Chapter 3

Special coordinates near generalized finite energy surfaces

The aim of this chapter is to construct a trivialization of the normal bundle of a finite energy surface into the manifold $\mathbb{R} \times M$. More precisely, we will prove the following theorem.

Theorem 3.1 ([12] Thm.4.7.)
Consider the immersed finite energy surface

$$\tilde{u} = (a, u) : \; (\dot{S}, j) \to (\mathbb{R} \times M, \tilde{J})$$

with the almost complex structure $\tilde{J}$ special or generalized. Assume the punctures are non-degenerate. Then for some neighbourhood $B_\varepsilon(0) \subset \mathbb{R}^2$ of the origin, there exists an immersion

$$\Phi : \; \dot{S} \times B_\varepsilon(0) \to \mathbb{R} \times M,$$

satisfying $\Phi(z, 0) = \tilde{u}(z)$ for all $z \in \dot{S}$. If $\tilde{u}$ is an embedding, then also Φ restricted to a sufficiently small neighbourhood of $\dot{S} \times \{0\}$ is an embedding.

Moreover, the almost complex structure

$$\bar{J} = T\Phi^{-1} \circ \tilde{J}(\Phi) \circ T\Phi$$

induced on $\dot{S} \times B_\varepsilon(0)$ splits along $\dot{S} \times \{0\}$ near the punctures,

$$\bar{J}(z, 0) = j(z) \oplus J_0 \; \in \mathcal{L}(T_z\dot{S} \times \mathbb{R}^2)$$

for all $z \in \dot{S}$ sufficiently close to a puncture.

Finally, introducing the cylindrical coordinates σ from (1.4) around a (positive) puncture $z_0 \in \Gamma \subset S$, the almost complex structure

$$\bar{J}_0(s, t, x, y) = (T\sigma \oplus \mathbb{1}_{\mathbb{R}^2})^{-1} \circ \bar{J}(\sigma(s, t), (x, y)) \circ (T\sigma \oplus \mathbb{1}_{\mathbb{R}^2})$$

converges as follows. For all derivatives D^α with respect to s, t, x and y we have

$$D^\alpha \Big(\bar{J}_0(s,t,x,y) - A(t,x,y)^{-1}\ \hat{J}_\infty(t,x,y)\ A(t,x,y) \Big) \overset{s\to\infty}{\longrightarrow} 0$$

uniformly on bounded sets of (t,x,y), where A and $\hat{J}_\infty$ are some smooth matrix functions.

In order to achieve the special behaviour of $\tilde{J}$ near the punctures, we will explicitly construct Φ near every puncture. Then we have to make sure that this is done in such a way that Φ can be extended to all of $\dot{S}$.

For the construction of a trivialization $\Phi_j : \dot{D}_j \times B_\varepsilon(0) \to \mathbb{R} \times M$ near a puncture $z_j \in \Gamma$ we use the special cylindrical holomorphic coordinates σ on $\dot{D}_j \subset \dot{S}$ introduced in (1.4). For simplicity we assume the puncture to be positive; negative punctures can be treated analogously.

According to the theorems 1.2 and 1.4 and since all punctures were assumed to be nondegenerate, we have $a(\sigma(s,\cdot)) \to \infty$ as $s \to \infty$, and $u(\sigma(s,\cdot))$ converging in $\mathcal{C}^\infty(S^1)$ to a nondegenerate periodic orbit of the Reeb vectorfield of M. The first convergence also implies that — after choosing D_j small enough — the almost complex structure $\tilde{J}$ has on all of $\tilde{u}(\dot{D}_j)$ the special form.

Furthermore, by the subsequent lemma, one has special coordinates in some tubular neighbourhood of the asymptotic orbit, and due to above convergence of u we can choose D_j sufficiently small such that $\tilde{u}(\dot{D}_j)$ lies within this tubular neighbourhood. That way we can use the following local coordinates of M for the construction of Φ_j.

Lemma 3.2 ([10] Lemma 2.3)
Let (M,λ) be a 3-dimensional contact manifold, and let $x(t)$ be a T-periodic solution of the corresponding Reeb vectorfield $\dot{x} = X_\lambda(x)$ on M. Let τ be the minimal period of x such that $T = k\tau$ for some positive integer k. Then there is an open neighbourhood $U \subset S^1 \times \mathbb{R}^2$ of $S^1 \times \{0\}$ and an open neighbourhood $V \subset M$ of $P = \{x(t) \mid t \in \mathbb{R}\}$ and a diffeomorphism $\psi : U \to V$ mapping $S^1 \times \{0\}$ onto P such that

$$\psi^*\lambda = f \cdot \lambda_0,$$

with $\lambda_0 = \mathrm{d}\theta + x\,\mathrm{d}y$ the canonical contact form on $S^1 \times \mathbb{R}^2$ and a positive smooth function $f : U \to \mathbb{R}$ satisfying

$$f(\theta,0,0) = \tau \qquad \text{and} \qquad df(\theta,0,0) = 0$$

for all $\theta \in S^1$.

Let us recall the properties of these coordinates in more detail from [10]. Without loss of generality we work on the covering $\mathbb{R} \times \mathbb{R}^2$ of $S^1 \times \mathbb{R}^2$ with coordinates (θ, x, y). Then the function f is 1-periodic in the variable θ and the periodic orbit is $x(t) = (kt + d, 0, 0)$, $t \in [0,1]$ for some $d \in \mathbb{R}$. The Reeb vector field X on M is represented by

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \frac{1}{f^2} \begin{pmatrix} f + x f_x \\ f_y - x f_\theta \\ -f_x \end{pmatrix}. \tag{3.1}$$

The contact plane ξ_m at $m = (\theta, x, y)$ is spanned by the vectors $(0,1,0)$ and $(-x, 0, 1)$ and the projection along X onto ξ is given by $\pi(v) = v - f\lambda_0(v)$.

From lemma 3.2 we also get local coordinates

$$\Psi := \mathbb{1}_{\mathbb{R}} \times \psi^{-1} \; : \; \mathbb{R} \times V \to \mathbb{R} \times U$$

of $\mathbb{R} \times M$ onto $\mathbb{R} \times S^1 \times \mathbb{R}^2$ or onto the covering $\mathbb{R}^4$. Using these and the cylindrical holomorphic coordinates σ on $\dot{S}$, we represent $\tilde{u}$ near the puncture by

$$\tilde{v} := \Psi \circ \tilde{u} \circ \sigma : \; (s_0, \infty) \times S^1 \to \mathbb{R} \times S^1 \times \mathbb{R}^2.$$

We will work on the covering $\mathbb{R}$ of S^1 and denote the coefficients of the map $\tilde{v}$ in the following way:

$$\begin{array}{rcll} \tilde{v}(s,t) & = & (a(s,t), v(s,t)) & \in \; \mathbb{R} \times \mathbb{R}^3 \\ & = & (a(s,t), \theta(s,t), z(s,t)) & \in \; \mathbb{R} \times \mathbb{R} \times \mathbb{R}^3. \end{array}$$

According to [10], the functions a and z are 1-periodic in t, whereas θ satisfies $\theta(s, t+1) = \theta(s,t) + k$.

We also represent the almost complex structure $\tilde{J}$ in the local coordinates: it is lifted to

$$\hat{J} = T\Psi \circ \tilde{J}(\Psi^{-1}) \circ T\Psi^{-1} : \; T(\mathbb{R} \times \mathbb{R}^3) \to T(\mathbb{R} \times \mathbb{R}^3)$$

and has on $\tilde{v}((s_0, \infty) \times S^1)$ the special form

$$\hat{J}(a, m)(h, w) = (-f(m)\lambda_0(w), hX(m) + J(m)\pi w),$$

for $(h, w) \in T_a\mathbb{R} \times T_m\mathbb{R}^3 \cong T_{(a,m)}(\mathbb{R} \times \mathbb{R}^3)$. Here $J : \xi \to \xi$ is the complex multiplication compatible with $d(f\lambda_0)$, that the given almost complex sturcutre (also denoted by J) on $\xi \in TM$ is lifted to by Ψ.

The differential equation (1.6) for $\tilde{u}$ now corresponds to

$$\hat{J}(\tilde{v})\tilde{v}_s = \tilde{v}_t \tag{3.2}$$

or — using the explicit form of $\hat{J}$ —

$$\begin{array}{rcl} -f\lambda_0(v_s) & = & a_t, \\ a_s & = & f\lambda_0(v_t), \\ J(v)v_s & = & \pi v_t. \end{array} \tag{3.3}$$

In order to formulate the condition of nondegeneracy of the asymptotic orbit x in local coordinates, we have to linearize the flow ϕ_{Tt}, $t \in [0,1]$ of the Reeb vector field and project it onto the contact planes along the orbit $x(t) = (kt + d, 0, 0)$, which are the (x, y)-planes. These contact planes are equipped with the almost complex structure

$$J(t) := J(kt + d, 0, 0) : \xi_{x(t)} \to \xi_{x(t)}, \tag{3.4}$$

and the local coordinates provide a trivialization of $x^*\xi$. Therefore, in analogy to the construction in chapter 1, the linearization yields an arc $\Phi := \mathrm{d}\phi_{Tt}(x_0)|_\xi : [0,1] \to Sp(\mathbb{R}^2, J)$ of matrices $\Phi(t)$ which are symplectic with respect to $J(t)$.

