A - \mathrm{d}_A \nabla_t\bigr) * F_A - \mathrm{d}_A * \nabla_t F_A\\
&= [*F_A, B_t] - *[B_s, *F_A] - *\mathrm{d}_A * \mathrm{d}_A B_s - [B_t, *F_A] - \mathrm{d}_A * \mathrm{d}_A B_t\\
&= -\Delta_A B_s - 3*[B_s, *F_A].
\end{aligned}$$

Fix any $z \in \Sigma$ and abbreviate $u^\nu := |B_s^\nu|^2$. This satisfies on $B_{\frac{1}{2}t^\nu}(s^\nu, t^\nu, z)$ with some constant C

$$\begin{aligned}\Delta u^\nu &= |\nabla u^\nu|^2 + 2\langle\, B_s\,,\, (\nabla_s^2 + \nabla_t^2 + \Delta_A) B_s\,\rangle \\ &= |\nabla u^\nu|^2 + 6\langle\, [B_s \wedge B_s]\,,\, F_A\,\rangle \\ &\geq -C(t^\nu)^{-2} u^\nu.\end{aligned}$$

Hence the a priori estimate in lemma 8.7 implies

$$u^\nu(s^\nu, t^\nu, z) \;\leq\; \tfrac{32}{\pi^2}\big(\tfrac{4}{9}\,C^2 + 16\big)(t^\nu)^{-4} \int_{B_{\frac{1}{2}t^\nu}(s^\nu,t^\nu,z)} u^\nu \;\leq\; C'(t^\nu)^{-4} E.$$

This proves the lemma since the constant C' is independent of $z \in \Sigma$. $\square$

Here we have used the following a priori estimate which is an adaptation of [DS2, Lemma 7.3] to 4 dimensions.

Lemma 8.7 *Let $u : \mathbb{R}^4 \supset B_r \to [0, \infty)$ be a $\mathcal{C}^2$-function such that for some constant $c > 0$*

$$\Delta u \geq -cu.$$

Then

$$\tfrac{\pi^2}{32}\, u(0) \leq \big(\tfrac{4}{9}\, c^2 + r^{-4}\big) \int_{B_r} u.$$

Proof: The general case $r > 0$ of this lemma can be reduced to $r = 1$ by rescaling, so it suffices to prove the lemma for $r = 1$. Consider the following function $f : [0, 1] \to \mathbb{R}$,

$$f(\rho) = (1 - \rho)^4 \sup_{B_\rho} u.$$

Choose $\rho^* \in [0, 1)$ at which f attains its maximum, let $\delta := \frac{1}{2}(1 - \rho^*)$, and choose $w^* \in B_{\rho^*}$ such that $\sup_{B_{\rho^*}} u = u(w^*) =: c^*$. Then for all $w \in B_{\rho^*+\delta}$ one has $u(w) \leq 16c^*$ since

$$\tfrac{1}{16}(1-\rho^*)^4 u(w) \;\leq\; (1-(\rho^*+\delta))^4 \sup_{B_{\rho^*+\delta}} u \;=\; f(\rho^*+\delta) \;\leq\; f(\rho^*) \;=\; (1-\rho^*)^4 c^*.$$

Now $\tilde{u}(w) := u(w) + 2cc^*|w - w^*|^2$ is a subharmonic function on $B_\delta(w^*)$, hence for all $0 < \rho \leq \delta$

$$c^* \;=\; \tilde{u}(w^*) \;\leq\; \frac{2}{\pi^2 \rho^4} \int_{B_\rho(w^*)} \tilde{u} \;=\; \tfrac{4}{3} cc^* \rho^2 + \frac{2}{\pi^2 \rho^4} \int_{B_\rho(w^*)} u.$$

In case $\frac{4}{3}cc^*\delta^2 > \frac{1}{2}$ choose $\rho^2 = \frac{3}{8}c^{-1} < \delta^2$ to obtain

$$u(0) \;\leq\; c^* \;\leq\; \frac{2}{\pi^2\rho^4}\int_{B_\rho(w^*)} u \;\leq\; \frac{128\,c^2}{9\,\pi^2}\int_{B_1} u.$$

In case $\frac{4}{3}cc^*\delta^2 \leq \frac{1}{2}$ choose $\rho = \delta$ to find that

$$u(0) \;=\; f(0) \;\leq\; f(\rho^*) \;=\; 16\,\delta^4 c^* \;\leq\; \frac{32}{\pi^2}\int_{B_\rho(w^*)} u.$$

Putting this together yields the claim for $r = 1$,

$$\tfrac{\pi^2}{32}u(0) \;\leq\; \left(\tfrac{4}{9}\,c^2 + 1\right)\int_{B_1} u.$$

□

Now assume that no instantons on S^4 bubble off at the boundary slice $\{0\} \times \Sigma$, then we have shown above that only holomorphic discs can result from the 2-dimensional rescaling. Under the additional assumption of a bound on F_{A^ν} near the boundary we now show that in fact one obtains a holomorphic halfplane with Lagrangian boudary conditions. Its finite energy should allow for a removal of the singularity at infinity, giving rise to a holomorphic disc. However, it is not yet clear whether this limit object is necessarily nontrivial – this would require a stronger convergence in the following lemma. [2]

Lemma 8.8 *Suppose that $p > 4$, $(s^\nu, t^\nu) \to 0$, and $0 < \varepsilon^\nu \to 0$ such that*

$$\|B_s^\nu(s^\nu, t^\nu)\|_{L^p(\Sigma)} = R^\nu \to \infty, \qquad \sup_{(s,t)\in B_{2\varepsilon^\nu}(s^\nu,t^\nu)} \|B_s^\nu(s,t)\|_{L^p(\Sigma)} \leq 4R^\nu$$

with $r^\nu := \varepsilon^\nu R^\nu \to \infty$ and $t^\nu R^\nu \to \Delta < \infty$. Moreover, assume that for all $T > 0$ there exists a constant C_T such that

$$\sup_{|s-s^\nu|,\, t \leq \frac{T}{R^\nu}} \|F_{A^\nu}(s,t)\|_{L^p(\Sigma)} \;\leq\; C_T \qquad \forall \nu \in \mathbb{N}.$$

Let $D^\nu := [-r^\nu, r^\nu] \times [0, r^\nu]$ and define the rescaled metrics $\tilde{g}^\nu$ and connections $\tilde{\Phi}^\nu \mathrm{d}s + \tilde{\Psi}^\nu \mathrm{d}t + \tilde{A}^\nu$ on $D^\nu \times \Sigma$ as in (8.3) with $t^\nu + \frac{1}{R^\nu}\tau$ replaced by $\frac{1}{R^\nu}\tau$. Then

[2] More precisely, one needs the convergence of B_s^ν in $\mathcal{C}^0(\mathbb{H}, L^p(\Sigma))$. This might follow from an analogon of the curvature estimates in [DS2, Theorem 7.1].

one finds a subsequence and a gauge such that $\tilde{\Phi}^\nu \to 0$ *and* $\tilde{\Psi}^\nu \to 0$ *converge* $W^{1,p}$*-weakly on every compact subset of* $\mathbb{H} := \{(s,t) \in \mathbb{R}^2 \mid t \geq 0\}$, *and* $\tilde{A}^\nu \to \tilde{A} \in W^{1,p}(\mathbb{H}, \mathcal{A}^{0,p}(\Sigma))$. *The limit* $\tilde{A}$ *is a holomorphic curve in* $\mathcal{A}^{0,2}(\Sigma)$ *with Lagrangian boundary conditions and finite energy,*

$$\begin{cases} \partial_s \tilde{A} + *\partial_t \tilde{A} = 0, \\ \qquad \tilde{A}|_{(s,0)\times\Sigma} \in \mathcal{L} \quad \forall s \in \mathbb{R}, \end{cases} \qquad \int_{\mathbb{H}} \|\partial_s \tilde{A}\|^2_{L^2(\Sigma)} \leq E.$$

Proof: Except for a shift in the t-variable, this rescaling is the same as in (8.3). One then has $\|(\partial_s \tilde{A}^\nu - \mathrm{d}_{\tilde{A}^\nu}\tilde{\Phi}^\nu)(0, t^\nu R^\nu)\|_{L^p(\Sigma)} = 1$ for all $\nu \in \mathbb{N}$, and the additional assumption on F_{A^ν} yields the following estimates instead of (8.4): For all $T > 0$

$$\begin{aligned} \|F_{\tilde{A}^\nu}\|_{L^\infty([-T,T]\times[0,T],L^p(\Sigma))} &\leq C_T, \\ \|\partial_s \tilde{A}^\nu - \mathrm{d}_{\tilde{A}^\nu}\tilde{\Phi}^\nu\|_{L^\infty([-T,T]\times[0,T],L^p(\Sigma))} &\leq 4, \\ \|\partial_t \tilde{\Phi}^\nu - \partial_s \tilde{\Psi}^\nu + [\tilde{\Psi}^\nu, \tilde{\Phi}^\nu]\|_{L^\infty([-T,T]\times[0,T],L^p(\Sigma))} &\leq C_T (R^\nu)^{-2} \underset{\nu\to\infty}{\longrightarrow} 0. \end{aligned}$$

Now Uhlenbeck's weak compactness theorem on noncompact manifolds (e.g. [We, Theorem 8.5]) applies and yields a subsequence and gauges such that those connections converge in the weak $W^{1,p}$-topology and the strong L^∞-topology on every compact subset of $\mathbb{H} \times \Sigma$. The limit $\tilde{\Phi}\,\mathrm{d}s + \tilde{\Psi}\,\mathrm{d}t + \tilde{A} \in \mathcal{A}^{1,p}_{\mathrm{loc}}(\mathbb{H}\times\Sigma)$ satisfies the following boundary value problem:

$$\begin{cases} \partial_s \tilde{A} - \mathrm{d}_{\tilde{A}}\tilde{\Phi} = - * \big(\partial_t \tilde{A} - \mathrm{d}_{\tilde{A}}\tilde{\Psi}\big), \\ \partial_t \tilde{\Phi} - \partial_s \tilde{\Psi} + [\tilde{\Psi}, \tilde{\Phi}] = 0, \\ \tilde{A}|_{(s,0)\times\Sigma} \in \mathcal{L} \qquad \forall s \in \mathbb{R}. \end{cases}$$

One finds a further gauge transformation in $W^{1,p}_{\mathrm{loc}}(\mathbb{H}, W^{1,p}(\Sigma, \mathrm{G}))$ that makes $\tilde{\Phi}$ and $\tilde{\Psi}$ vanish and preserves the convergence of $\tilde{\Phi}^\nu, \tilde{\Psi}^\nu$. This gauge moreover transforms $\tilde{A}$ into a connection $\tilde{A} \in W^{1,p}_{\mathrm{loc}}(\mathbb{H}, \mathcal{A}^{0,p}(\Sigma))$ that is holomorphic with Lagrangian boundary conditions as claimed. However, the convergence of $\tilde{A}^\nu$ only is locally in the weak $W^{1,p}(\mathbb{H}, L^p(\Sigma))$- and the strong $L^\infty(\mathbb{H}, L^p(\Sigma))$-topology. Moreover, this rescaling and gauging preserves the energy bound

$$\begin{aligned} \int_{\mathbb{H}} \|\partial_s \tilde{A}\|^2_{L^2(\Sigma)} &\leq \limsup_{\nu\to\infty} \int_H \|\partial_s \tilde{A}^\nu - \mathrm{d}_{\tilde{A}^\nu}\tilde{\Phi}^\nu\|^2_{L^2(\Sigma)} \\ &\leq \limsup_{\nu\to\infty} \int_{B_{2\varepsilon^\nu}(s^\nu,t^\nu)\times\Sigma} |\partial_s \tilde{A}^\nu - \mathrm{d}_{\tilde{A}^\nu}\tilde{\Phi}^\nu|^2 \;\leq\; E. \end{aligned}$$

□

Appendix A

Gauge theory

In order to set up notation and state some general facts that are used in this thesis we give a short introduction to connections and curvature on principal bundles.

We consider a **principal** G-**bundle** $\pi : P \to M$, that is a manifold P with a free right action $P \times \mathrm{G} \to P$, $(p, g) \mapsto pg$ of a Lie group G such that the orbits of this action are the fibres $P_x = \pi^{-1}(x) \cong \mathrm{G}$ of a locally trivial fibre bundle $\pi : P \to M$. So M is a smooth manifold, the G-action preserves the fibres, $\pi(pg) = \pi(p)$, and there exists a bundle atlas $M = \bigcup_{\alpha \in A} U_\alpha$ with equivariant local trivializations

$$\Phi_\alpha : \begin{array}{ccc} \pi^{-1}(U_\alpha) & \longrightarrow & U_\alpha \times \mathrm{G} \\ p & \longmapsto & (\pi(p), \phi_\alpha(p)) \end{array} .$$

More precisely, the Φ_α are diffeomorphisms and their second component is equivariant, $\phi_\alpha(pg) = \phi_\alpha(p)g$.