Now in analogy to proposition 2.37, the orbit x is nondegenerate if and only if the boundary value problem for $z : [0,1] \to \mathbb{R}^2$,

$$\begin{cases} \dot{z}(t) &= F_\infty(t) z(t) \\ z(0) &= z(1) \end{cases}, \qquad \text{with} \quad F_\infty(t) = \dot{\Phi}(t)\Phi(t)^{-1}$$

admits only the trivial solution $z(t) \equiv 0$, or equivalently, if 0 is not included in the spectrum $\sigma(A_\infty)$ of the *asymptotic operator*

$$A_\infty = -J(t)\tfrac{\mathrm{d}}{\mathrm{d}t} + J(t)F_\infty(t) : W^{1,2}(S^1, \mathbb{R}^2) \to L^2(S^1, \mathbb{R}^2).$$

In order to calculate F_∞, we first remark that due to the reparametrization to $t \in [0,1]$ we now have $\frac{\mathrm{d}}{\mathrm{d}t}\phi_{Tt}(x_0) = T \cdot X(\phi_{Tt}(x_0))$ and $\phi_{Tt}(x_0) = x(t)$, hence linearization yields

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\mathrm{d}\phi_{Tt}(x_0) = T \cdot DX(x(t))\,\mathrm{d}\phi_{Tt}(x_0).$$

From this we obtain

$$\begin{aligned} F_\infty(t) &= \left(\tfrac{\mathrm{d}}{\mathrm{d}t}\mathrm{d}\phi_{Tt}(x_0)\right)\left(\mathrm{d}\phi_{Tt}(x_0)\right)^{-1}\big|_{\xi(x(t))} \\ &= T \cdot DX(x(t))|_{\xi(x(t))} \\ &= T \cdot DY(kt + d, 0, 0), \end{aligned} \tag{3.5}$$

where we have abbreviated $Y = (X_2, X_3)$.

The asymptotic operator is needed for the following theorem that describes the exponential convergence of $\tilde{v}$ to the asymptotic orbit.

Theorem 3.3 ([10] Thm.2.8.)
Consider a non-degenerate finite energy surface $\tilde{u} : \dot{S} \to \mathbb{R} \times M$ and assume $z_j \in \Gamma$ is a positive puncture. Near z_j, the map $\tilde{v} : \mathbb{R} \times S^1 \to \mathbb{R}^4$ defined as above, has, for large s, the following properties.

Either there exist constants $c, d \in \mathbb{R}$ such that

$$\tilde{v}(s,t) = (Ts + c, kt + d, 0, 0),$$

for all $(s,t) \in (s_0, \infty) \times S^1$*, or we have for some* $c, d \in \mathbb{R}$

$$\begin{aligned} a(s,t) &= Ts + c + \hat{a}(s,t), \\ \theta(s,t) &= kt + d + \hat{\theta}(s,t), \\ z(s,t) &= \mathrm{e}^{\int_{s_0}^{s} \mu(\rho)\mathrm{d}\rho}(e(t) + \hat{z}(s,t)). \end{aligned}$$

The functions $\hat{a}(s,\cdot)$ *and* $\hat{\theta}(s,\cdot)$ *converge to* 0 *exponentially as* $s \to \infty$ *in the following sense: for all derivatives* D^α, $\alpha \in \mathbb{N}_0 \times \mathbb{N}_0$*, with respect to* s *and* t *we have*

$$\begin{aligned} |D^\alpha \hat{a}(s,t)| &\le M_\alpha \mathrm{e}^{-\lambda s}, \\ |D^\alpha \hat{\theta}(s,t)| &\le M_\alpha \mathrm{e}^{-\lambda s} \end{aligned}$$

with constants M_α *and a constant* $\lambda > 0$*. Moreover,*

$$D^\alpha \hat{z}(s,t) \overset{s\to\infty}{\longrightarrow} 0$$

uniformly in t *for all derivatives* $\alpha \in \mathbb{N}_0 \times \mathbb{N}_0$*, and for all* $j \in \mathbb{N}_0$ *we have*

$$D^j(\mu(\rho) - \mu) \overset{\rho\to\infty}{\longrightarrow} 0,$$

where the number μ *is a negative eigenvalue of the asymptotic operator* A_∞ *and* $e(t) \in \mathbb{R}^2$ *is a corresponding eigenfunction.*

Actually, in [10] it is only proven that all derivatives of μ are bounded and that $\lim_{\rho\to\infty} \mu(\rho) = \mu$. But in view of the subsequent remark, this suffices in order to have all derivatives of μ converging to zero.

Remark 3.4 *Let* $\mu : (a, \infty) \to \mathbb{R}$ *with* $a \in \mathbb{R}$ *or* $a = -\infty$ *be a smooth function satisfying*

$$\mu(x) \overset{x\to\infty}{\longrightarrow} \mu \in \mathbb{R}$$

and

$$\left| \frac{\mathrm{d}^k \mu}{\mathrm{d}x^k}(x) \right| \le C_k$$

for all $x \in (a, \infty)$, $k \in \mathbb{N}$ *and some finite constants* C_k*. Then it follows that*

$$\frac{\mathrm{d}^k}{\mathrm{d}x^k}[\mu(x) - \mu] \overset{x\to\infty}{\longrightarrow} 0, \tag{3.6}$$

for all $k \in \mathbb{N}_0$.

Proof: Arguing by contradiction, we assume that there exists an $m \in \mathbb{N}$ such that (3.6) holds for $k = m - 1$ but does not hold for $k = m$. (Note that (3.6) holds for $k = 0$ by assumption.) Hence there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset (a+1, \infty)$ with $x_n \overset{n\to\infty}{\longrightarrow} \infty$ and some $d > 0$ such that we have $|\mu^{(m)}(x_n)| \ge d$ for all $n \in \mathbb{N}$. Since $\mu^{(m+1)}$ is uniformly bounded, we find some $\varepsilon \in (0,1)$ such that $|\mu^{(m)}(x)| \geq \frac{1}{2}d$ holds for all $x \in \bigcup_{n\in\mathbb{N}}[a_n, b_n]$, where $a_n : s = x_n - \varepsilon, \beta_n := x_n + \varepsilon \in (a, \infty)$. Therefore, $\mu^{(m)}$ can not change its sign on any of the intervals $[a_n, b_n]$, and thus we have for all $n \in \mathbb{N}$

$$
\begin{aligned}
\left|\mu^{(m-1)}(b_n) - \mu^{(m-1)}(a_n)\right| &= \left|\int_{a_n}^{b_n} \mu^{(m)}(x)\,\mathrm{d}x\right| \\
&= \int_{a_n}^{b_n} |\mu^{(m)}(x)|\,\mathrm{d}x \\
&\geq \varepsilon \cdot d.
\end{aligned}
$$

On the other hand, (3.6) for $k = m - 1$ implies that

$$
\begin{aligned}
0 < \varepsilon \cdot d &\leq \left|\mu^{(m-1)}(b_n) - \mu^{(m-1)}(a_n)\right| \\
&\leq \left|\frac{\mathrm{d}^{m-1}}{\mathrm{d}x^{m-1}}\bigg|_{b_n} (\mu(x) - \mu)\right| + \left|\frac{\mathrm{d}^{m-1}}{\mathrm{d}x^{m-1}}\bigg|_{a_n} (\mu(x) - \mu)\right| \stackrel{x\to\infty}{\longrightarrow} 0,
\end{aligned}
$$

a contradiction. □

In both cases of theorem 3.3, the periodic orbit is obviously given by $x(t) = (kt + d, 0, 0)$.

The first alternative of the theorem is fully characterized since we have the following proposition for pseudoholomorphic curves, i.e. solutions of (1.3), in analogy to the theory of (classical) holomorphic maps.

Proposition 3.5 *If two pseudoholomorphic curves are defined on a connected surface, then they are identical if and only if they are identical on a subset which contains a cluster point.*

For a proof see §5 in [9].

This propositon implies that in the case of $\tilde{v}(s,t) = (Ts + c, kt + d, 0, 0)$ on the whole neighbourhood of a puncture, the finite energy surface $\tilde{u}$ is actually a part of the finite energy cylinder

$$
\begin{aligned}
\mathbb{R} \times S^1 &\to \mathbb{R} \times M \\
(s,t) &\mapsto (Ts + c, x(t))
\end{aligned}
$$

over a periodic orbit x. This very special case is of no interest in the theory and thus is left out of consideration in the following.

In the second alternative, we have $\pi T u(z) \neq 0$ for z sufficiently close to z_j, so we can choose D_j even smaller such that we have $\pi T u(z) \neq 0$ on all of $\dot{D}_j$. Using this we can define an explicit basis of the normal bundle of $\tilde{u}(\dot{D}_j)$ in $\mathbb{R} \times M$. In the case of $\tilde{u}$ only being an immersion, $\tilde{u}(\dot{D}_j)$ might not be a submanifold of $\mathbb{R} \times M$ and hence the normal bundle might not be well-defined. But for any $z \in \dot{D}_j$ we can find a neighbourhood $\mathcal{U} \subset \dot{D}_j$ on which $\tilde{u}$ is an embedding and thus a normal bundle of $\tilde{u}(\mathcal{U})$ exists, for which we can ask for a basis. We make this more precise by the following lemma, for which we recall from (1.2) the $\tilde{J}$-invariant metric $\langle \cdot, \cdot \rangle_J$ on $\mathbb{R} \times M$ and its associated norm $|\cdot|_J$.

Lemma 3.6 *Assume that z_j is a (positive) puncture, for which the second alternative of theorem 3.3 holds. Then in a sufficiently small punctured neighbourhood $\dot{D}_j$ of z_j we can define the following vectors in $T_{\tilde{u}(z)}(\mathbb{R} \times M) \cong \mathbb{R} \times T_{u(z)}M$,*

$$\begin{aligned} n(z) &= \frac{1}{|\tilde{u}_s|_J} \left(|\pi u_s|_J \, , \, -\lambda(u_t) \frac{\pi u_s}{|\pi u_s|_J} + \lambda(u_s) \frac{\pi u_t}{|\pi u_t|_J} \right), \\ m(z) &= \tilde{J}(\tilde{u}(z))\, n(z), \end{aligned}$$

such that for all $z \in \dot{D}_j$ we have

(i) $|n(z)|_J = |m(z)|_J = 1$,

(ii) $\langle n(z), m(z) \rangle_J = 0$,

(iii) $n(z), m(z) \in (T_{\tilde{u}(z)}\tilde{u}(\mathcal{U}))^\perp \subset T_{\tilde{u}(z)}(\mathbb{R} \times M)$ *for some neighbourhood* $\mathcal{U} \subset \dot{D}_j$ *of z, on which $\tilde{u}$ is an embedding.*

Moreover, let $|\cdot| = |\Psi^* \cdot|_J$ and

$$\hat{e}(t) := \frac{e(t)}{|e(t)|}, \qquad \hat{f}(t) := J(t)\hat{e}(t),$$

where $e(t) \in \xi_{x(t)} \equiv \mathbb{R}^2$ is the nonzero eigenfunction of the asymptotic operator from theorem 3.3 and $J(t)$ is given by (3.4). Then for the normal vectors in local coordinates,

$$\begin{aligned} n(s,t) &:= T_{\tilde{u}(\sigma(s,t))}\Psi\, n(\sigma(s,t)), \\ m(s,t) &:= T_{\tilde{u}(\sigma(s,t))}\Psi\, m(\sigma(s,t)) \;=\; \hat{J}(\tilde{v}(s,t))\, n(s,t), \end{aligned}$$

we have

$$D^\alpha \left(n(s,t) - (0,0,\hat{e}(t))\right) \overset{s\to\infty}{\longrightarrow} 0$$

and

$$D^\alpha \left(m(s,t) - (0,0,\hat{f}(t))\right) \overset{s\to\infty}{\longrightarrow} 0$$

uniformly in t for all derivatives D^α with respect to s and t.

Proof: The metric $\langle \cdot, \cdot \rangle_J$ on $\mathbb{R} \times M$ induces a metric in local coordinates,

$$\begin{aligned} \langle (a,v), (b,w) \rangle &= \langle \Psi^*(a,v), \Psi^*(b,w) \rangle_J \\ &= a \cdot b + \psi^*\lambda(v) \cdot \psi^*\lambda(w) + \psi^* \mathrm{d}\lambda(\pi v, (\psi^* J)\pi w) \\ &= a \cdot b + f^2 \cdot \lambda_0(v) \cdot \lambda_0(w) + f \cdot \mathrm{d}\lambda_0(\pi v, J\pi w) \end{aligned}$$

for $(a,v),(b,w) \in T(\mathbb{R}\times U) \subset T(\mathbb{R}^4)$, where we have used the fact that $\xi = \ker(f\lambda_0) = \ker(\lambda_0)$ and hence

$$\psi^*\mathrm{d}\lambda|_\xi = \mathrm{d}\psi^*\lambda|_\xi = \mathrm{d}(f\lambda_0)|_\xi = f\cdot \mathrm{d}\lambda_0|_\xi + \mathrm{d}f\wedge\lambda_0|_\xi = f\cdot\mathrm{d}\lambda_0|_\xi$$

and we write J for ψ^*J as before. The associated norm on $\mathbb{R}^4$ is denoted by $|\cdot|$. Moreover, in the following we will denote by $B(s,t)$ various functions having all partial derivatives uniformly bounded and by $R(s,t)$ various functions which satisfy $D^\alpha R(s,t) \overset{s\to\infty}{\longrightarrow} 0$ uniformly in t.

Abbreviating by $z=(x,y)$ the $\mathbb{R}^2$-components of $v(s,t)$ and using theorem 3.3 and the fact that all partial derivatives of the function f from lemma 3.2 are uniformly bounded, we obtain as a first step

$$\begin{aligned} f(v(s,t)) &= f(\theta,0) + \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\rho} f(\theta,\rho z)\,\mathrm{d}\rho \\ &= f(\theta,0) + \left[\int_0^1 Df(\theta,\rho z)\,\mathrm{d}\rho\right]\cdot z \\ &= \tau + \mathrm{e}^{\int_{s_0}^s \mu(\rho)\mathrm{d}\rho}\left[\int_0^1 Df(\theta,\rho z)\,\mathrm{d}\rho\right]\cdot[e(t)+\hat{z}(s,t)] \\ &= \tau + \mathrm{e}^{\int_{s_0}^s \mu(\rho)\mathrm{d}\rho} B(s,t), \end{aligned} \tag{3.7}$$

where we have moreover used the fact that e is a smooth function on S^1, hence $e(t)=B(s,t)$. Analogously, we calculate for the Reeb vectorfield that is represented in (3.1) by partial derivatives of f,

$$\begin{aligned} X_1(v(s,t)) &= X_1(\theta,0) + \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\rho} X_1(\theta,\rho z)\,\mathrm{d}\rho \\ &= \frac{1}{\tau} + \left[\int_0^1 DX_1(\theta,\rho z)\,\mathrm{d}\rho\right]\cdot z \\ &= \frac{1}{\tau} + \mathrm{e}^{\int_{s_0}^s \mu(\rho)\mathrm{d}\rho}\left[\int_0^1 DX_1(\theta,\rho z)\mathrm{d}\rho\right]\cdot[e(t)+\hat{z}(s,t)] \\ &= \frac{1}{\tau} + \mathrm{e}^{\int_{s_0}^s \mu(\rho)\mathrm{d}\rho} B(s,t). \end{aligned} \tag{3.8}$$

For the component Y of the Reeb vector field, we have to take into account its connection (3.5) with the asymptotic operator. For this purpose we introduce

$$F(s,t) := \int_0^1 DY(\theta(s,t),\rho z(s,t))\,\mathrm{d}\rho$$

and remark that due to theorem 3.3 and the representation (3.1) of Y by partial derivatives of f, we have

$$F(s,t) = \int_0^1 DY(kt+d,0)\,\mathrm{d}\rho \; + R(s,t) \;=\; \tfrac{1}{T}F_\infty(t) + R(s,t).$$

Thus we obtain

$$
\begin{aligned}
Y(v(s,t)) &= Y(\theta,0) + \left[\int_0^1 DY(\theta,\rho z)\mathrm{d}\rho\right]\cdot z \\
&= \mathrm{e}^{\int_{s_0}^s \mu(\rho)\mathrm{d}\rho} F(s,t)[e(t)+\hat{z}(s,t)] \\
&= \mathrm{e}^{\int_{s_0}^s \mu(\rho)\mathrm{d}\rho}\left[\tfrac{1}{T}F_\infty(t)e(t)+R(s,t)\right] \qquad (3.9)\\
&= \mathrm{e}^{\int_{s_0}^s \mu(\rho)\mathrm{d}\rho} B(s,t). \qquad (3.10)
\end{aligned}
$$