This atlas gives rise to **transition functions** $\phi_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{G}$ defined by $\Phi_\alpha \circ \Phi_\beta^{-1}(x, g) = (x, \phi_{\alpha\beta}(x)g)$ for $x \in U_\alpha \cap U_\beta$, i.e. $\phi_{\alpha\beta}(x) = \phi_\alpha(p)\phi_\beta(p)^{-1}$ for all $p \in \pi^{-1}(x)$.

Now for any other manifold F with a representation $\sigma : \mathrm{G} \to \mathrm{Diff}(F)$ the **associated bundle** $\mathbf{P} \times_\sigma \mathbf{F}$ is the set of equivalence classes $\lfloor p, f \rfloor$ in $P \times F$, where the equivalence is given by σ, i.e. $\lfloor p, f \rfloor \sim \lfloor pg, \sigma(g^{-1})f \rfloor$ for all $g \in \mathrm{G}$. (Here we write $\lfloor \cdot, \cdot \rfloor$ for the equivalence classes in order to distinguish this notation from the Lie bracket $[\cdot, \cdot]$.) With the projection $\tilde{\pi}\lfloor p, g \rfloor = \pi(p)$ this is a principal bundle over M with fibre F. A local trivialization $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathrm{G}$ of P naturally induces the following local trivialization of $P \times_\sigma F$:

$$\tilde{\Phi}_\alpha : \begin{array}{ccc} \tilde{\pi}^{-1}(U_\alpha) & \longrightarrow & U_\alpha \times F \\ \lfloor p, f \rfloor & \longmapsto & (\pi(p), \sigma(\phi_\alpha(p))f) \end{array} .$$

We will, for example, encounter the associated bundle where F is the group G itself and the representation is the conjugation $c : \mathrm{G} \to \mathrm{Aut}(\mathrm{G})$ given by

$$c_g(h) = ghg^{-1} \qquad \forall\, g, h \in \mathrm{G}.$$

Another example is $\mathfrak{g}_\mathbf{P} := \mathbf{P} \times_{\mathrm{Ad}} \mathfrak{g}$. Here $\mathfrak{g} = \mathrm{T}_{1\!\!1}\mathrm{G}$ is the Lie algebra of G and $\mathrm{Ad} : \mathrm{G} \to \mathrm{End}\,\mathfrak{g}$, $g \mapsto \mathrm{Ad}_g = \mathrm{d}_{1\!\!1} c_g$ is the adjoint representation of G on its Lie algebra $\mathfrak{g}$,

$$\mathrm{Ad}_g(\xi) := g\xi g^{-1} \qquad \forall \xi \in \mathfrak{g}, g \in G.$$

This uses the following notation: For $\xi \in \mathfrak{g}$ and $g \in \mathrm{G}$

$$g\xi \;:=\; \mathrm{d}_{1\!\!1} L_g(\xi) \;=\; \tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}\, g \exp(t\xi) \;\in \mathrm{T}_g\mathrm{G}.$$

Here L_g denotes left multiplication by g and $\exp : \mathfrak{g} \to \mathrm{G}$ is the usual exponential map (with respect to any metric on G). The notation ξg is defined analogously by right multiplication. This makes particular sense when $\mathrm{G} \subset \mathbb{C}^{n \times n}$ is a matrix group since then $g\xi$ can be understood as matrix multiplication.

Moreover, the adjoint representation of $\mathfrak{g}$ on $\mathfrak{g}$ is given by the Lie bracket of vector fields as follows. We identify the Lie algebra elements $\xi \in \mathfrak{g}$ with left invariant vector fields $g \mapsto g\xi$ on G, then for $\xi, \zeta \in \mathfrak{g}$

$$\mathrm{ad}_\xi(\zeta) \;:=\; \mathrm{d}_{1\!\!1}\mathrm{Ad}(\xi)\; \zeta \;=\; \tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \exp(t\xi)\zeta \exp(t\xi)^{-1} \;=\; \mathcal{L}_\xi \zeta(1\!\!1) \;=\; [\xi, \zeta].$$

In the case of a matrix group note that the Lie bracket is given by the commutator $[\xi, \zeta] = \xi\zeta - \zeta\xi$.

Coming back to the principal bundle let $\Theta : \mathrm{G} \to \mathrm{Diff}(P)$, $g \mapsto \Theta_g$ denote the action of G on P. Then for $v \in \mathrm{T}_p P$, $g \in \mathrm{G}$ we write

$$vg := \mathrm{d}_p \Theta_g(v).$$

Moreover, the **infinitesimal action** is defined for $\xi \in \mathfrak{g}$ and $p \in P$ by

$$p\xi \;:=\; \mathrm{d}_{1\!\!1}\Theta(\xi)\, p \;=\; \tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} p \exp(t\xi).$$

Furthermore, a G-**bundle isomorphism** is a bundle isomorphism that preserves the action of the Lie group G, and such isomorphic bundles are usually identified. So when studying a fixed bundle we also have to consider its G-bundle automorphisms, i.e. diffeomorphisms $\psi : P \to P$ such that $\pi \circ \psi = \psi$ and that are equivariant, $\psi(pg) = \psi(p)g$ for all $p \in P, g \in \mathrm{G}$. Every such automorphism is given by $\psi(p) = p\,u(p)$, where the smooth map $u : P \to \mathrm{G}$ is a unique element of the **gauge group** $\mathcal{G}(\mathbf{P})$. This means that u is equivariant,

$$u(pg) = g^{-1}u(p)g \qquad \forall\, p \in P,\ g \in \mathrm{G}.$$

Obviously, composition $\psi_2 \circ \psi_1$ of G-bundle isomorphisms corresponds to group multiplication $u_1 u_2$ of the corresponding gauge transformations. Moreover, the gauge group is isomorphic to the group of sections of the associated bundle $P \times_c \mathrm{G}$. Let $u \in \mathcal{G}(P)$, then the corresponding section $\bar{u} : M \to P \times_c \mathrm{G}$ is given by

$$\bar{u}(\pi(p)) = \lfloor p, u(p) \rfloor \qquad \forall\, p \in P.$$

In the local trivialization a gauge transformation $u \in \mathcal{G}(P)$ is represented by $u_\alpha = \tilde{\phi}_\alpha \circ \bar{u} : M \to \mathrm{G}$ and acts by $(x, g) \mapsto (x, g\, u_\alpha(x))$ on $U_\alpha \times \mathrm{G}$. Here $\tilde{\phi}_\alpha(\lfloor p, g \rfloor) = \phi_\alpha(p)\, g\, \phi_\alpha(p)^{-1}$ is the second component of the trivialization $\tilde{\Phi}_\alpha$ of $P \times_c \mathrm{G}$. Thus $u_\alpha(x) = \phi_\alpha(p)u(p)\phi_\alpha(p)^{-1}$ for all $p \in \pi^{-1}(x)$, and hence on $U_\alpha \cap U_\beta$ one has the transition identity

$$u_\beta = \phi_{\alpha\beta}^{-1} u_\alpha \phi_{\alpha\beta}.$$

Finally, to introduce connections we first note that the G-bundle P has a canonical vertical subbundle $V \subset \mathrm{T}P$ given as follows. For every $p \in P$ the vertical space $V_p = \ker(\mathrm{d}_p\pi) \subset \mathrm{T}_pP$ is composed of all tangencies $p\xi$, $\xi \in \mathfrak{g}$ to the orbits of G through p. Every complement of V_p is isomorphic to $\mathrm{im}(\mathrm{d}_p\pi) = \mathrm{T}_{\pi(p)}M$, but there is no canonical choice of this horizontal space in T_pP. Now a connection of P defines such an equivariant horizontal distribution $H \subset \mathrm{T}P$ as follows.