Now we consider the projections of v_s and v_t onto the contact planes, that are explicitly given by

$$
\begin{aligned}
\pi v_s &= v_s - f(v)\lambda_0(v_s)X(v) \\
&= v_s + a_t X \\
&= (\theta_s + a_t X_1, z_s + a_t Y)
\end{aligned}
$$

and

$$
\begin{aligned}
\pi v_t &= v_t - f(v)\lambda_0(v_t)X(v) \\
&= v_t - a_s X \\
&= (\theta_t - a_s X_1, z_t - a_s Y),
\end{aligned}
$$

where we have used (3.3). Recalling $\mu(s) = \mu + R(s,t)$ and $a = Ts + c + R(s,t)$ from theorem 3.3, we calculate for the z-component of v_s

$$
\begin{aligned}
z_s + a_t Y &\overset{(3.10)}{=} \mathrm{e}^{\int_{s_0}^s \mu(\rho)\mathrm{d}\rho}[\mu(s)(e(t)+\hat{z}(s,t))+\hat{z}_s(s,t)] + a_t \mathrm{e}^{\int_{s_0}^s \mu(\rho)\mathrm{d}\rho} B(s,t) \\
&= \mathrm{e}^{\int_{s_0}^s \mu(\rho)\mathrm{d}\rho}[\mu\, e(t) + R(s,t)], \qquad (3.11)
\end{aligned}
$$

and for the z- component of v_t we obtain

$$
\begin{aligned}
z_t - a_s Y &\overset{(3.9)}{=} \mathrm{e}^{\int_{s_0}^s \mu(\rho)\mathrm{d}\rho}[\dot{e}(t)+\hat{z}_t(s,t)] \\
&\quad -[T+R(s,t)]\mathrm{e}^{\int_{s_0}^s \mu(\rho)\mathrm{d}\rho}[\tfrac{1}{T}F_\infty(t)e(t)+R(s,t)] \\
&= \mathrm{e}^{\int_{s_0}^s \mu(\rho)\mathrm{d}\rho}[\dot{e}(t) - F_\infty(t)e(t) + R(s,t)] \\
&= \mathrm{e}^{\int_{s_0}^s \mu(\rho)\mathrm{d}\rho}[J(t)\mu\, e(t) + R(s,t)]. \qquad (3.12)
\end{aligned}
$$

Here we have used in the last step $J(t)^2 = -\mathbb{1}$ and the fact that e is the eigenfunction of A_∞ corresponding to the eigenvalue μ, i.e. we have

$$
\mu\, e(t) = -J(t)[\dot{e}(t) - F_\infty(t)e(t)]. \qquad (3.13)
$$

Noting that for $x \in \xi$ the norm simplifies to $|x|^2 = f\cdot \mathrm{d}\lambda_0(x, Jx)$ and that $\mathrm{d}\lambda_0 = \mathrm{d}x \wedge \mathrm{d}y$ only acts on the z-components, we now calculate using (3.11) and (3.12)

$$
\begin{aligned}
|\pi v_s|^2 &= f(v)\cdot \mathrm{d}\lambda_0(\pi v_s, J(v)\pi v_s)\\
&\overset{(3.3)}{=} f(v)\cdot \mathrm{d}\lambda_0(\pi v_s, \pi v_t)\\
&= f(v)\cdot \mathrm{d}\lambda_0(z_s + a_t Y, z_t - a_s Y)\\
&= f(v)\cdot \mathrm{e}^{2\int_{s_0}^{s}\mu(\rho)\mathrm{d}\rho}[\,\mathrm{d}\lambda_0(\,\mu\, e(t), J(t)\mu\, e(t)\,) + R(s,t)\,]\\
&= \mathrm{e}^{2\int_{s_0}^{s}\mu(\rho)\mathrm{d}\rho}[\mu^2|e(t)|^2 + R(s,t)],
\end{aligned}
$$

which implies in view of $\mu < 0$

$$
|\pi v_s| = \mathrm{e}^{\int_{s_0}^{s}\mu(\rho)\mathrm{d}\rho}(-\mu)|e(t)|[1 + R(s,t)] = R(s,t). \tag{3.14}
$$

Here we also used $e(t) = B(s,t)$. Moreover, from the differential equation (3.13) it is clear that either $|e(t)| \neq 0$ for all $t \in S^1$ or $e \equiv 0$, in which case e would not be an eigenvector of A_∞. From this we see that $\pi v_s \neq 0$ for s sufficiently large, and hence also $\pi u_s \neq 0$ and $\pi u_t = J\pi u_s \neq 0$, so that n and m can be defined as stated.

In order to check (i) to (iii), we first remark that because of the $\tilde{J}$-invariance of the metric we have for all $v, w \in T(\mathbb{R}\times M)$

$$
\langle v, \tilde{J}w\rangle_J = \langle \tilde{J}v, \tilde{J}^2 w\rangle_J = \langle -\tilde{J}v, w\rangle_J
$$

and thus $\tilde{J}^{ad} = -\tilde{J}$ for the adjoint with respect to this metric. Now (ii) simply follows from

$$
\langle n(z), m(z)\rangle_J = \langle n(z), \tilde{J}n(z)\rangle_J = \langle -\tilde{J}n(z), n(z)\rangle_J = -\langle n(z), m(z)\rangle_J\,.
$$

(i) is easily checked for n by the subsequent calculation and then follows for m since the metric is $\tilde{J}$-invariant.

$$
\begin{aligned}
&|n(z)|_J^2\\
&= \frac{1}{|\tilde{u}_s|_J^2}\Bigg[|\pi u_s|_J^2 + \mathrm{d}\lambda\Bigg(-\lambda(u_t)\frac{\pi u_s}{|\pi u_s|_J} + \lambda(u_s)\frac{\pi u_t}{|\pi u_t|_J}\,,\\
&\qquad\qquad J(u)\left\{-\lambda(u_t)\frac{\pi u_s}{|\pi u_s|_J} + \lambda(u_s)\frac{\pi u_t}{|\pi u_t|_J}\right\}\Bigg)\Bigg]\\
&\overset{(1.7)}{=} \frac{1}{|\tilde{u}_s|_J^2}\Bigg[|\pi u_s|_J^2 + \mathrm{d}\lambda\Bigg(-a_s\frac{\pi u_s}{|\pi u_s|_J} + \lambda(u_s)\frac{\pi u_t}{|\pi u_t|_J}\,,\\
&\qquad\qquad -a_s\frac{\pi u_t}{|\pi u_t|_J} + \lambda(u_s)\frac{-\pi u_s}{|\pi u_t|_J}\Bigg)\Bigg]\\
&= \frac{1}{|\tilde{u}_s|_J^2}\Bigg[|\pi u_s|_J^2 + a_s^2\frac{1}{|\pi u_s|_J^2}\mathrm{d}\lambda(\pi u_s, \pi u_t) + \lambda(u_s)^2\frac{1}{|\pi u_t|_J^2}\mathrm{d}\lambda(\pi u_t, -\pi u_s)\Bigg]\\
&\overset{(1.7)}{=} \frac{1}{|\tilde{u}_s|_J^2}\Bigg[|\pi u_s|_J^2 + a_s^2\frac{1}{|\pi u_s|_J^2}\mathrm{d}\lambda(\pi u_s, J(u)\pi u_s)
\end{aligned}
$$

$$+\lambda(u_s)^2\frac{1}{|\pi u_t|_J^2}\mathrm{d}\lambda(\pi u_t, J(u)\pi u_t)\Big]$$

$$= \quad \frac{1}{|\tilde{u}_s|_J^2}\left[|\pi u_s|_J^2 + a_s^2 + \lambda(u_s)^2\right] \;=\; 1.$$

For (iii) we calculate

$$\begin{aligned}
&\langle n(z), \tilde{u}_s(z)\rangle \\
&= \quad \frac{1}{|\tilde{u}_s|_J}\left[|\pi u_s|_J\cdot a_s + \mathrm{d}\lambda\left(-\lambda(u_t)\frac{\pi u_s}{|\pi u_s|_J} + \lambda(u_s)\frac{\pi u_t}{|\pi u_t|_J}\,,\; J(u)\pi u_s\right)\right] \\
&\overset{(1.7)}{=} \quad \frac{1}{|\tilde{u}_s|_J}\left[|\pi u_s|_J\cdot\lambda(u_t) - \frac{\lambda(u_t)}{|\pi u_s|_J}\mathrm{d}\lambda\left(\pi u_s, J(u)\pi u_s\right)\right] = 0
\end{aligned}$$

and we analogously obtain $\langle n(z), \tilde{u}_t(z)\rangle = 0$, hence $n(z) \in (T_{\tilde{u}(z)}\tilde{u}(\mathcal{U}))^{\perp}$. Now the statement for $m(z)$ follows simply from the fact that $T_{\tilde{u}(z)}\tilde{u}(\mathcal{U})$ is $\tilde{J}$-invariant (since $\tilde{J}\tilde{u}_s = \tilde{u}_t$ holds by (1.6)) — and thus so is its orthogonal complement, i.e. for all $w \in T_{\tilde{u}(z)}\tilde{u}(\mathcal{U})$ we have

$$\langle m(z), w\rangle_J = \langle \tilde{J}(v(z))n(z), w\rangle = -\langle n(z), \tilde{J}(v(z))w\rangle = 0$$

and hence $m(z) = \tilde{J}(v(z))n(z) \in N_{\tilde{u}(z)}\tilde{u}(\mathcal{U})$.