A **connection** on P is an equivariant $\mathfrak{g}$-valued 1-form with fixed values in the vertical direction, i.e. $A \in \Omega^1(P; \mathfrak{g})$ satisfies

$$\begin{aligned} A_{pg}(vg) &= g^{-1}A_p(v)g && \forall\, v \in \mathrm{T}_pP,\ g \in \mathrm{G}, \\ A_p(p\xi) &= \xi && \forall\, p \in P,\ \xi \in \mathfrak{g}. \end{aligned}$$

We denote the set of smooth connections by $\mathcal{A}(\mathbf{P})$. Now every connection $A \in \mathcal{A}(P)$ corresponds to a splitting $\mathrm{T}P = V \oplus H$, where the horizontal distribution H is defined by $H_p = \ker A_p$.

Again, this can be formulated equivalently in terms of an associated bundle. If we fix one connection $\tilde{A} \in \mathcal{A}(P)$ then the space of connections is the affine space $\mathcal{A}(P) = \tilde{A} + \Omega^1_{\mathrm{Ad}}(P;\mathfrak{g})$. Here $\Omega^k_{\mathrm{Ad}}(P;\mathfrak{g})$ denotes the space of equivariant horizontal k-forms, i.e. $\tau \in \Omega^k(P;\mathfrak{g})$ that satisfy

$$\begin{aligned} &\Theta_g^* \tau = g^{-1}\tau g && \forall\, g \in \mathrm{G}, \\ &\iota_{p\xi}\tau_p = 0 && \forall\, p \in P,\ \xi \in \mathfrak{g}. \end{aligned}$$

Now this space is isomorphic to the space $\Omega^k(M;\mathfrak{g}_P)$ of k-forms on M with values in the associated bundle $\mathfrak{g}_P = P \times_{\mathrm{Ad}} \mathfrak{g}$. Indeed, for $\tau \in \Omega^k_{\mathrm{Ad}}(P;\mathfrak{g})$ the corresponding $\bar{\tau} \in \Omega^k(M;\mathfrak{g}_P)$ is uniquely defined by

$$\lfloor p, \tau_p(Y_1, \ldots Y_k)\rfloor = \bar{\tau}_{\pi(p)}(\mathrm{d}_p\pi(Y_1), \ldots \mathrm{d}_p\pi(Y_k)) \qquad \forall\, Y_1, \ldots Y_k \in \mathrm{T}_pP.$$

Consequently, in the local trivialization every $\tau \in \Omega^k_{\mathrm{Ad}}(P;\mathfrak{g})$ is represented by $\tau_\alpha = \tilde{\phi}_\alpha \circ \bar{\tau} \in \Omega^k(U_\alpha;\mathfrak{g})$. Here $\tilde{\phi}_\alpha(\lfloor p, \xi\rfloor) = \phi_\alpha(p)\,\xi\,\phi_\alpha(p)^{-1}$ is the second component of the associated trivialization of $\mathfrak{g}_P$. On the intersection $U_\alpha \cap U_\beta$ of two charts these k-forms satisfy

$$\tau_\beta = \phi_{\alpha\beta}^{-1}\,\tau_\alpha\,\phi_{\alpha\beta}. \tag{A.1}$$

In the case of connections this local representation depends on the chosen connection $\tilde{A}$ and there is no canonical choice for this reference connection. However, locally on $\pi^{-1}(U_\alpha)$ a natural choice of the reference connection is $\tilde{A}_\alpha = \phi_\alpha^{-1}\mathrm{d}\phi_\alpha$. This corresponds to the pullback of the splitting under $\Phi_\alpha : P \to U_\alpha \times \mathrm{G}$. The local representative $A_\alpha \in \Omega^1(U_\alpha;\mathfrak{g})$ of $A \in \mathcal{A}(P)$ is then given by

$$A_\alpha(\mathrm{d}_p\pi(Y)) = \phi_\alpha(p)\,A(Y)\,\phi_\alpha(p)^{-1} - \mathrm{d}_p\phi_\alpha(Y)\,\phi_\alpha(p)^{-1} \qquad \forall\, Y \in \mathrm{T}_pP.$$

Note that the transition between different coordinate charts is different from (A.1) since the reference connection is not globally defined. On $U_\alpha \cap U_\beta$ one has

$$A_\beta = \phi_{\alpha\beta}^{-1} A_\alpha \phi_{\alpha\beta} + \phi_{\alpha\beta}^{-1}\mathrm{d}\phi_{\alpha\beta}.$$

One can think of this as the effect of a local gauge transformation. So next, we discuss the action of the gauge group. Consider a gauge transformation

$u \in \mathcal{G}(P)$ and the corresponding G-bundle automorphism $\psi : p \mapsto p\,u(p)$. Their action on a connection $A \in \mathcal{A}(P)$ is given by

$$u^*A \;:=\; \psi^*A \;=\; u^{-1}Au + u^{-1}\mathrm{d}u.$$

This is the connection on $\psi^*P \cong P$ that corresponds to the connection A on P. Hence u^*A and A are viewed as equivalent connections – they are **gauge equivalent**. Finally, the local formula for the gauge action is

$$(u^*A)_\alpha = u_\alpha^{-1}A_\alpha u_\alpha + u_\alpha^{-1}\mathrm{d}u_\alpha.$$

This shows that locally a gauge transformation can also be thought of as a change of the trivialization.

Connections also induce covariant derivatives on associated vector bundles. In particular, a connection $A \in \mathcal{A}(P)$ defines the following covariant derivative on $\mathfrak{g}_P$:

$$\nabla_A : \begin{array}{ccc} \Gamma(\mathfrak{g}_P) & \longrightarrow & \Gamma(\mathrm{T}^*M \otimes \mathfrak{g}_P) \\ s & \longmapsto & \mathrm{d}s + [A, s]. \end{array}$$

Here and throughout, $\Gamma(\cdot)$ denotes the set of smooth sections of a bundle. For $X \in \mathrm{T}_xM$ with $Y \in \mathrm{T}_pP$ such that $\mathrm{d}_p\pi(Y) = X$ this evaluates as

$$\nabla_A s(X) \;=\; \lfloor\, p\,,\, \mathrm{d}_p s(Y) + [A(Y), s(p)]\,\rfloor \;\in (\mathfrak{g}_P)_x,$$

where on the right hand side $s \in \Gamma(\mathfrak{g}_P)$ is understood as map from P to $\mathfrak{g}$. By a standard construction this covariant derivative can then be extended to $\nabla_A : \Gamma(\otimes^k\mathrm{T}^*M \otimes \mathfrak{g}_P) \to \Gamma(\otimes^{k+1}\mathrm{T}^*M \otimes \mathfrak{g}_P)$ for all $k \in \mathbb{N}$ as follows. Let ∇ be the Levi-Civita connection on M, then for $\alpha \in \Gamma(\otimes^k\mathrm{T}^*M \otimes \mathfrak{g}_P)$ and $X_0, \ldots, X_k \in \Gamma(\mathrm{T}M)$

$$\begin{aligned} \nabla_A\alpha(X_0, \ldots, X_k) = \nabla_A\left(\alpha(X_1, \ldots, X_k)\right)(X_0) - \alpha(\nabla_{X_0}X_1, X_2, \ldots, X_k) \\ - \ldots - \alpha(X_1, \ldots, X_{k-1}, \nabla_{X_0}X_k). \end{aligned} \tag{A.2}$$

But the covariant derivative on $\mathfrak{g}_P$ can also be understood as the special case $k = 0$ of the exterior derivative

$$\mathrm{d}_A : \begin{array}{ccc} \Omega^k_{\mathrm{Ad}}(P; \mathfrak{g}) & \longrightarrow & \Omega^{k+1}_{\mathrm{Ad}}(P; \mathfrak{g}) \\ \tau & \longmapsto & \mathrm{d}\tau + [A \wedge \tau]. \end{array}$$

Here $[A\wedge\tau]$ denotes the wedge product of the two forms with the Lie bracket used to combine the values in $\mathfrak{g}$. For example, for $A, B \in \Omega^1_{\mathrm{Ad}}(P;\mathfrak{g})$ and $X, Y \in \mathrm{T}_pP$

$$[A\wedge B](X,Y) = [A(X),B(Y)] - [A(Y),B(X)].$$

Now ${\mathrm{d}_A}^2$ does not vanish in general, but we obtain $\mathrm{d}_A\mathrm{d}_A\tau = [F_A\wedge\tau]$ for all $\tau\in\Omega^k_{\mathrm{Ad}}(P;\mathfrak{g})$, with the **curvature**

$$F_A \;=\; \mathrm{d}A + \tfrac{1}{2}[A\wedge A] \;\in \Omega^2_{\mathrm{Ad}}(P;\mathfrak{g}).$$

The curvature satisfies the Bianchi identity (see e.g. [KN, II,Theorem 5.4])

$$\mathrm{d}_A F_A = 0.$$

Locally, the exterior derivative d_A on $\tau\in\Omega^k_{\mathrm{Ad}}(P;\mathfrak{g})$ is represented by

$$(\mathrm{d}_A\tau)_\alpha = \mathrm{d}\tau_\alpha + [A_\alpha\wedge\tau_\alpha].$$

Thus for the curvature in terms of the local representatives A_α of the connection we obtain the same formula as globally,

$$(F_A)_\alpha = \mathrm{d}A_\alpha + \tfrac{1}{2}[A_\alpha\wedge A_\alpha].$$

In coordinates $(x^1,\dots,x^k)$ of U_α and dropping the subscript α one has the following formula for the components $F_{ij} := F_A(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j})$ of the curvature,

$$F_{ij} = \tfrac{\partial A_i}{\partial x^j} - \tfrac{\partial A_j}{\partial x^i} + [A_i, A_j].$$

A change of the trivialization has the effect that on $U_\alpha\cap U_\beta$ the local representatives of the curvature satisfy $(F_A)_\beta = \phi_{\alpha\beta}^{-1}(F_A)_\alpha\phi_{\alpha\beta}$. Analogously, gauge transformations act on the curvature by the adjoint action,

$$F_{u^*A} = u^{-1}F_A u \qquad \forall\, A\in\mathcal{A}(P), u\in\mathcal{G}(P).$$

A connection $A\in\mathcal{A}(P)$ is called **flat** if its curvature vanishes, $F_A = 0$. This is equivalent to $\mathrm{d}_A\circ\mathrm{d}_A = 0$, and moreover one has the following characterization, see e.g. [DK, Section 2.2].

Theorem A.1 *A connection $A\in\mathcal{A}(P)$ is flat if and only if the associated horizontal distribution $H = \mathrm{kerA}\subset \mathrm{T}P$ is locally integrable. That is given a fixed trivialization over a simply connected domain $U_\alpha\subset M$ there exists a local gauge transformation $u\in\mathcal{G}(P|_{U_\alpha})$ such that $(u^*A)_\alpha = 0$, or equivalently if one changes the trivialization by u_α, then $H = \mathrm{T}U_\alpha\times\{0\}$.*

Every connection $A \in \mathcal{A}(P)$ also defines parallel transport in P. Along a path $\gamma : [0,1] \to M$ the parallel transport

$$\Pi_\gamma : P_{\gamma(0)} \to P_{\gamma(1)}$$

is given by $s(0) \mapsto s(1)$, where $s : [0,1] \to P$ is a horizontal lift of γ, i.e. $\pi \circ s = \gamma$, such that $\dot{s}(t) \in H_{s(t)}$ for all $t \in [0,1]$. The parallel transport is equivariant in the sense that $\Pi_\gamma(pg) = \Pi_\gamma(p)g$, so Π_γ is determined by its value on one fixed $p \in P_{\gamma(0)}$. For loops γ starting at a fixed x and with fixed $p \in P_x$, the parallel transport can be identified with group elements h_γ via $\Pi_\gamma(p) = ph_\gamma$. However, if p is allowed to vary, then h_γ is welldefined only up to conjugation.