Finally, we will prove the convergence of

$$n(s,t) \overset{(3.3)}{=} \frac{1}{|\tilde{v}_s|}\left(|\pi v_s|\,,\; -a_s\frac{\pi v_s}{|\pi v_s|} - a_t\frac{\pi v_t}{|\pi v_t|}\right).$$

We start by noting that

$$|\tilde{v}_s| = T[1 + R(s,t)] \tag{3.15}$$

follows from theorem 3.3 and (3.14)

$$\begin{aligned}
|\tilde{v}_s|^2 \quad &= \quad a_s^2 + \lambda(v_s)^2 + |\pi v_s|^2 \\
&\overset{(3.3)}{=} \quad a_s^2 + a_t^2 + |\pi v_s|^2 \\
&= \quad T^2 + R(s,t).
\end{aligned}$$

Using again (3.14) we thus obtain for the $\mathbb{R}$-component of $n(s,t)$

$$\frac{|\pi v_s|}{|\tilde{v}_s|} = \frac{\mathrm{e}^{\int_{s_0}^{s}\mu(\rho)\mathrm{d}\rho}(-\mu)|e(t)|[1 + R(s,t)]}{T[1 + R(s,t)]} = \mathrm{e}^{\int_{s_0}^{s}\mu(\rho)\mathrm{d}\rho}B(s,t) = R(s,t),$$

as claimed. In order to show the convergence for the M-component of $n(s,t)$, we first calculate

$$\begin{aligned}
a_t(s,t) \quad &\overset{(3.3)}{=} \quad -f(v)\lambda_0(v_s) \\
&\overset{(3.7)}{=} \quad -\tau(\theta_s + xy_s) - \mathrm{e}^{\int_{s_0}^{s}\mu(\rho)\mathrm{d}\rho}B(s,t)(\theta_s + xy_s) \\
&= \quad -\tau\theta_s + \mathrm{e}^{\int_{s_0}^{s}\mu(\rho)\mathrm{d}\rho}R(s,t) \;=\; R(s,t),
\end{aligned}$$

and from this obtain for the θ-component of πv_s

$$\begin{aligned}\theta_s + a_t X_1 &\overset{(3.8)}{=} \theta_s + \frac{a_t}{\tau} + a_t e^{\int_{s_0}^s \mu(\rho)\mathrm{d}\rho} B(s,t) \\ &= e^{\int_{s_0}^s \mu(\rho)\mathrm{d}\rho} R(s,t). \end{aligned} \tag{3.16}$$

From (3.11), (3.14) and (3.16) we now obtain with respect to the splitting into the θ- and z-component

$$\begin{aligned}\frac{\pi v_s}{|\pi v_s|} &= \left(\frac{R(s,t)}{(-\mu)|e(t)|[1+R(s,t)]}, \frac{\mu\, e(t) + R(s,t)}{(-\mu)|e(t)|[1+R(s,t)]}\right) \\ &= (R(s,t)\,,\, -\hat{e}(t) + R(s,t)) \\ &= -\hat{e}(t) + R(s,t) \qquad (3.17) \\ &= B(s,t). \qquad (3.18)\end{aligned}$$

Due to (3.15) and theorem 3.3 we have

$$\frac{-a_s}{|\tilde{v}_s|} = \frac{-T + R(s,t)}{T[1+R(s,t)]} = -1 + R(s,t),$$

and together with (3.17) we infer

$$\frac{-a_s}{|\tilde{v}_s|}\frac{\pi v_s}{|\pi v_s|} = \hat{e}(t) + R(s,t)$$

for the first part of the M-component of $n(s,t)$.

Furthermore, we remark that $\hat{J}$ is a smooth function of (θ, z) and we thus have in view of theorem 3.3

$$\hat{J}(\tilde{v}(s,t)) = \hat{J}(kt+d, 0) + R(s,t) \tag{3.19}$$

mapping $(b, w) \in T_{\tilde{v}(s,t)}(\mathbb{R} \times \mathbb{R}^3)$ to

$$\begin{aligned}&\Big(-f(kt+d,0)\cdot\lambda_0(w)\,,\, b\cdot X(kt+d,0) + J(kt+d,0)\,\pi w\Big) + R(s,t) \\ =\ &\Big(-\tau\cdot\lambda_0(w)\,,\, b\cdot(\tau^{-1},0) + J(t)\,\pi w\Big) + R(s,t).\end{aligned}$$

Considering the second part, $-a_t\frac{\pi v_t}{|\pi v_t|}$, of the M-component of $n(s,t)$, we first remark that due to (3.3) and the $\hat{J}$-invariance of the norm we have $|\pi v_t| = |\hat{J}(v)\pi v_s| = |\pi v_s|$. Thus, using (3.15), (3.18) and (3.19), we infer

$$\begin{aligned}\frac{-a_t}{|\tilde{v}_s|}\frac{\pi v_t}{|\pi v_t|} &= \frac{R(s,t)}{T[1+R(s,t)]}(\hat{J}(kt+d,0) + R(s,t))|_\xi\, B(s,t) \\ &= R(s,t)[J(t)B(s,t) + R(s,t)] \\ &= R(s,t),\end{aligned}$$

so alltogether we have with respect to the $\mathbb{R} \times \mathbb{R}^3$-splitting

$$n(s,t) = (R(s,t), \hat{e}(t) + R(s,t))$$

as claimed. The convergence of $m(s,t)$ now follows easily from (3.19):

$$\begin{aligned}
m(s,t) &= \hat{J}(\tilde{v}(s,t))n(s,t) \\
&= [\hat{J}(kt+d,0) + R(s,t)]\ (\ R(s,t)\ ,\ \hat{e}(t) + R(s,t)\ ) \\
&= (\ -\tau \cdot \lambda_0(R(s,t))\ ,\ R(s,t) \cdot (\tau^{-1}, 0) + J(t)[\hat{e}(t) + R(s,t)]\ ) \\
&= (\ R(s,t)\ ,\ J(t)\hat{e}(t) + R(s,t)\ ) \\
&= (\ R(s,t)\ ,\ \hat{f}(t) + R(s,t)\ )\ .
\end{aligned}$$

□

Having established this lemma, we can now — as a start for the construction of Φ_j — define an immersion $\Gamma : (s_0, \infty) \times S^1 \times B_\varepsilon(0) \to \mathbb{R}^4$ for some $\varepsilon > 0$ by

$$\Gamma(s,t,x,y) = \tilde{v}(s,t) + x\,\tilde{n}(s,t) + y\,\tilde{m}(s,t),$$

where

$$\begin{aligned}
\tilde{n}(s,t) &= \mathrm{e}^{wt\hat{J}(\tilde{v}(s,t))}n(s,t) \\
&= \cos(wt)\,n(s,t) + \sin(wt)\,m(s,t), \\
\tilde{m}(s,t) &= \mathrm{e}^{wt\hat{J}(\tilde{v}(s,t))}m(s,t) \\
&= -\sin(wt)\,n(s,t) + \cos(wt)\,m(s,t).
\end{aligned}$$

In order to see that this is indeed an immersion, we first calculate the tangent map $T_p\Gamma$ abbreviating $p = (s,t,x,y)$. It has the following columns:

$$T_p\Gamma = (\ \tilde{v}_s + x\tilde{n}_s + y\tilde{m}_s \mid \tilde{v}_t + x\tilde{n}_t + y\tilde{m}_t \mid \tilde{n} \mid \tilde{m}\ ), \tag{3.20}$$

where all functions are evaluated at the point (s,t). From theorem 3.3 and the previous lemma we know that

$$\begin{aligned}
\tilde{v}_s(s,t) &\overset{s\to\infty}{\longrightarrow} (T,0,0) \\
\tilde{v}_t(s,t) &\overset{s\to\infty}{\longrightarrow} (0,k,0) \\
\tilde{n}(s,t) &\overset{s\to\infty}{\longrightarrow} (0,0,\cos(wt)\hat{e}(t) + \sin(wt)J(t)\hat{e}(t)) \\
\tilde{m}(s,t) &\overset{s\to\infty}{\longrightarrow} (0,0,-\sin(wt)\hat{e}(t) + \cos(wt)J(t)\hat{e}(t))
\end{aligned} \tag{3.21}$$

uniformly in t, thus for $x = y = 0$, the columns of $T_p\Gamma$ converge to linearly independent vectors as $s \to \infty$. Hence, if we choose s_0 sufficiently large, then on $(s_0, \infty) \times S^1 \times \{0\}$ we have $\det(T_p\Gamma)$ bounded away from zero. Moreover,