If the connection is flat, i.e. H is locally integrable, then the parallel transport is invariant under homotopies of the path with fixed endpoints. Thus for fixed $x \in M$ and $p \in P_x$ every flat connection determines a representation of $\pi_1(M,x)$ in G, the holonomy based at x,

$$\rho_{x,p}(A) : \begin{array}{ccc} \pi_1(M,x) & \longrightarrow & \mathrm{G} \\ \gamma & \longmapsto & h_\gamma . \end{array}$$

Variation of x in a connected component or another choice of p results in a conjugation of $\rho_{x,p}$. In the case of a trivial bundle $P = M \times \mathrm{G}$, there is a natural choice $p = (x, \mathbb{1})$ which leads to a natural welldefined based holonomy

$$\rho_x : \begin{array}{ccc} \mathcal{A}(P) & \longrightarrow & \mathrm{Hom}(\pi_1(M,x), \mathrm{G}) \\ A & \longmapsto & \rho_x(A). \end{array}$$

If M is connected then the conjugacy class of $\rho := \rho_{x,p}$ is welldefined independently of the choice of x and p. Moreover, gauge transformations also act on the holonomy by conjugation,

$$\rho_{x,p}(u^*A) = u(p)^{-1}\rho_{x,p}(A)\, u(p).$$

One even finds that two flat connections are gauge equivalent if and only if their holonomies are conjugate. Conversely, given a representation of the fundamental group in G one can construct a G-bundle and a connection on it that has this holonomy map. So denote the space of flat connections on a G-bundle over a fixed base manifold M by $\mathcal{A}_{\text{flat}}(M)$, the group of gauge transformations on such bundles by $\mathcal{G}(M)$, and the conjugacy equivalence relation by $\sim$, then this leads to the following observation (see e.g. [DK, Proposition 2.2.3]).

Theorem A.2 *Let M be connected, then the holonomy induces a natural bijection of sets*

$$\mathcal{A}_{\text{flat}}(M)/\mathcal{G}(M) \cong \text{Hom}(\pi_1(M), \mathrm{G})/\sim .$$

Next, one would like to have a gauge invariant quantity measuring the nonflatness of a connection. Note that the norm of the curvature is indeed gauge invariant if $\mathfrak{g}$ is equipped with an inner product that is invariant under the adjoint action of G. So from now on we restrict ourselves to compact Lie groups G because of the following proposition. Its proof can be found in [K, Proposition 4.24].

Proposition A.3 *Let* G *be a compact Lie group and let $\mathfrak{g}$ be its Lie algebra. Then there exists an inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ that is invariant under the adjoint action of the Lie group,*

$$\langle\, g\xi g^{-1}, g\zeta g^{-1} \,\rangle = \langle\, \xi, \zeta \,\rangle \qquad \forall \xi, \zeta \in \mathfrak{g}, g \in \mathrm{G}. \tag{A.3}$$

Remark A.4

(i) The G-invariant inner product on $\mathfrak{g}$ moreover satisfies for all $\xi, \zeta, \eta \in \mathfrak{g}$

$$\langle\, [\xi, \eta], \zeta \,\rangle = \langle\, \xi, [\eta, \zeta] \,\rangle.$$

(ii) The G-invariant inner product on $\mathfrak{g}$ induces a metric on G by

$$\langle\, X \,, \, Y \,\rangle_G := \langle\, g^{-1}X \,, \, g^{-1}Y \,\rangle \qquad \forall X, Y \in \mathrm{T}_g\mathrm{G}.$$

In this metric the left and right multiplications are isometries of G. Denote by $\exp_g$ the exponential map with base point $g \in \mathrm{G}$ and set $\exp := \exp_{\mathbb{1}}$, then for all $\xi \in \mathfrak{g}$ and $g \in \mathrm{G}$

$$\exp_g(g\xi) = g \exp(\xi), \qquad \exp(g^{-1}\xi g) = g^{-1} \exp(\xi) g.$$

Moreover, the geodesics are the flow lines of the left invariant vector fields, hence they are 1-parameter subgroups: For all $s, t \in \mathbb{R}$ and $\xi \in \mathfrak{g}$

$$\exp((s + t)\xi) = \exp(s\xi) \exp(t\xi).$$

Here (i) follows from differentiating (A.3) with $g = \exp(t\eta)$. In (ii) one has used that for all $\xi \in \mathfrak{g}$ and $g \in \mathrm{G}$ both $t \mapsto \exp_g(tg\xi)$ and $t \mapsto g\exp(t\xi)$ are geodesics with identical initial values. The same holds for $t \mapsto \exp(tg^{-1}\xi g)$ and $t \mapsto g^{-1}\exp(t\xi)g$.

A flow line $\gamma(t)$ satisfies $\dot\gamma(t) = \gamma(t)\xi$ for some $\xi \in \mathfrak{g}$. One checks the geodesic equation $\nabla_{\dot\gamma}\dot\gamma = 0$ with $Z(t) = \gamma(t)\eta$ for all $\eta \in \mathfrak{g}$,

$$\begin{aligned} g(\nabla_{\dot\gamma}\dot\gamma, Z) &= \mathcal{L}_{\dot\gamma}\, g(\dot\gamma, \gamma\eta) - \tfrac{1}{2}\mathcal{L}_Z\, g(\dot\gamma, \dot\gamma) - g(\dot\gamma, [\dot\gamma, \gamma\eta]) \\ &= \mathcal{L}_{\dot\gamma}\langle\, \xi\, ,\, \eta\,\rangle - \tfrac{1}{2}\mathcal{L}_Z\langle\, \xi\, ,\, \xi\,\rangle - \langle\, \xi\, ,\, [\xi, \eta]\,\rangle \;=\; 0. \end{aligned}$$

Throughout this thesis every compact Lie group G is equipped with the metric from proposition A.3 and remark A.4. Furthermore, fix a metric on M. This defines a volume element dvol_M and the Hodge operator $*$ on differential forms. Together with the inner product of $\mathfrak{g}$ this moreover defines an inner product on the fibres of $\otimes^k \mathrm{T}^*M \otimes \mathfrak{g}_P$ for all $k \in \mathbb{N}_0$ as follows. In the first component of the fibre, $\otimes^k \mathrm{T}^*_x M$, use the standard inner product on T^*M in each factor. On $(\mathfrak{g}_P)_x = \{\lfloor p, \xi\rfloor \,\big|\, p \in \pi^{-1}(x), \xi \in \mathfrak{g}\}$ the G-invariant inner product of $\mathfrak{g}$ induces the welldefined

$$\langle\, \lfloor p, \xi\rfloor\, ,\, \lfloor p, \zeta\rfloor\,\rangle_{\mathfrak{g}_P} := \langle\, \xi\, ,\, \zeta\,\rangle.$$