$\tilde{n}_s, \tilde{m}_s, \tilde{n}_t$ and $\tilde{m}_t$ are all uniformly bounded on $(s_0, \infty) \times S^1$, since we have the following uniform convergences:

$$\begin{aligned}
\tilde{n}_s(s,t) &= \cos(wt)n_s(s,t) + \sin(wt)m_s(s,t) \stackrel{s\to\infty}{\longrightarrow} 0, \\
\tilde{m}_s(s,t) &= -\sin(wt)n_s(s,t) + \cos(wt)m_s(s,t) \stackrel{s\to\infty}{\longrightarrow} 0, \\
\tilde{n}_t(s,t) &= -\sin(wt)n(s,t) + \cos(wt)m(s,t) \\
&\quad + \cos(wt)n_t(s,t) + \sin(wt)m_t(s,t) \\
&\stackrel{s\to\infty}{\longrightarrow} -\sin(wt)\hat{e}(t) + \cos(wt)\hat{f}(t) \\
&\quad + \cos(wt)\hat{e}'(t) + \sin(wt)\hat{f}'(t), \qquad (3.22) \\
\tilde{m}_t(s,t) &= -\cos(wt)n(s,t) - \sin(wt)m(s,t) \\
&\quad - \sin(wt)n_t(s,t) + \cos(wt)m_t(s,t) \\
&\stackrel{s\to\infty}{\longrightarrow} -\cos(wt)\hat{e}(t) - \sin(wt)\hat{f}(t) \\
&\quad - \sin(wt)\hat{e}'(t) + \cos(wt)\hat{f}'(t).
\end{aligned}$$

Therefore the perturbation of $\det(T_p\Gamma)$ by $(x, y) \in B_\varepsilon(0)$ is bounded on all of $(s_0, \infty) \times S^1$, so if we choose $\varepsilon > 0$ sufficiently small, then Γ indeed is an immersion.

Finally, since σ and Ψ are diffeomorphisms, we can use them to lift Γ to the required immersion

$$\Phi_j := \Psi^{-1} \circ \Gamma \circ (\sigma^{-1} \times \mathbb{1}_{\mathbb{R}^2}) \; : \; \dot{D}_j \times B_\varepsilon(0) \to \mathbb{R} \times M$$

that is explicitly given by

$$\Phi_j(\sigma(s,t), (x,y)) = \Psi^{-1}\left[\tilde{v}(s,t) + x\,\tilde{n}(s,t) + y\,\tilde{m}(s,t)\right].$$

The winding $\mathrm{e}^{wt\hat{J}}$ in $\tilde{n}$ and $\tilde{m}$ has been introduced in order to adjust Φ_j by choosing $w \in \mathbb{Z}$, so that it can be extended to all of $\dot{S}$. This construction moreover obviously meets

$$\Phi_j(z, 0) = \Psi^{-1}[\tilde{v}(\sigma^{-1}(z))] = \tilde{u}(z) \qquad (3.23)$$

for all $z \in \dot{D}_j$.

In the case of $\tilde{u}$ being an embedding, note that also $\tilde{v}$ is an embedding and thus — since Ψ is a diffeomorphism — Φ_j is an embedding on some neighbourhood of $\dot{D}_j \times \{0\}$. This can only be a neighbourhood of the form $\dot{D}_j \times B_\varepsilon(0)$, if the asymptotic orbit is simply covered; if it is multiply covered, then the embedded neighbourhood of $\dot{D}_j \times \{0\}$ has to shrink to zero at the puncture.

We have thus found disks $\dot{D}_j$ around all punctures $z_j \in \Gamma$, on which we constructed trivializations Φ_j as required, leaving some freedom in order to make sure that these maps can be extended to all of $\dot{S}$. Moreover, since Γ

is finite, we can choose the disks small enough to not intersect one another. Now the punctured surface $\dot{S} = S \setminus \Gamma$ can be decomposed as follows:

$$\dot{S} = \tilde{S} \,\dot{\cup} \dot{\bigcup_{z_j \in \Gamma}} \dot{D}_j \qquad \text{where} \qquad \tilde{S} = S \setminus \dot{\bigcup_{z_j \in \Gamma}} D_j.$$

As next step we will prove that — for the surface $\tilde{S}$ away from the punctures — there exists a trivialization $\tilde{S} \times B_\varepsilon(0) \to \mathbb{R} \times M$ with the required properties.

Lemma 3.7 *With the previous assumptions and notation there exists for some $\varepsilon > 0$ an immersion*

$$\tilde{\Phi} : \tilde{S} \times B_\varepsilon(0) \to \mathbb{R} \times M$$

satisfying $\tilde{\Phi}(z, 0) = \tilde{u}(z)$ for all $z \in \tilde{S}$. Moreover, if $\tilde{u}$ is an embedding, then also $\tilde{\Phi}$ restricted to a sufficiently small neighbourhood of $\tilde{S} \times \{0\}$ is an embedding.

Proof: We introduce a Hermitian vector bundle over $\tilde{S}$,

$$E := \{(z, \xi) \mid z \in \tilde{S}, \xi \in N_{\tilde{u}(z)}\tilde{u}(U)\},$$

where $U \in \tilde{S}$ is a neighbourhood of z on which $\tilde{u}$ is an embedding, and $N_{\tilde{u}(z)}\tilde{u}(U)$ is the normal space of $\tilde{u}(U)$ in $\mathbb{R} \times M$ with respect to the metric $\langle \cdot, \cdot \rangle_J$ defined in (1.2). With the projection $(z, \xi) \mapsto z$ this obviously is a vector bundle over $\tilde{S}$. Furthermore, the almost complex structure $\tilde{J}$ on $\mathbb{R} \times M$ induces an almost complex structure on E. Indeed, as shown in the proof of lemma 3.6, $N_{\tilde{u}(z)}\tilde{u}(U)$ is invariant under $\tilde{J}$, and thus induces the almost complex structure

$$\begin{array}{rccl} J : & E & \to & E \\ & (z, \xi) & \mapsto & (z, \tilde{J}(\tilde{u}(z))\xi) =: J(z)\,(z, \xi). \end{array}$$

According to [17] proposition 2.61, there exists a symplectic form on E that is compatible with J, hence E can be viewed as a Hermitian vector bundle. Moreover, $\tilde{S}$ is a compact surface since S was assumed to be compact and the disks D_j are open, and it has nonempty boundary since there is at least one puncture $z_1 \in \Gamma$, hence $\partial D_1 \subset \partial \tilde{S}$. Now in view of [17] proposition 2.64, there exists a unitary trivialization of E, i.e. we have a smooth bundle isomorphism

$$\begin{array}{rccl} \phi^E : & \tilde{S} \times \mathbb{R}^2 & \to & E \\ & (z, \vec{v}) & \mapsto & (z, A(z)\vec{v}) \end{array}$$

such that the linear maps $A(z)$ satisfy

$$A(z)^* J(z) = J_0, \tag{3.24}$$

where J_0 is the canonical almost complex structure on $\mathbb{R}^2$.

If $\tilde{u}$ is an embedding, then E is just the lifting of the normal bundle $N\tilde{u}(\tilde{S})$ of $\tilde{u}(\tilde{S})$ in $\mathbb{R} \times M$, and in the above we could have used $N\tilde{u}(\tilde{S})$ instead of E. Therefore, identifying $E = \tilde{u}^* N\tilde{u}(\tilde{S})$ with $N\tilde{u}(\tilde{S})$, we have a diffeomorphism $\phi^E : \tilde{S} \times \mathbb{R}^2 \to N\tilde{u}(\tilde{S})$. Then by the tubular neighbourhood theorem (e.g. [7] §4 Thm.6.3), a neighbourhood $\mathcal{V}$ of the zero section of $N\tilde{u}(\tilde{S})$ is diffeomorphic to a neighbourhood of $\tilde{u}(\tilde{S})$ in $\mathbb{R} \times M$. Since $\tilde{S}$ is compact, there will be an $\varepsilon > 0$ such that $\phi^E(\tilde{S} \times B_\varepsilon(0)) \subset \mathcal{V}$, so composing the tubular neighbourhood diffeomorphism ψ with ϕ^E yields a diffeomorphism

$$\tilde{\Phi} := \psi \circ \phi^E : \tilde{S} \times B_\varepsilon(0) \to \mathcal{U} \subset \mathbb{R} \times M$$

onto some neighbourhood $\mathcal{U} \subset \mathbb{R} \times M$ of $\tilde{u}(\tilde{S})$. This map satisfies the requirement $\tilde{\Phi}(z,0) = \tilde{u}(z)$ for all $z \in \tilde{S}$ since

$$\phi^E(z,0) = (z, A(z)0) = (z,0) \cong 0 \in N_{\tilde{u}(z)}\tilde{u}(\tilde{S})$$

and the tubular neighbourhood diffeomorphism ψ leaves the zero section of $N\tilde{u}(\tilde{S})$ (that is identified with $\tilde{S}$) fix.