For $\sigma, \tau \in \Omega^k(M; \mathfrak{g}_P)$ this pointwise inner product equals the inner product of the local representatives $\sigma_\alpha, \tau_\alpha$ in every trivialization over $U_\alpha \subset M$,

$$\langle\, \sigma\, ,\, \tau\,\rangle_{\Lambda^k \mathrm{T}^*M \otimes \mathfrak{g}_P} \;=\; *\langle\, \sigma \wedge *\tau\,\rangle_{\mathfrak{g}_P} \;=\; *\langle\, \sigma_\alpha \wedge *\tau_\alpha\,\rangle_{\mathfrak{g}} \;=\; \langle\, \sigma_\alpha\, ,\, \tau_\alpha\,\rangle_{\Lambda^k \mathrm{T}^*M \otimes \mathfrak{g}}.$$

In the second and third expression the values of the differential forms are paired by the inner product idicated by the subscript. For example, for $\mathfrak{g}$-valued 1-forms $\sigma = \sigma_1 \mathrm{d}x^1 + \sigma_2 \mathrm{d}x^2$ and $\tau = \tau_1 \mathrm{d}x^1 + \tau_2 \mathrm{d}x^2$ on $\mathbb{R}^2$ one has

$$\langle\, \sigma \wedge \tau\,\rangle_{\mathfrak{g}} = \big(\langle\, \sigma_1, \tau_2\,\rangle_{\mathfrak{g}} - \langle\, \sigma_2, \tau_1\,\rangle_{\mathfrak{g}}\big)\mathrm{d}x^1 \wedge \mathrm{d}x^2.$$

Usually the inner product is clear from the context, so we drop all subscripts. Furthermore, $*$ denotes the obvious Hodge operator on $\mathfrak{g}_P$- or $\mathfrak{g}$-valued differential forms. (When written in local coordinates as a sum of products of sections (or $\mathfrak{g}$-valued functions) and differential forms $\mathrm{d}x_{i_1} \wedge \ldots \mathrm{d}x_{i_k}$ the Hodge operator only operates on the differential form.)

Now the curvature F_A can be viewed as a section of $\otimes^2 \mathrm{T}^*M \otimes \mathfrak{g}_P$, so the norm induced by above inner product defines a function $|F_A| : M \to \mathbb{R}$ that can be integrated to give the **Yang-Mills energy**

$$\mathcal{YM}(A) = \int_M |F_A|^2\, \mathrm{dvol}_M.$$

Due to the invariance of the metric (A.3) this functional on $\mathcal{A}(P)$ is gauge invariant,

$$\mathcal{YM}(u^*A) = \mathcal{YM}(A) \qquad \forall\, u \in \mathcal{G}(P).$$

Its extrema solve the weak Yang-Mills equation

$$\int_M \langle\, F_A\, ,\, \mathrm{d}_A\beta \,\rangle = 0 \qquad \forall\, \beta \in \Omega^1(M; \mathfrak{g}_P). \tag{A.4}$$

Here $\Omega^1(\cdot)$ denotes the smooth 1-forms. When the base manifold M is compact and has no boundary then for smooth connections (A.4) is equivalent to the usual Yang-Mills equation $\mathrm{d}_A^* F_A = 0$. If the base manifold is allowed to have boundary then (A.4) for smooth connections is equivalent to the following boundary value problem:

$$\begin{cases} \mathrm{d}_A^* F_A = 0, \\ *F_A|_{\partial M} = 0. \end{cases} \tag{A.5}$$

Here the operator $\mathrm{d}_A^* : \Omega^k(M; \mathfrak{g}_P) \to \Omega^{k-1}(M; \mathfrak{g}_P)$ is the formally adjoint differential operator of $\mathrm{d}_A : \Omega^{k-1}(M; \mathfrak{g}_P) \to \Omega^k(M; \mathfrak{g}_P)$ defined in the usual sense: For $\omega \in \Omega^k(M; \mathfrak{g}_P)$ and all $\beta \in \Omega^{k-1}(M; \mathfrak{g}_P)$ compactly supported in the interior of M

$$\int_M \langle\, \mathrm{d}_A^*\omega\, ,\, \beta \,\rangle = \int_M \langle\, \omega\, ,\, \mathrm{d}_A\beta \,\rangle.$$

From this one sees that $\mathrm{d}_A^* = -(-1)^{(n-k)(k-1)} * \mathrm{d}_A *$ on $\Omega^k(M; \mathfrak{g}_P)$, where $n = \dim M$, and locally for all $\omega \in \Omega^k(M; \mathfrak{g}_P)$

$$(\mathrm{d}_A^*\omega)_\alpha = \mathrm{d}^*\omega_\alpha - (-1)^{(n-k)(k-1)} * [A_\alpha \wedge *\omega_\alpha]. \tag{A.6}$$

The weak and strong Yang-Mills equation are preserved under gauge transformations. For the weak equation this is obvious from the gauge invariance of the Yang-Mills functional – the extrema come in gauge orbits. For the strong (i.e. pointwise) equation (A.5) one can check that $\mathrm{d}_{u^*A}^* F_{u^*A} = u^{-1}(\mathrm{d}_A^* F_A)u$.

So far this is all well defined since we only considered smooth connections. The Yang-Mills energy might however be infinite if the base manifold M is not compact. In that case there also are no natural Sobolev spaces of connections. So for the rest of this appendix we assume both M and G to be compact and explain how above concepts generalize to the appropriate Sobolev spaces.

Firstly, for $1 \leq p < \infty$ and $k \in \mathbb{N}$ the Sobolev space of connections

$$\mathcal{A}^{k,p}(P) := \tilde{A} + W^{k,p}(M, \mathrm{T}^*M \otimes \mathfrak{g}_P),$$

is independent of the choice of a smooth reference connection $\tilde{A} \in \mathcal{A}(P)$. Only the corresponding Sobolev norm $\|\cdot\|_{W^{k,p}}$ on $W^{k,p}(M, \mathrm{T}^*M \otimes \mathfrak{g}_P)$ depends on $\tilde{A}$ unless $k = 0$. This norm is defined as usual – using the Levi-Civita connection on M and the above norm on $\otimes^{\ell}\mathrm{T}^*M \otimes \mathfrak{g}_P$. The Sobolev space then is the completion of the space of smooth connections with respect to this norm. In a local trivialization over $U \subset M$ the connections in $\mathcal{A}^{1,p}(P)$ are represented by 1-forms in

$$\mathcal{A}^{1,p}(U) := W^{1,p}(U, \mathrm{T}^*U \otimes \mathfrak{g}),$$

and the corresponding Sobolev norm is the usual $W^{1,p}$-norm on this space. The gauge action can also be defined for a suitable Sobolev space of gauge transformations for $kp > n$. (The definition of this Sobolev space requires the choice of an atlas for G, but for $kp > n$ it is independent of this choice.)

$$\mathcal{G}^{k,p}(P) := W^{k,p}(M, P \times_c \mathrm{G}).$$

This space consist of all those gauge transformations $u = s \cdot \exp(\xi)$, where $s \in \mathcal{G}(P)$ is smooth and $\xi \in W^{k,p}(M, \mathfrak{g}_P)$ is understood as equivariant map $\xi : P \to \mathfrak{g}$. In a trivialization over $U \subset M$ gauge transformations in $\mathcal{G}^{k,p}(P|_U)$ are represented by maps $u \in \mathcal{G}^{k,p}(U)$, i.e. $u = s \cdot \exp(\xi) : U \to \mathrm{G}$ with $s \in \mathcal{C}^\infty(U, \mathrm{G})$ and $\xi \in W^{k,p}(U, \mathfrak{g})$. For more details on these Sobolev spaces see e.g. [We, Appendix B].