For the case of $\tilde{u}$ being only an immersion, we will to describe the construction of an analogue of the tubular neighbourhood in detail. By the Whitney embedding theorem (see e.g. [7] §1 Thm.3.5) we can think of $\mathbb{R} \times M$ as a submanifold of $\mathbb{R}^8$. We then consider the normal bundle $N(\mathbb{R} \times M) \subset T\mathbb{R}^8 \cong \mathbb{R}^8$ of $\mathbb{R} \times M$ with respect to the Euclidean metric on $\mathbb{R}^8$ and construct a diffeomorphism between a neighbourhood $\mathcal{V}$ of the zero section of this bundle and a tubular neighbourhood $\mathcal{U}$ of $\mathbb{R} \times M$ in $\mathbb{R}^8$:

$$\begin{aligned} \tau : N(\mathbb{R} \times M) \supset \mathcal{V} &\to \mathcal{U} \subset \mathbb{R}^8 \\ v \in N_p(\mathbb{R} \times M) &\mapsto p + v. \end{aligned}$$

According to [7], this actually is a diffeomorphism that leaves $\mathbb{R} \times M$ (that is identified with the zero section of $N(\mathbb{R} \times M)$) fix. We now find a neighbourhood $\mathcal{W} \subset E$ of the zero section, such that for all $(z,\xi) \in \mathcal{W}$ we have $\tilde{u}(z) + \xi \in \mathcal{U}$. Then using τ we can define

$$\begin{aligned} \mathcal{T} : \qquad E \supset \mathcal{W} &\to \mathbb{R} \times M \\ (z,\xi) &\mapsto \mathrm{pr}(\tau^{-1}(\tilde{u}(z) + \xi)) \end{aligned},$$

where $\tilde{u}(z)$ and ξ are identified with vectors in $\mathbb{R}^8$, and pr is the bundle projection of $N(\mathbb{R} \times M)$.

Identifiying the tangent space of $N_{\tilde{u}(z)}\tilde{u}(U)$ in 0 with itself, we obtain for every $z \in \tilde{S}$ the splitting

$$T_{(z,0)}E = T_z\tilde{S} \oplus N_{\tilde{u}(z)}\tilde{u}(U). \tag{3.25}$$

We now want to determine how $T_{(z,0)}\mathcal{T}$ acts on this space. On the zero section of E (that is identified with $\tilde{S}$), we have for all $z \in \tilde{S}$

$$\mathcal{T}(z,0) = \mathrm{pr}(\tau^{-1}(\tilde{u}(z))) = \mathrm{pr}(0_{N_{\tilde{u}(z)}(\mathbb{R}\times M)}) = \tilde{u}(z)$$

and hence

$$T_{(z,0)}\mathcal{T}\big|_{T_z\tilde{S}} = T_z\tilde{u}.$$

Along the fibres $N_{\tilde{u}(z)}\tilde{u}(U)$ of E we obtain

$$T_{(z,0)}\mathcal{T}\big|_{N_{\tilde{u}(z)}\tilde{u}(U)} = \mathbb{1}.$$

In order to show this, we have to consider any tangent vector of E in $(z,0)$. This tangent vector is represented by some path $(-\varepsilon,\varepsilon) \ni t \mapsto (z, t\xi)$, where $\xi \in N_{\tilde{u}(z)}\tilde{u}(U)$ can also be seen as the tangent vector itself. We first note that the differential of $(z,\xi) \mapsto \tilde{u}(z) + \xi$ maps the above tangent vector to $\xi \in N_{\tilde{u}(z)}\tilde{u}(U) \subset T_{\tilde{u}(z)}(\mathbb{R}\times M)$. Moreover, since τ and hence also $\mathrm{pr}\circ\tau^{-1}$ leaves $\mathbb{R}\times M$ fix, we have $T_p(\mathrm{pr}\circ\tau^{-1})\big|_{T_p(\mathbb{R}\times M)} = \mathbb{1}$ for all $p \in \mathbb{R}\times M$.

Alltogether we obtain

$$T_{(z,0)}\mathcal{T} = T_z\tilde{u} \oplus \mathbb{1}$$

with respect to the splitting (3.25) of $T_{(z,0)}E$ and mapping onto

$$T_{\tilde{u}(z)}\tilde{u}(U) \oplus N_{\tilde{u}(z)}\tilde{u}(U) = T_{\tilde{u}(z)}(\mathbb{R}\times M).$$

Now $\tilde{u}$ was assumed to be an immersion, therefore the differential of $\mathcal{T}$ is nondegenerate all along $\tilde{S}\times\{0\}$ and hence $\mathcal{T}$ is an immersion on some neighbourhood of $\tilde{S}\times\{0\}$. As in the embedded case we can now define $\tilde{\Phi} := \mathcal{T}\circ\phi^E$, and we restrict this map to $\tilde{S}\times B_\varepsilon(0)$ for some $\varepsilon > 0$ such that $\mathcal{T}$ is an immersion on $\phi^E(\tilde{S}\times B_\varepsilon(0))$ (we have $\varepsilon > 0$ since $\tilde{S}$ is compact). This $\tilde{\Phi}$ is indeed an immersion and it meets

$$\tilde{\Phi}(z,0) = \mathcal{T}(z,0) = \tilde{u}(z)$$

as claimed. □

Of course, the $\tilde{\Phi}$ constructed above can still be homotoped in the set of immersions $\tilde{S}\times B_\varepsilon(0) \to \mathbb{R}\times M$ that are equal to $\tilde{u}$ on $\dot{S}\times\{0\}$. We will use this freedom to make $\tilde{\Phi}$ match Φ_j on ∂D_j for all $z_j \in \Gamma$. For this purpose, we only have to make sure that restricted to $\partial D_j \times B_\varepsilon(0)$, the maps Φ_j and $\tilde{\Phi}$ are homotopic within the set of immersions equalling $\tilde{u}$ on $\partial D_j\times\{0\}$. The homotopy class of $\tilde{\Phi}|_{\partial D_j\times B_\varepsilon(0)}$ is determined by its linearization on $\partial D_j \times \{0\}$, that is

$$\begin{aligned} T_{(z,0)}\tilde{\Phi} &= T_{(z,0)}\mathcal{T}\circ T_{(z,0)}\phi^E \\ &= (T_z\tilde{u}\oplus\mathbb{1})\circ(\mathbb{1}\oplus A(z)) \\ &= T_z\tilde{u}\oplus A(z). \end{aligned}$$

for $z=\sigma(s_0,t)$, $t\in S^1$.

For all Φ in the set that we want to homotope in, we have $\Phi(\cdot,0)=\tilde{u}(\cdot)$ and hence $T_{(z,0)}\Phi\big|_{T_zS}=T_z\tilde{u}$, so for the homotopy degree only $T_{(\sigma(s_0,\cdot),0)}\Phi\big|_{T_0\mathbb{R}^2}$ is relevant.

From the construction we know that $T_{(\sigma(s_0,t),0)}\tilde{\Phi}|_{T_0\mathbb{R}^2}=A(\sigma(s_0,t))$ maps onto $N_{\tilde{u}(\sigma(s_0,t))}\tilde{u}(U)$, where U is a neighbourhood of $\sigma(s_0,t)$, and so does $T_{(\sigma(s_0,t),0)}\Phi_j\big|_{T_0\mathbb{R}^2}$. Indeed, we calculate from (3.23)

$$\begin{aligned}
T_{(\sigma(s_0,t),0)}\Phi_j\,\frac{\partial}{\partial x} &= T_{\tilde{v}(s_0,t)}\Psi\ \mathrm{e}^{wt\hat{J}(\tilde{v}(s_0,t))}n(s_0,t)\\
&= \mathrm{e}^{wt\tilde{J}(\tilde{u}(\sigma(s_0,t)))}n(\sigma(s_0,t))\\
&= [\,\cos(wt)\mathbb{1}+\sin(wt)\tilde{J}(\tilde{u}(\sigma(s_0,t))\,]\;n(\sigma(s_0,t))\\
&= \cos(wt)\,n(\sigma(s_0,t))+\sin(wt)\,m(\sigma(s_0,t)),\\
T_{(\sigma(s_0,t),0)}\Phi_j\,\frac{\partial}{\partial y} &= T_{\tilde{v}(s_0,t)}\Psi\ \mathrm{e}^{wt\hat{J}(\tilde{v}(s_0,t))}m(s_0,t)\\
&= \mathrm{e}^{wt\tilde{J}(\tilde{u}(\sigma(s_0,t)))}m(\sigma(s_0,t))\\
&= [\,\cos(wt)\mathbb{1}+\sin(wt)\tilde{J}(\tilde{u}(\sigma(s_0,t))\,]\;m(\sigma(s_0,t))\\
&= \cos(wt)\,m(\sigma(s_0,t))-\sin(wt)\,n(\sigma(s_0,t)).
\end{aligned}$$

Therefore, we can determine the required homotopy class by choosing the bases $(\frac{\partial}{\partial x},\frac{\partial}{\partial y}=J_0\frac{\partial}{\partial x})$ of $T_0\mathbb{R}^2$ and $(n(z),m(z)=\tilde{J}(\tilde{u}(z))n(z))$ of $N_{\tilde{u}(z)}\tilde{u}(U)$.

From (3.24) we have $\tilde{J}(\tilde{u}(z))A(z)=A(z)J_0$, hence $A(z)$ preserves the orientation of above bases, and thus $T_{(\sigma(s_0,\cdot),0)}\tilde{\Phi}|_{T_0\mathbb{R}^2}=A(\sigma(s_0,\cdot))$ is represented by a loop of matrices in $Gl^+(\mathbb{R}^2)$. Hence it is homotopic to

$$S^1\ni\ t\mapsto\begin{pmatrix}\cos(kt) & -\sin(kt)\\ \sin(kt) & \cos(kt)\end{pmatrix}$$

for some $k\in\mathbb{Z}$, but this is just $T_{(\sigma(s_0,\cdot),0)}\Phi_j\big|_{T_0\mathbb{R}^2}$ when we choose $w=k$. So we have shown that $\tilde{\Phi}$ can be homotoped in a small neighbourhood of each of the boundary components ∂D_j of $\tilde{S}$ such that it can be extended by Φ_j to $\dot{D}_j$. That way we obtain a map $\Phi:\dot{S}\times B_{\bar{\varepsilon}}(0)\to\mathbb{R}\times M$, where $\bar{\varepsilon}>0$ is the finite minimum of the $\varepsilon>0$ for which $\tilde{\Phi}$ and the Φ_j were constructed. As all of its constituents, Φ is an immersion, and in the case of $\tilde{u}$ being an embedding, it is an embedding on a small neighbourhood of $\dot{S}\times\{0\}$. Moreover, the above homotopy left $\tilde{\Phi}$ fix on $\dot{S}\times\{0\}$ and hence we also have $\Phi(z,0)=\tilde{u}(z)$ for all $z\in\dot{S}$. This proves the first part of the theorem.

The rest of the theorem describes the behaviour of the almost complex structure near the punctures in our special coordinates Φ. Hence considering the neighbourhood of $z_j\in\Gamma$, we have Φ explicitly given by Φ_j. Now we calculate in this neighbourhood

$$\bar{J}(z,0)\ =\ (T_{(z,0)}\Phi_j)^{-1}\circ\tilde{J}(\tilde{u}(z))\circ T_{(z,0)}\Phi_j$$

$$
\begin{aligned}
&= ((T_z\sigma^{-1})^{-1} \times 1\!\!1) \circ (T_{(\sigma^{-1}(z),0)}\Gamma)^{-1} \circ T_{\tilde v(\sigma^{-1}(z))}\Psi \circ \tilde J(\tilde u(z)) \\
&\quad \circ (T_{\tilde v(\sigma^{-1}(z))}\Psi)^{-1} \circ T_{(\sigma^{-1}(z),0)}\Gamma \circ (T_z\sigma^{-1} \times 1\!\!1) \\
&= ((T_z\sigma^{-1})^{-1} \times 1\!\!1) \circ (T_{(\sigma^{-1}(z),0)}\Gamma)^{-1} \circ \hat J(\tilde v(\sigma^{-1}(z))) \\
&\quad \circ T_{(\sigma^{-1}(z),0)}\Gamma \circ (T_z\sigma^{-1} \times 1\!\!1) \\
&= ((T_z\sigma^{-1})^{-1} \times 1\!\!1) \circ (J_0 \times J_0) \circ (T_z\sigma^{-1} \times 1\!\!1) \\
&= j(z) \oplus J_0.
\end{aligned}
$$

Here we used the definition of $\hat J$, the fact that σ is holomorphic (see (1.5)) and

$$
(T_{(s,t,0)}\Gamma)^{-1} \circ \hat J(\tilde v(s,t)) \circ T_{(s,t,0)}\Gamma = J_0 \times J_0
$$

as an isomorphism on $T_{(s,t)}((s_0,\infty) \times S^1) \times T_0\mathbb{R}^2$. The latter holds since from (3.20) and (3.2) we obtain

$$
\begin{array}{lllllll}
T_{(s,t,0)}\Gamma\frac{\partial}{\partial s} &=& \tilde v_s(s,t) &=& -\hat J(\tilde v(s,t))\tilde v_t(s,t) &=& -\hat J(\tilde v(s,t))T_{(s,t,0)}\Gamma\frac{\partial}{\partial t}, \\
T_{(s,t,0)}\Gamma\frac{\partial}{\partial t} &=& \tilde v_t(s,t) &=& \hat J(\tilde v(s,t))\tilde v_s(s,t) &=& \hat J(\tilde v(s,t))T_{(s,t,0)}\Gamma\frac{\partial}{\partial s}, \\
T_{(s,t,0)}\Gamma\frac{\partial}{\partial x} &=& \tilde n(s,t) &=& -\hat J(\tilde v(s,t))\tilde m(s,t) &=& -\hat J(\tilde v(s,t))T_{(s,t,0)}\Gamma\frac{\partial}{\partial y}, \\
T_{(s,t,0)}\Gamma\frac{\partial}{\partial y} &=& \tilde m(s,t) &=& \hat J(\tilde v(s,t))\,\tilde n(s,t) &=& \hat J(\tilde v(s,t))T_{(s,t,0)}\Gamma\frac{\partial}{\partial x}.
\end{array}
$$

For the last claim of theorem 3.1 we first note that the convergences in (3.21) and (3.22) even hold for all derivatives D^α with respect to s and t of the left and right hand side. Thus in view of (3.20), all entries of $T_{(s,t,x,y)}\Gamma$ converge with all derivatives with repect to s and t and uniformly in t to some smooth functions as $s \to \infty$. Moreover, x and y enter $T_{(s,t,x,y)}\Gamma$ only with linear dependence. Thus there is a smooth matrix function $A(t,x,y)$ such that

$$
D^\alpha(T_{(s,t,x,y)}\Gamma - A(t,x,y)) \ \stackrel{s\to\infty}{\longrightarrow}\ 0
$$

converges uniformly on bounded sets of (t,x,y) for all derivatives D^α with respect to s, t, x and y. The same holds, as $T_{(s,t,x,y)}\Gamma$ is bounded away from the noninvertible matrices, for $(T_{(s,t,x,y)}\Gamma)^{-1}$. Finally, in the same sense as above, the asymptotic behaviour of the map Γ itself is

$$
\Gamma(s,t,x,y) \ \stackrel{s\to\infty}{\longrightarrow}\ (\ \infty\,,\ kt+d\,,\ x\,\mathrm{e}^{wtJ(t)}\hat e(t) + y\,\mathrm{e}^{wtJ(t)}\hat f(t)\ )\ .
$$

Since the almost complex structure $\hat J$ does not depend on the first coordinate a, and depends smoothly on all other components, we deduce the existence of a smooth matrix function $\hat J_\infty(t,x,y)$ such that in the above sense of convergence

$$
\hat J(\Gamma(s,t,x,y)) \ \stackrel{s\to\infty}{\longrightarrow}\ \hat J_\infty(t,x,y).
$$

This finally proves the claim since

$$
\begin{aligned}
\bar J_0 &= (T\sigma \oplus 1\!\!1_{\mathbb{R}^2})^{-1} \circ \bar J(\sigma \times 1\!\!1_{\mathbb{R}^2}) \circ (T\sigma \oplus 1\!\!1_{\mathbb{R}^2}) \\
&= (T\sigma \oplus 1\!\!1_{\mathbb{R}^2})^{-1} \circ (T\Phi)^{-1} \circ \tilde J(\Phi) \circ T\Phi \circ (T\sigma \oplus 1\!\!1_{\mathbb{R}^2}) \\
&= (T\Gamma)^{-1} \circ T\Psi \circ \tilde J(\Phi) \circ (T\Psi)^{-1} \circ T\Gamma \\
&= (T\Gamma)^{-1} \circ \hat J(\Gamma) \circ T\Gamma.
\end{aligned}
$$

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