These sets are Banach manifolds in the topological space of continuous gauge transformations and they are actual groups with continuous group operations. A proof of the following lemma can for example be found in [We, Appendix A].

Lemma A.5 *Let $k \in \mathbb{N}$ and $1 \leq p < \infty$ be such that $kp > n$. Then group multiplication and inversion are continuous maps on $\mathcal{G}^{k,p}(P)$, and the gauge action is a continuous map*

$$\begin{array}{ccc} \mathcal{G}^{k,p}(P) \times \mathcal{A}^{k-1,p}(P) & \longrightarrow & \mathcal{A}^{k-1,p}(P) \\ (u, A) & \longmapsto & u^*A. \end{array}$$

If the base manifold M is noncompact, then one considers the Sobolev spaces $\mathcal{A}^{k,p}_{\mathrm{loc}}(P)$ and $\mathcal{G}^{k,p}_{\mathrm{loc}}(P)$ of locally $W^{k,p}$-regular connections and gauge transformations respectively – these are required to be of class $W^{k,p}$ on every compact subset of M.

Bibliography

[A1] M.F. Atiyah, Instantons in two and four dimensions, *Comm. Math. Phys.* **93** (1984), 437-451.

[A2] M.F. Atiyah, New invariants of three and four dimensional manifolds, *Proc. Symp. Pure Math.* **48** (1988).

[Ad] R.A.Adams, *Sobolev Spaces*, Academic Press, 1978.

[AB] M.F.Atiyah, R.Bott, The Yang Mills equations over Riemann surfaces, *Phil. Trans. R. Soc. Lond. A* **308** (1982), 523–615.

[B] T.Bröcker, T.tom Dieck, *Representations of Compact Lie Groups*, Springer, 1985.

[D1] S.K. Donaldson, Connections, cohomology and the intersection forms of four manifolds, *J. Diff. Geom.* **24** (1986), 275–341.

[D2] S.K. Donaldson, Polynomial invariants of smooth four-manifolds, *Topology* **29** (1990), 257–315.

[D3] S.K.Donaldson, Boundary value problems for Yang-Mills fields, *Journal of Geometry and Physics* **8** (1992), 89-122.

[D4] S.K.Donaldson *Floer Homology Groups in Yang-Mills Theory*, Cambridge University Press, 2002.

[DK] S.K.Donaldson, P.B.Kronheimer, *The Geometry of Four-Manifolds*, Oxford Science Publications, 1990.

[DS1] S.Dostoglou, D.A.Salamon, Instanton Homology and Symplectic Fixed Points, Symplectic Geometry, edited by D. Salamon, Proceedings of a Conference, LMS Lecture Notes Series 192, Cambridge University Press, 1993, 57-94.

[DS2] S.Dostoglou, D.A.Salamon, Self-dual instantons and holomorphic curves, *Annals of Mathematics* **139** (1994), 581–640.

[F1] A.Floer, An instanton invariant for 3-manifolds, *Comm. Math. Phys.* **118** (1988), 215–240.

[F2] A.Floer, Morse theory for Lagrangian intersections, *J. Diff. Geom.* **28** (1988), 513–547.

[F3] A. Floer, Symplectic fixed points and holomorphic spheres, *Comm. Math. Phys.* **120** (1989), 575–611.

[Fu1] K.Fukaya, Floer homology for 3-manifolds with boundary I, Preprint 1997. `http://www.kusm.kyoto-u.ac.jp/~fukaya/fukaya.html`

[Fu2] K.Fukaya, Anti-self-dual equation on 4-manifolds with degenerate metric, *GAFA* **8** (1998), 466–528.

[Gr] M.Gromov, Pseudo holomorphic curves in symplectic manifolds, *Invent. Math.* **82** (1985), 307–347.

[GT] D.Gilbarg & N.S.Trudinger, *Elliptic partial differential equations of second order*, Springer, 1977.

[HZ] H. Hofer, E. Zehnder, *Symplectic Invariants and Hamiltonian Dynamics*, Birkhäuser, 1994.

[J] J. Jost *Riemannian Geometry and Geometric Analysis*, Springer, 1995.

[K] A.W.Knapp, *Lie Groups beyond an Introduction*, Birkhäuser, 1996.

[KN] S.Kobayashi, K.Nomizu, *Foundations of Differential Geometry I*, Wiley Classics Library, 1996.

[L] S.Lang, *Analysis II*, Addison-Wesley, 1969.

[MS1] D.McDuff, D.Salamon, *Introduction to Symplectic Topology*, Oxford University Press, 1995.

[MS2] D.McDuff, D.Salamon, *J-holomorphic Curves and Quantum Cohomology*, new edition, in preparation.

[Sa1] D.A.Salamon, Lagrangian intersections, 3-manifolds with boundary, and the Atiyah–Floer conjecture, *Proceedings of the ICM, Zürich, 1994*, Birkhäuser, Basel, 1995, Vol. 1, 526–536.

[Sa2] D.A.Salamon, Lectures on Floer Homology, *Park City Mathematics Series* **7** (1999), 145 –229.

[T] C.H. Taubes, Casson's invariant and gauge theory, *J. Diff. Geom.* **31** (1990), 547–599.

[U1] K.K.Uhlenbeck, Removable singularities in Yang-Mills fields, *Comm. Math. Phys.* **83** (1982), 11-29.

[U2] K.K.Uhlenbeck, Connections with L^p-bounds on curvature, *Comm. Math. Phys.* **83** (1982), 31-42.

[Wa] F.W.Warner, *Foundations of Differentiable Manifolds and Lie Groups*, Scott, Foresman and Company, 1971.

[We] K.Wehrheim, *Uhlenbeck Compactness*, textbook, in preparation.

[Z] E.Zeidler, *Applied Functional Analysis – Main Principles and their Applications*, Springer, 1995.

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